MATH 223, Linear Algebra
2005 Final Silutions

1. We row reduce the given matrix:

$$\begin{pmatrix} 1 & 1+i & 0 & 3 & 0 & -3i \\ 2i & -2+2i & 1 & 2+5i & -i & 7+2i \\ 1+i & 2i & -2i & 1-i & -2 & 1-5i \\ 3 & 3+3i & 1-i & 10-3i & -1-i & 3-8i \end{pmatrix} \sim \begin{pmatrix} 1 & 1+i & 0 & 3 & 0 & -3i \\ 0 & 0 & 1 & 2-i & -i & 1+2i \\ 0 & 0 & -2i & -2-4i & -2 & -2-2i \\ 0 & 0 & 1-i & 1-3i & -1-i & 3+i \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1+i & 0 & 3 & 0 & -3i \\ 0 & 0 & 1 & 2-i & -i & 1+2i \\ 0 & 0 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1+i & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2-i & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By looking at the pivot entries, we deduce that the first three rows of the row-reduced matrix are a basis of the row space, and that columns 1, 3, and 6 of the *original* matrix are a basis of its column space (recall that in general, the columns of the row-reduced matrix are *not* a basis–or even members–of the column space of the original matrix). As for the nullspace, we seek a basis for the solutions to the homogeneous system:

$$\begin{aligned} x_6 &= 0 \\ x_3 + (2-i)x_4 - ix_5 &= 0 \\ x_1 + (1+i)x_2 + 3x_4 &= 0 \end{aligned}$$

in which the free variables are clearly x_5, x_4, and x_2. Writing the solution space in parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = a \begin{bmatrix} 0 \\ 0 \\ i \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ i-2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1-i \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

gives a basis for the null space (i.e. the three column vectors occurring above). As a quick check, we easily compute that these vectors all satisfy the given equations.

2. Let $W = \text{Span}\{\begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}\}$ be the given subspace of $\mathcal{R}^4$. We first find a basis of $W^\perp$. Recall that $W^\perp$ is the set of all column vectors $x \in \mathcal{R}^4$ that satisfy $v^T \cdot x = 0$ for each basis vector v of W. Equivalently, $W^\perp$ is the null space of

$$\begin{pmatrix} 2 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}.$$

Row reducing this matrix yields

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & 0 \end{pmatrix}.$$

from which we deduce (in the usual way) that a basis for the nullspace–and hence $W^\perp$–is given by

$$\{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix}\}$$

To find orthonormal bases of W and $W^\perp$, we apply Gram-Schmidt to the bases we already have. For W, we find that

$$v_1' := v_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \quad v_2' := v_2 - \frac{\langle v_2, v_1' \rangle}{\langle v_1', v_1' \rangle} v_1' = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} - \frac{6}{10} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -1 \\ -3 \\ 7 \\ -1 \end{pmatrix}$$

is an orthogonal basis of W and hence, normalizing, that

$$\frac{1}{\|v_1'\|} v_1' = \frac{1}{\sqrt{10}} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \quad \frac{1}{\|v_2'\|} v_2' = \frac{1}{\sqrt{60}} \begin{pmatrix} -1 \\ -3 \\ 7 \\ -1 \end{pmatrix}$$

is an orthonormal basis for W. Applying the same method to $W^\perp$, we find that

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{12}} \begin{pmatrix} 1 \\ -3 \\ -1 \\ 1 \end{pmatrix}$$

is an orthonormal basis for $W^\perp$.

To find the projections $\text{proj}_W(v)$ and $\text{perp}_W(V) := \text{proj}_{W^\perp}(v)$, we recall that if w_1, w_2 is an orthonormal basis of W then

$$\text{proj}_W(v) = \langle v, w_1 \rangle w_1 + \langle v, w_2 \rangle w_2.$$

Using our basis for W determined above, gives

$$\begin{aligned} \text{proj}_W(v) &= \left(\begin{pmatrix} 0 \\ 4 \\ 2 \\ 2 \end{pmatrix} \cdot \frac{1}{\sqrt{10}} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right) \frac{1}{\sqrt{10}} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} + \left(\begin{pmatrix} 0 \\ 4 \\ 2 \\ 2 \end{pmatrix} \cdot \frac{1}{\sqrt{60}} \begin{pmatrix} -1 \\ -3 \\ 7 \\ -1 \end{pmatrix} \right) \frac{1}{\sqrt{60}} \begin{pmatrix} -1 \\ -3 \\ 7 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}. \end{aligned}$$

Using that (for any v) we have $v = \text{proj}_W(v) + \text{proj}_{W^\perp}(v)$ (by the definition of projection), we conclude that

$$\text{proj}_{W^\perp}(v) = v - \text{proj}_W(v) = \begin{pmatrix} 0 \\ 4 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

3.

