MATH 223, Linear Algebra
Fall 2004
Solution to the Midterm exam

1. (a) Find all solutions in the complex numbers to the equation $z^3 = \frac{1}{8}$.

   Solution: Clearly $\frac{1}{2}$ is the only solution in the real numbers. But there are two more in the complex numbers. They are $\frac{1}{2}e^{\frac{2\pi i}{3}} = \frac{1}{2}(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -\frac{1}{4} + i\frac{\sqrt{3}}{4}$ and $\frac{1}{2}e^{\frac{4\pi i}{3}} = \frac{1}{2}(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = -\frac{1}{4} - i\frac{\sqrt{3}}{4}$. This is because $(e^{\frac{2\pi i}{3}})^3 = (e^{\frac{4\pi i}{3}})^3 = e^{2\pi i} = 1$. Another way to see this is to factor $z^3 - \frac{1}{8}$ as $(z - \frac{1}{2})(z^2 + \frac{1}{2}z + \frac{1}{4})$ and to complete the square to find the roots of $z^2 + \frac{1}{2}z + \frac{1}{4}$. This quadratic factors as $(z - (-\frac{1}{4} - i\frac{\sqrt{3}}{4}))(z - (-\frac{1}{4} + i\frac{\sqrt{3}}{4}))$ and we get the roots that way.

   (b) Let $A = \begin{pmatrix} 3-i & \frac{1}{1+i} \\ 2i & 2 \end{pmatrix}$. Show that A is invertible; let $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the largest and smallest of the numbers $|a|$, $|b|$, $|c|$ and $|d|$.

   Solution: First notice that $\frac{1}{1+i} = (\frac{1}{1+i})(\frac{1-i}{1-i}) = \frac{1-i}{2}$. Now the determinant of A is $(3-i)(2) - (\frac{1-i}{2})(2i) = 5 - 3i \neq 0$, so the matrix is invertible. Its inverse is $A^{-1} = \frac{1}{5-3i}\begin{pmatrix} 2 & -\frac{1-i}{2} \\ -2i & 3-i \end{pmatrix}$ and $|a| = |\frac{2}{5-3i}| = \frac{2}{\sqrt{34}}$, $|b| = |\frac{-1+i}{10-6i}| = \frac{2}{\sqrt{68}}$, $|c| = |\frac{-2i}{5-3i}| = \frac{2}{\sqrt{34}}$ and $|d| = |\frac{3-i}{3-5i}| = \frac{\sqrt{10}}{\sqrt{34}}$. $|b|$ is the smallest of these and $|d|$ is the largest.

   (c) Solve the system of linear equations $\begin{array}{rcrcl} (3-i)x_1 & + & \frac{1}{1+i}x_2 & = & 2 \\ 2ix_1 & + & 2x_2 & = & 1-i \end{array}$.

   Solution: It's usually easier to do this by row-reduction, but since we have the inverse handy we just find $A^{-1}\vec{v} = \frac{1}{5-3i}\begin{pmatrix} 2 & -\frac{1-i}{2} \\ -2i & 3-i \end{pmatrix}\begin{pmatrix} 2 \\ 1-i \end{pmatrix} = \frac{1}{5-3i}\begin{pmatrix} 4+i \\ 2-8i \end{pmatrix} = \begin{pmatrix} \frac{1+i}{2} \\ 1-i \end{pmatrix}$. $x_1 = \frac{1+i}{2}$ and $x_2 = 1 - i$.

2. Let $A = \begin{pmatrix} 0 & 0 & 2 & 6 & -1 \\ -1 & -2 & 1 & 4 & -1 \\ 0 & 0 & 1 & 3 & 0 \\ 2 & 4 & 0 & -2 & 0 \end{pmatrix}$.

   (a) Find a basis for each of the column space, row space, and null space of A.

