

1. Find a basis for each of the row space, the column space, and the null space of the following matrix. What is its rank?

$$\begin{pmatrix} 1 & 1+i & 0 & 3 & 0 & -3i \\ 2i & -2+2i & 1 & 2+5i & -i & 7+2i \\ 1+i & 2i & -2i & 1-i & -2 & 1-5i \\ 3 & 3+3i & 1-i & 10-3i & -1-i & 3-8i \end{pmatrix}.$$

2. Let  $W = \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}\right\}$  be a subspace of  $\mathcal{R}^4$ . Find an orthonormal basis for each of  $W$  and  $W^\perp$ . Also find the projections  $\text{proj}_W \vec{v}$  and  $\text{perp}_W \vec{v}$  of  $\vec{v}$  onto  $W$  and onto  $W^\perp$ , where  $\vec{v} = \begin{pmatrix} 0 \\ 4 \\ 2 \\ 2 \end{pmatrix}$ .

3. Let  $P_3(t)$  be the vector space of polynomials over the reals with degree at most three. Let  $T : P_3(t) \rightarrow P_3(t)$  be defined by  $Tf(t) = t^2 f''(t) - 3t f'(t) + 3f(t)$ .

- (a) Find a basis for the kernel of  $T$  and for the image (range) of  $T$ .  
 (b) Find the matrix  $[T]_{\mathcal{B}}$  of  $T$  with respect to the standard ordered basis  $\mathcal{B}$  of  $P_3(t)$ . Do the same (i.e., find  $[T]_{\mathcal{C}}$ ) for the nonstandard ordered basis  $\mathcal{C} = (1, 1+t, 1+t+t^2, 1+t+t^2+t^3)$ .

4. Suppose that  $A$  and  $B$  are similar matrices, and  $\lambda$  is an eigenvalue of  $A$ . Show that  $\lambda$  is also an eigenvalue of  $B$ , and that the dimensions of the corresponding eigenspaces are the same for  $A$  and  $B$ . [Hint?]

5. Let  $A = \begin{pmatrix} 8 & 18 & 8 \\ -4 & -9 & -4 \\ 1 & 2 & 1 \end{pmatrix}$ . Find (explicitly)  $A^{25}$ . Maybe first find  $P$  such that  $P^{-1}AP$  is diagonal.

6. Let  $V$  be the vector space of all continuous real-valued functions on the interval  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ . For  $f, g \in V$ , define  $\langle f, g \rangle = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x f(x) g(x) dx$ .

- (a) Verify that this gives an inner product on  $V$ .  
 (b) Show that, for any  $f \in V$ ,  $(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x f(x) dx)^2 \leq 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x f(x)^2 dx$ .  
 [Hint: Let  $g(x) = 1$  for all  $x$ .]

7. Let  $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$  have determinant  $2+i$ , and  $B = \begin{pmatrix} k & \ell & m \\ d & e & f \\ g & h & j \end{pmatrix}$  have determinant  $4-i$ . Find the determinant of  $\begin{pmatrix} a + 2ik & d - g & 3g \\ b + 2i\ell & e - h & 3h \\ c + 2im & f - j & 3j \end{pmatrix}$ .

8. For each of the following quadratic forms  $Q(x_1, x_2, x_3)$ , find a symmetric matrix  $A$  such that  $Q(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$  and identify the shape of the graph of  $Q(x_1, x_2, x_3) = 1$ .

(a)  $Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 4x_1x_3 + x_3^2$ ;

(b)  $Q(x_1, x_2, x_3) = 3x_1^2 + 4x_1x_2 + 2x_2^2 - 4x_1x_3 + 4x_3^2$ .