1. To find a basis for the kernel of T and for the image of T we first find the matrix of T with respect to the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of $P_3(t)$. From the formula for T, we compute
$$T(1) = 3 \quad T(t) = 0 \quad T(t^2) = -t^2 \quad T(t^3) = 0,$$
and hence
$$M := [T]_{\mathcal{B}} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
We easily read off bases for the kernel and image of M:
$$\ker(M) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \operatorname{im}(M) = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$$
and since these are the coordinate vectors (with respect to $\mathcal{B}$) of bases of the kernel and image of T (respectively), we find that $\ker(T)$ has basis $\{t, t^3\}$ and that $\operatorname{im}(T)$ has basis $\{3, -t^2\}$ (equivalently $\{1, t^2\}$).

2. We found the matrix $[T]_{\mathcal{B}}$ in the first part. To find $[T]_{\mathcal{C}}$ for the nonstandard ordered basis $\mathcal{C} = (1, 1+t, 1+t+t^2, 1+t+t^2+t^3)$, we use linearity of T and our computations above to determine that
$$\begin{aligned} &T(1) = 3 \\ &T(1+t) = 3 \\ &T(1+t+t^2) = 3 - t^2 = 3 + (1+t) - (1+t+t^2) \\ &T(1+t+t^2+t^3) = 3 - t^2 = 3 + (1+t) - (1+t+t^2), \end{aligned}$$
and hence that
$$[T]_{\mathcal{C}} = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. Let A and B are similar $n \times n$ matrices with entries in a field K. By hypothesis, there is an invertible matrix P such that $B = PAP^{-1}$. Setting $V = K^n$ and viewing A and B as linear transformations on V, we claim that for any scalar $\lambda \in K$, the invertible linear transformation $P : V \to V$ carries the λ-eigenspace of A into the λ-eigenspace of B (these spaces may, or course, be zero). Indeed, if $v \in V$ is an eigenvector of A with eigenvalue λ then
$$B(Pv) = PAP^{-1}(Pv) = PAv = P\lambda v = \lambda(Pv)$$

so Pv is an eigenvector of B with eigenvalue λ. Since similarity is symmetric, the same argument shows that P^{-1} carries the λ-eigenspace of B in to the λ-eigenspace of A. Since $PP^{-1} = P^{-1}P$ is the identity, it follows that the λ-eigenspaces of A and B are isomorphic (via P). This gives the claim.

5. We first present the "usual" solution to this type of problem. Let $A = \begin{pmatrix} 8 & 18 & 8 \\ -4 & -9 & -4 \\ 1 & 2 & 1 \end{pmatrix}$. We diagonalize A. The characteristic polynomial is easily found to be $t^3 - t = t(t-1)(t+1)$. To determine each eigenspace, we compute:

$$E_0 = \text{nullsp} \begin{pmatrix} 8 & 18 & 8 \\ -4 & -9 & -4 \\ 1 & 2 & 1 \end{pmatrix} = \text{nullsp} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\},$$

$$E_1 = \text{nullsp} \begin{pmatrix} 7 & 18 & 8 \\ -4 & -10 & -4 \\ 1 & 2 & 0 \end{pmatrix} = \text{nullsp} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \right\},$$

$$E_{-1} = \text{nullsp} \begin{pmatrix} 9 & 18 & 8 \\ -4 & -8 & -4 \\ 1 & 2 & 2 \end{pmatrix} = \text{nullsp} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

Thus, setting

$$P := \begin{pmatrix} 1 & 4 & 2 \\ 0 & -2 & -1 \\ -1 & 1 & 0 \end{pmatrix},$$

we know that

$$P^{-1}AP = D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and hence that

$$(P^{-1}AP)^{25} = D^{25} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

But since $(P^{-1}AP)^{25} = P^{-1}A^{25}P$, we deduce that

$$A^{25} = PD^{25}P^{-1} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & -2 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ -2 & -5 & -2 \end{pmatrix} = \begin{pmatrix} 8 & 18 & 8 \\ -4 & -9 & -4 \\ 1 & 2 & 1 \end{pmatrix},$$

where the inverse of P was computed using any method of your choice (e.g. by viewing P as a change of basis matrix from the standard basis to the basis given by the eigenvectors of A, and then by determining the inverse change of basis matrix).