   Solution: We row-reduce the augmented matrix $(A|\vec{b})$ where $\vec{b} =$

$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ to get (using $R_4 \mapsto R_4 + 2R_2$ and $R_1 \mapsto R_1 - 2R_3$)

$$\left(\begin{array}{ccccccc} 0 & 0 & 0 & 0 & -1 & | & b_1 - 2b_3 \\ -1 & -2 & 1 & 4 & -1 & | & b_2 \\ 0 & 0 & 1 & 3 & 0 & | & b_3 \\ 0 & 0 & 2 & 6 & -2 & | & b_4 + 2b_2 \end{array}\right).$$

Next we do $R_4 \mapsto R_4 - 2R_3$, $R_2 \mapsto R_2 - R_3$ and $R_1 \mapsto -R_1$ to get $\left(\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 1 & | & -b_1 + 2b_2 \\ -1 & -2 & 0 & 1 & -1 & | & b_2 - b_3 \\ 0 & 0 & 1 & 3 & 0 & | & b_3 \\ 0 & 0 & 0 & 0 & -2 & | & b_4 + 2b_2 - 2b_3 \end{array}\right)$. Finally, we do $R_4 \mapsto R_2 + 2R_1$, $R_2 \mapsto R_2 + R_1$, then $R_2 \mapsto -R_2$ and switch the rows to get the RREF form. (With the extra column) it is $\left(\begin{array}{ccccccc} 1 & 2 & 0 & -1 & 0 & | & b_1 - b_2 - b_3 \\ 0 & 0 & 1 & 3 & 0 & | & b_3 \\ 0 & 0 & 0 & 0 & 1 & | & -b_1 + 2b_3 \\ 0 & 0 & 0 & 0 & 0 & | & b_4 - 2b_1 + 2b_2 + 2b_3 \end{array}\right)$. Ignoring the extra column for now, we find that a basis for the row space is $\{\begin{pmatrix} 1 & 2 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}\}$. Looking at the corresponding columns of the original matrix, we see that a basis for the column space is $\{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}\}$. For the null space we have $x_1 + 2x_2 - x_4 = 0$, $x_3 + 3x_4 = 0$ and $x_5 = 0$ Setting $x_4 = t$ and $x_2 = s$, an arbitrary element of the null space is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}$. A basis for the null space is then $\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}\}$.

(b) Find a system of linear equations (possibly just a single equation) such that the set of solutions to the system is just the column space of A.

Solution: This is what the extra vector $\vec{b}$ was for. A vector $\vec{b}$ is in the column space if and only if you can solve the system $A\vec{x} = \vec{b}$. From the reduced matrix, it is clear that this occurs if and only if $b_4 - 2b_1 + 2b_3 + 2b_2 = 0$; this is the equation we seek. (You can easily directly check that all the columns of A satisfy this equation.)

3. (a) Let V be the vector space K^3 over K, where K is either $\mathcal{R}$ (the reals) or $\mathcal{C}$ (the complexes) as the case may be. For each subset $U \subseteq V$, determine whether or not U is a subspace of V; justify your answers.

   i. $U = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathcal{R}, xz = 0\}$; $K = \mathcal{R}$.

   Solution: This is not a subspace, because it is not closed under addition. For instance, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are in U, but their sum $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ isn't.

   ii. $U = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathcal{C}, x^2 + y^2 + z^2 = 0\}$; $K = \mathcal{C}$.

   Solution: Again, this is not a subspace, and for basically the same reason. $\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$ are both in U, but their sum $\begin{pmatrix} 1 \\ 1+i \\ i \end{pmatrix}$ is not. $(1^2 + (1+i)^2 + i^2 = 2i \neq 0.)$

   iii. $U = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathcal{R}, x^2 + y^2 + z^2 = 0\}$; $K = \mathcal{R}$.

   Solution: The only *real* vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfying $x^2 + y^2 + z^2 = 0$ is the zero vector; we must have $x = y = z = 0$. So $U = \{\vec{0}\}$ which is the trivial subspace. (Incidentally, yes it IS a subspace.)

   (b) Let V be a finite-dimensional vector space and $U \subseteq V$ a subspace of V. Show that $dim(U) \leq dim(V)$. Show further that if $dim(U) = dim(V)$, then $U = V$.

