MATH 223, Linear Algebra
Fall 2004
Midterm exam, Wednesday, October 20, 2004

Instructions: No notes, books or calculators permitted.

This exam has six questions. All questions carry the same weight.

Do all your work on the sheets provided. Do not separate sheets that have been stapled together.

The questions have been divided into two parts, purely to facilitate marking. Make sure you have a "white" and a "blue" set of questions. Make sure that your name, student number, and section number are on both parts. (If your instructor is Jim Loveys, you are in section 1; if your instructor is Peter Russell, you are in section 2.)

1. (a) Find all solutions in the complex numbers to the equation $z^3 = \frac{1}{8}$.

(b) Let $A = \begin{pmatrix} 3-i & \frac{1}{1+i} \\ 2i & 2 \end{pmatrix}$. Show that A is invertible; let $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the largest and smallest of the numbers $|a|$, $|b|$, $|c|$ and $|d|$.

(c) Solve the system of linear equations $\begin{array}{rcrcl} (3-i)x_1 & + & \frac{1}{1+i}x_2 & = & 2 \\ 2ix_1 & + & 2x_2 & = & 1-i \end{array}$.

2. Let $A = \begin{pmatrix} 0 & 0 & 2 & 6 & -1 \\ -1 & -2 & 1 & 4 & -1 \\ 0 & 0 & 1 & 3 & 0 \\ 2 & 4 & 0 & -2 & 0 \end{pmatrix}$.

(a) Find a basis for each of the column space, row space, and null space of A.

(b) Find a system of linear equations (possibly just a single equation) such that the set of solutions to the system is just the column space of A.

3. (a) Let V be the vector space K^3 over K, where K is either $\mathcal{R}$ (the reals) or $\mathcal{C}$ (the complexes) as the case may be. For each subset $U \subseteq V$, determine whether or not U is a subspace of V; justify your answers.

i. $U = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathcal{R}, xz = 0\}$; $K = \mathcal{R}$.

ii. $U = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathcal{C}, x^2 + y^2 + z^2 = 0\}$; $K = \mathcal{C}$.

iii. $U = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathcal{R}, x^2 + y^2 + z^2 = 0\}$; $K = \mathcal{R}$.

(b) Let V be a finite-dimensional vector space and $U \subseteq V$ a subspace of V. Show that $dim(U) \leq dim(V)$. Show further that if $dim(U) = dim(V)$, then $U = V$.

4. Let $V = P_2(t)$, the space of polynomials of degree ≤ 2. Then $B = (p_1, p_2, p_3)$ is an ordered basis for V, where $p_1(t) = (t-1)^2$, $p_2(t) = t-1$ and $p_3(t) = 1$. Let $q_1(t) = (t+1)^2$, $q_2(t) = (t-1)^2 + t$ and $q_3(t) = t - 1$.

(a) Show that $B' = (q_1, q_2, q_3)$ is also a basis for V, and find the change of basis matrix from B to B' and from B' to B.

(b) Compute $[p]_{B'}$, where $p(t) = a_0 + a_1(t-1) + a_2(t-1)^2$.

5. Let $V = M_{2,2}(\mathcal{R})$ be the vector space of real 2×2 matrices. Let $C = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. Define the transformation $L : V \longrightarrow V$ by
$L(A) = CA - 2A^T$, where A^T is the transpose of A.

(a) Show that L is linear.

(b) Let $B = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$; you may assume that B is a basis for V. Find $[L]_B$.

(c) Is L singular or nonsingular? Justify your answer.

6. (a) Suppose that $det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 5$ and $det\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} = -2$. Find $det\begin{pmatrix} 3a+2a' & 2c-9a-6a' \\ 3b+2b' & 2d-9b-6b' \end{pmatrix}$.

(b) Let A be a 4×4 matrix of the form $\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, where B, C and D are 2×2 matrices, and 0 represents the 2×2 zero matrix. Show that $det(A) = det(B)det(D)$.