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MATHEMATICS 223, FALL 2005
LINEAR ALGEBRA
MIDTERM EXAMINATION Solutions

1. (a) Express the following matrix A (over the complex numbers) as a product of elementary matrices. $A = \begin{pmatrix} 1+i & -2+2i \\ 3i & -4 \end{pmatrix}$.

   (b) For the above A, find A^{-1}.

   Solution: Let's do these two parts simultaneously. Start with $(A|I) = \begin{pmatrix} 1+i & -2+2i & | & 1 & 0 \\ 3i & -4 & | & 0 & 1 \end{pmatrix}$. With $E_1 = \begin{pmatrix} \frac{1}{1+i} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{1}{2}i & 0 \\ 0 & 1 \end{pmatrix}$, we have $(E_1A|E_1) = \begin{pmatrix} 1 & 2i & | & \frac{1}{2} - \frac{1}{2}i & 0 \\ 3i & -4 & | & 0 & 1 \end{pmatrix}$. Let $E_2 = \begin{pmatrix} 1 & 0 \\ -3i & 1 \end{pmatrix}$, so $(E_2E_1A|E_2E_1) = \begin{pmatrix} 1 & 2i & | & \frac{1}{2} - \frac{1}{2}i & 0 \\ 0 & 2 & | & -\frac{3}{2} - \frac{3}{2}i & 1 \end{pmatrix}$. With $E_3 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $(E_3E_2E_1A|E_3E_2E_1) = \begin{pmatrix} 1 & 2i & | & \frac{1}{2} - \frac{1}{2}i & 0 \\ 0 & 1 & | & -\frac{3}{4} - \frac{3}{4}i & \frac{1}{2} \end{pmatrix}$. Finally with $E_4 = \begin{pmatrix} 1 & -2i \\ 0 & 1 \end{pmatrix}$, $(E_4E_3E_2E_1A|E_4E_3E_2E_1) = \begin{pmatrix} 1 & 0 & | & -1+i & -i \\ 0 & 1 & | & -\frac{3}{4} - \frac{3}{4}i & \frac{1}{2} \end{pmatrix}$. $A^{-1} = \begin{pmatrix} -1+i & -i \\ -\frac{3}{4} - \frac{3}{4}i & \frac{1}{2} \end{pmatrix}$, and this is $E_4E_3E_2E_1$. So $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{pmatrix} 1+i & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2i \\ 0 & 1 \end{pmatrix}$.

   (c) Solve the system of linear equations $\begin{array}{cccc} (1+i)x_1 & + & (-2+2i)x_2 & = & -1+3i \\ 3ix_1 & + & -4x_2 & = & i \end{array}$.

   Solution: We could of course row-reduce again (or we could have carried along an extra column in the above steps), but let's just notice that the system is essentially $A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1+3i \\ i \end{pmatrix}$. We multiply both sides by A^{-1} and get $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1-4i \\ 3+\frac{7}{2}i \end{pmatrix}$.

2. Let $V = M_3(\mathcal{R})$ be the vector space of 3×3 matrices over the real numbers. For each of the following subsets of V, decide whether or not it is a subspace of V. Justify your answers briefly.

   (a) $S_1 = \{A \in V : A^T = A \text{ and } A\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \vec{0}\}$.

   Solution: This is a subspace. The zero matrix clearly satisfies both conditions. Suppose that A_1 and A_2 do; then

$(A_1+A_2)^T = A_1^T + A_2^T = A_1 + A_2$; also $A_1 \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = A_2 \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \vec{0}$, so $(A_1 + A_2) \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \vec{0} + \vec{0} = \vec{0}$. (So S_1 is closed under addition.)

Also if A is in S_1 and α is a scalar, then $(\alpha A)^T = \alpha A^T = \alpha A$ and $(\alpha A) \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \alpha(A \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \alpha(\vec{0} = \vec{0}$; so S_1 is closed under scalar multiplication.

(b) $S_3 = \{A \in V : A\vec{v} = \vec{v}$ for some nonzero vector $\vec{v}\}$.
Solution: This is not a subspace. For example, the identity matrix I is in S_2, but $2I$ is not. ($(2I)\vec{v} = 2\vec{v} \neq \vec{v}$ as long as $\vec{v} \neq \vec{0}$.)

3. Find a basis for each of the row space, column space and null space of the following matrix. What is its rank?

$$\begin{pmatrix} 1 & 0 & 3 & 1 & -4 \\ 2 & 1 & -1 & -1 & 3 \\ 6 & 2 & 4 & 0 & -2 \end{pmatrix}.$$

Solution: The row-reduction is very quick and yields the RREF matrix

$$\begin{pmatrix} 1 & 0 & 3 & 1 & -4 \\ 0 & 1 & -7 & -3 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

One basis for the row space is then $\{\begin{pmatrix} 1 & 0 & 3 & 1 & -4 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -7 & -3 & 11 \end{pmatrix}\}$.
For the column space we can use the first two columns of the original matrix; a basis is $\{\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\}$.