   Solution: Suppose that $\{\vec{u}_1, \ldots, \vec{u}_m\} = B$ is a basis for U, so that U is m-dimensional. This set of vectors in V is independent, and thus can be expanded to a basis for V, which therefore has a basis with at least m elements. That is, $dim(V) \geq dim(U)$.

In case the dimensions are equal, we don't need to (and *can't*) add any more vectors to B to get a basis for V, so $Span(B) = V$. But $Span(B)$ is U, so $U = V$.

4. Let $V = P_2(t)$, the space of polynomials of degree ≤ 2. Then $B = (p_1, p_2, p_3)$ is an ordered basis for V, where $p_1(t) = (t-1)^2$, $p_2(t) = t-1$ and $p_3(t) = 1$. Let $q_1(t) = (t+1)^2$, $q_2(t) = (t-1)^2 + t$ and $q_3(t) = t-1$.

   (a) Show that $B' = (q_1, q_2, q_3)$ is also a basis for V, and find the change of basis matrix from B to B' and from B' to B.

   Solution: Since B' has the right number (three) of vectors, we just need to check that it's independent. But if $a_1q_1(t) + a_2q_2(t) + a_3q_3(t) = 0$ — where the zero on the right refers to the zero polynomial — then $(a_1+a_2)t^2 + (2a_1-a_2+a_3)t + (a_1+a_2-a_3) = 0$. This is for *every* t; so we must have $a_1+a_2 = 2a_1-a_2+a_3 = a_1+a_2-a_3 = 0$. Adding the last two equations, we see that $3a_1 = 0$, so $a_1 = 0$ and from the first equation $a_2 = 0$ and then from either of the last two equations, a_3 has to be 0, too. Thus no nontrivial linear combo of q_1, q_2 and q_3 can be zero; thus B' is independent, and hence a basis of $P_2(t)$.

   $q_1(t) = ((t-1)+2)^2 = (t-1)^2 + 4(t-1) + 4$, so the first column of what I call ${}_BP_{B'}$ (the change-of-basis matrix from B to B') has 1,4 and 4 in it (in that order). $q_2(t) = 1(t-1)^2 + 1(t-1) + 1$, so the second column is all 1's. And the third column is 0,1,0. The matrix is ${}_BP_{B'} = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 1 & 1 \\ 4 & 1 & 0 \end{pmatrix}$. ${}_{B'}P_B$ (the change-of -basis matrix from B' to B) is the inverse of this; it is $\begin{pmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ \frac{4}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & -1 \end{pmatrix}$. [This can also be found directly.]

   (b) Compute $[p]_{B'}$, where $p(t) = a_0 + a_1(t-1) + a_2(t-1)^2$.

   Solution: $[p]_B = \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix}$, so $[p]_{B'} =_{B'} P_B[p]_B = \begin{pmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ \frac{4}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} =$ $\begin{pmatrix} -\frac{1}{3}a_2 + \frac{1}{3}a_0 \\ \frac{4}{3}a_2 - \frac{1}{3}a_0 \\ a_1 - a_0 \end{pmatrix}$.

5. Let $V = M_{2,2}(\mathcal{R})$ be the vector space of real 2×2 matrices. Let $C = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. Define the transformation $L : V \longrightarrow V$ by $L(A) = CA - 2A^T$, where A^T is the transpose of A.

(a) Show that L is linear.

Solution: The particular numbers in C are irrelevant here.

$$T(A_1+A_2) = C(A_1+A_2)-2(A_1+A_2)^T = CA_1+CA_2-2A_1^T-2A_2^T = T(A_1)+T(A_2)$$

for any 2×2 matrices A_1 and A_2; also for any 2×2 A and scalar r, $T(rA) = C(rA) - 2(rA)^T = r(CA - 2A^T) = rTA$. This verifies that T is linear.

(b) Let $B = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$; you may assume that B is a basis for V. Find $[L]_B$.