We could have been *far* more slick in this particular problem. Note that since the characteristic polynomial of A is $t^3 - t$ and since this polynomial divides $t^{25} - t$ (the latter clearly has roots 0, 1, -1, which forces such divisibility), we deduce that $A^{25} - A = 0$ since already $A^3 - A = 0$ by Cayley-Hamilton. Thus, $A^{25} = A$.

6. Let V be the vector space of all continuous real-valued functions on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. For $f, g \in V$, define $< f, g >= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(x) f(x) g(x) dx$.

1. To see that this gives a real inner product on V, we must show that it is symmetric, linear in the first coordinate, and positive definite. If $f, g \in V$, we clearly have $f(x)g(x) = g(x)f(x)$ as multiplication in $\mathcal{R}$ is commutative, so
$$\langle f, g\rangle = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(x) f(x) g(x) dx = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(x) g(x) f(x) dx = \langle g, f\rangle,$$
so we have symmetry. For linearity in the first variable, let $\alpha \in \mathcal{R}$; then
$$\begin{aligned} \langle \alpha f_1 + f_2, g\rangle &= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(x)(\alpha f_1(x) + f_2) g(x) dx \\ &= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \alpha \cos(x) f_1(x) g(x) + \cos(x) f_2(x) g(x) dx \\ &= \alpha \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(x) f_1(x) g(x) dx + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(x) f_2(x) g(x) dx \\ &= \alpha \langle f_1, g\rangle + \langle f_2, g\rangle \end{aligned}$$
where we have used linearity of the definite integral in the third equality. As for positive-definiteness, we have
$$\langle f, f\rangle = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(x) f(x)^2 dx,$$
and since $f(x)^2$ and $\cos(x)$ are nonnegative on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the value of the integral must be nonnegative also. Moreover, since the integrand is nonnegative, this integral is zero if and only if $\cos(x) f(x)^2 = 0$ for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, i.e. if and only if $f(x) = 0$ for all such x. Positive-definiteness now follows.

2. Now let $f \in V$ be arbitrary, and set $g(x) = 1$ for all x. By Cauchy-Schwartz, we have
$$\langle f, g\rangle^2 \leq \langle f, f\rangle \langle g, g\rangle.$$
Using the definition of the inner product and the function g, we deduce that
$$\left(\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(x) f(x) dx \right)^2 \leq \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(x) f(x)^2 dx \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx.$$
The last integral on the right is equal to 2, so the desired inequality follows.

7. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ have determinant 2+i, and $B = \begin{pmatrix} k & \ell & m \\ d & e & f \\ g & h & j \end{pmatrix}$ have determinant 4-i. Since the determinant remains unchanged if we add a scalar multiple of any column to any other column, we

have (adding 1/3 or the third column to the second column)

$$\det\begin{pmatrix} a+2ik & d-g & 3g \\ b+2i\ell & e-h & 3h \\ c+2im & f-j & 3j \end{pmatrix} = \det\begin{pmatrix} a+2ik & d & 3g \\ b+2i\ell & e & 3h \\ c+2im & f & 3j \end{pmatrix}.$$

By linearity of the determinant in the first column, the above determinant is equal to

$$\det\begin{pmatrix} a & d & 3g \\ b & e & 3h \\ c & f & 3j \end{pmatrix} + \det\begin{pmatrix} 2ik & d & 3g \\ 2i\ell & e & 3h \\ 2im & f & 3j \end{pmatrix}.$$

Using linearity again, this is equal to

$$3\det\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & j \end{pmatrix} + 6i\det\begin{pmatrix} k & d & g \\ \ell & e & h \\ m & f & j \end{pmatrix} = 3\det A^T + 6i\det B^T.$$

Finally, since the determinant of the transpose of a matrix is equal to the determinant of that matrix, the above is equal to

$$3(2+i) + 6i(4-i) = 12 + 27i.$$

8. Let

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

Then if $x = (x_1, x_2, x_3)$ we have $xAx^T = Q(x)$ for the first quadratic form and $xBx^T = Q(x)$ for the second. Since A and B are symmetric, they can be diagonalized, so we can change variables to diagonalize both quadratic forms. Just computing eigenvalues, we know that a change of coordinates will bring the first quadratic form to the diagonal form $-y_1^2 + 2y_2^2 + 3y_3^2$ and the second quadratic form to $6y_2^2 + 3y_3^2$. Thus, the shape of $Q = 1$ for the first form is of a hyperboloid with one sheet, and the shape of $Q = 1$ in the second case is of a cylinder with elliptical cross-section.