Using the RREF to solve the homogeneous system we get $x_1 = -3r - s + 4t$, $x_2 = 7r + 3s - 11t$, $x_3 = r$, $x_4 = s$, $x_5 = t$. A basis for the null space is $\{\begin{pmatrix} -3 \\ 7 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -11 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$.

4. Let $W_1 = Span\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}\}$ and $W_2 = Span\{\begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 6 \\ 1 \end{pmatrix}\}$ be subspaces of $\mathcal{R}^4$.

Find a basis for each of $W_1 + W_2$ and $W_1 \cap W_2$.

Solution: We row-reduce $\begin{pmatrix} 1 & 2 & | & 1 & 5 \\ 0 & 1 & | & 2 & 3 \\ 1 & 2 & | & 2 & 6 \\ 0 & 1 & | & 0 & 1 \end{pmatrix}$, getting $\begin{pmatrix} 1 & 0 & | & 0 & 2 \\ 0 & 1 & | & 0 & 1 \\ 0 & 0 & | & 1 & 1 \\ 0 & 0 & | & 0 & 0 \end{pmatrix}$.

The first three columns are independent, but not all four; a basis for $W_1 + W_2$ is then (returning to the orginal matrix) $\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}\}$.

From the modular law, we see that $dim(W_1 \cap W_2) = dim(W_1) + dim(W_2) - dim(W_1 + W_2) = 2 + 2 - 3 = 1$. From the RREF we see that the fourth column is twice the first column, plus the second, plus the third. Labelling the columns of the original matrix C_1, C_2, C_3 and C_4, we have $C_4 = 2C_1 + C_2 + C_3$, so $2C_1 + C_2 = C_4 - C_3 = \begin{pmatrix} 4 \\ 1 \\ 4 \\ 1 \end{pmatrix}$ is in the intersection. A basis for $W_1 \cap W_2$ is then $\{\begin{pmatrix} 4 \\ 1 \\ 4 \\ 1 \end{pmatrix}\}$.

5. (a) Verify that $\mathcal{B} = (1, 1 + t, (1 + t)^2, (1 + t)^3)$ is a basis for $P_3(t)$, the vector space of polynomials over the reals with degree at most 3.

   Solution: If $\alpha_1(1) + \alpha_2(1 + t) + \alpha_3(1 + t)^2 + \alpha_4(1 + t)^4 = 0$ (where the α_j's are real scalars), then $\alpha_4 = 0$ (by considering the coefficient of t^3). Then by considering the coefficient of t^2, we have $\alpha_3 = 0$; by considering the coefficient of t, we get $\alpha_2 = 0$, and finally $\alpha_1 = 0$. Only the trivial linear combination of the given polynomials yields the zero polynomial, so $\mathcal{B}$ is independent. (Or you can row-reduce the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ to get the same conclusion.) Since it has four vectors, and $P_3(t)$ is a four-dimensional space, $\mathcal{B}$ is a basis for $P_3(t)$.

   (b) Find the coordinates of f with respect to $\mathcal{B}$ (in the given order), where $f(t) = t^3$.

   Solution: If $\alpha_1(1) + \alpha_2(1 + t) + \alpha_4(1 + t)^2 + \alpha_4(1 + t)^3 = t^3$, we see that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$, $\alpha_2 + 2\alpha_3 + 3\alpha_4 = 0$, $\alpha_3 + 3\alpha_4 = 0$ and $\alpha_4 = 1$. So $\alpha_3 = -3$, $\alpha_2 = 3$ and $\alpha_1 = -1$. The coordinates are (in order) -1,3,-3 and 1. (Id est, $(f)_{\mathcal{B}} = [f]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix}$.)

6. Suppose that V is a vector space, and each of $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ and $C = \{\vec{w}_1, \ldots, \vec{w}_\ell\}$ is an independent subset of V, but $B \cup C$ is not independent.

   Letting $U = Span(B)$ and $W = Span(C)$, show that $U \cap W \neq \{\vec{0}\}$.

   Solution: If $B \cup C$ is dependent, there are scalars $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_\ell$ — not all zero — such that

$$\alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k + \beta_1\vec{w}_1 + \cdots + \beta_\ell\vec{w}_\ell = \vec{0},$$

$$\text{so } \alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k = -\beta_1\vec{w}_1 - \cdots - \beta_\ell\vec{w}_\ell.$$

   If we call this vector $\vec{u}$, then it's in U by looking at the left-hand side and it's in W by considering the right-hand side.

   If $\vec{u} = \vec{0}$, then by the independence of B, we must have $\alpha_1 = \ldots = \alpha_k = 0$. By the independence of C, we would also have $\beta_1 = \ldots = \beta_\ell = 0$. But these scalars are not all zero, so $\vec{u} \neq \vec{0}$; and $\vec{u} \in U \cap W$.