Solution: First note that the space is 4-dimensional, so $[L]_B$ will be a 4×4 matrix. If we label the matrices in B as E_1, E_2, E_3 and E_4 (in that order) we have $LE_1 = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0E_1+0E_2+1E_3+0E_4$. Similarly, $LE_2 = \begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix} = 0E_1+2E_2-2E_3+1E_4$, $LE_3 = \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix} = -1E_1-2E_2+1E_3+0E_4$ and $LE_4 = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} = 0E_1-1E_2+0E_3-1E_4$ and thus $[L]_B$ is $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$.

(c) Is L singular or nonsingular? Justify your answer.

Solution: The determinant of $[L]_B$ is $1det\begin{pmatrix} 0 & -1 & 0 \\ 2 & -2 & -1 \\ 1 & 0 & -1 \end{pmatrix}$ (expanding down the first column), and expanding along the first row, this is $1(-(-1))det\begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$, which $-1 \neq 0$. The matrix is nonsingular, so L is, too.

6. (a) Suppose that $det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 5$ and $det\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} = -2$. Find $det\begin{pmatrix} 3a+2a' & 2c-9a-6a' \\ 3b+2b' & 2d-9b-6b' \end{pmatrix}$.

Solution: First we take the transpose of that last matrix; this doesn't affect the determinant. $A^T = \begin{pmatrix} 3a+2a' & 3b+2b' \\ 2c-9a-6a' & 2d-9b-6b' \end{pmatrix}$. Next we perform the row operation $R_2 \mapsto R_2 + 3R_1$, again with no change of determinant. We get $\begin{pmatrix} 3a+2a' & 3b+2b' \\ 2c & 2d \end{pmatrix}$. The determinant of this baby is $det\begin{pmatrix} 3a & 3b \\ 2c & 2d \end{pmatrix} + \begin{pmatrix} 2a' & 2b' \\ 2c & 2d \end{pmatrix}$. The first of *these* determinants is (3)(2)(5) and of the second is (2)(2)(−2), (The first of these comes from factoring 3 from the first row, and 2 from the second row; the other one is similar.) As 30-8=22, the determinant called for is 22.

(b) Let A be a 4×4 matrix of the form $\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, where B, C and D are 2×2 matrices, and 0 represents the 2×2 zero matrix. Show that $det(A) = det(B)det(D)$.

Solution: This can be done directly by expanding down the first column, and then calculating the two 3×3 determinants you need by expanding down *their* first columns. Tedious, but effective.

The right way to look at this, though, is in terms of row operations.If $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $a \neq 0$, we turn B into an upper triangular matrix by a single row operation $R_2 \mapsto R_2 - \frac{c}{a}R_1$, not affecting he determinant. Performing the same operation on A again doesn't change the determinant and produces $\begin{pmatrix} B' & C \\ 0 & D \end{pmatrix}$, where $B' = \begin{pmatrix} a & b \\ 0 & d' \end{pmatrix}$. In case $a = 0$ and $c \neq 0$, we have to do a row-switch to both B and A to get the same kind of result. (Changing the sign of both determinants.) If $a = c = 0$, of course both $det(A)$ and $det(B)$ are zero.

Next, if $D = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and $e \neq 0$, we row-reduce D to get $D' = \begin{pmatrix} e & f \\ 0 & h' \end{pmatrix}$; using the corresponding row-operation on A doesn't change *its* determinant. (Again, if $e = 0$ but $g \neq 0$ we do a row-switch on both D and A.) After we do this we have upper-triangular matrices A', B' and D' and it is clear that $det(A') = ad'eh' = det(B')det(D')$.

(Small point: if $e = g = 0$ then both $det(A)$ and $det(D)$ are zero.) The reason this is the right way to look at this problem is that it generalizes — if B is an $n \times n$ matrix, D an $m \times m$ matrix, C an $n \times m$ matrix and we just write 0 for the $m \times n$ zero matrix, then $\begin{pmatrix} B & C \\ 0 & D \end{pmatrix} = det(B)det(D)$; essentially the same reasoning applies.