MATH 223, Linear Algebra, Fall 2004, Final Examination Questions

For your convenience, on the two sides of this sheet, we list all the problems on this exam. You may keep this sheet, but do not write anything on it during the exam. All the problems are of equal weight.

1. Let $A = \begin{pmatrix} 1 & 1 & 5 & -2 & 1 \\ 2 & 2 & 10 & -3 & 3 \\ 4 & 4 & 20 & -9 & 3 \end{pmatrix}$.
   (a) Find a basis for each of the row space, the column space, and the null space of A.
   (b) Find an invertible matrix Q such that QA is in reduced row-echelon form.

2. For each of the following matrices A, find the characteristic polynomial χ_A and the minimal polynomial min_A. Find the eigenvalues, and a basis for each eigenspace. Decide in each case whether the matrix is diagonalizable over the reals. $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 0 \end{pmatrix}$; $A = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$; $A = \begin{pmatrix} 6 & 0 & 12 \\ 0 & 4 & 0 \\ -3 & 0 & -6 \end{pmatrix}$.

3. Let V be the vector space $P_5(t)$ of polynomials over the reals of degree ≤ 5. Define $T : V \longrightarrow V$ by

$$T(p(t)) = t^2 p''(t) - 2tp'(t) + 2p(t).$$

   (a) Show that T is linear.
   (b) Show that each of 1, t, t^2, t^3, t^4 and t^5 is an eigenvector of T. In each case, give the corresponding eigenvalue.
   (c) Find a basis for the kernel $ker(T)$ and the dimension $dim(Im(T))$ of the image of T.

4. Suppose that V is a finite-dimensional vector space, and $T : V \longrightarrow V$ is a linear operator on V such that $T^2 = T$, but T is not the zero operator or the identity operator.
   (a) Give an example of such a T. (You may let $V = \mathcal{R}^2$.)
   (b) Show that the minimal polynomial of T is $t^2 - t$. Show that T is diagonalizable.
   (c) Letting W_0 be the eigenspace corresponding to 0, and W_1 the eigenspace corresponding to 1, show that $V = W_0 \oplus W_1$; show also that $ker(T) = W_0$ and $Im(T) = W_1$.

(d) In case $dim(V) = 3$, list all possible diagonal matrices which are $[T]_B$ for some ordered basis B of V.

5. (a) Find
$$det \begin{pmatrix} a & -1 & 0 & 0 \\ -1 & a & -1 & 0 \\ 0 & -1 & a & -1 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

   (b) For which (complex) values of a is this matrix singular?

6. In this problem we use the standard Hermitian inner product on $\mathcal{C}^3$. We let $U \subseteq \mathcal{C}^3$ be the subspace spanned by $\{\vec{u}_1, \vec{u}_2\}$, where $\vec{u}_1 = \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix}$.

   (a) Find an orthonormal basis for U.

   (b) If $\vec{v} = \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}$, find the vector $\vec{u}$ in U such that the norm $||\vec{v} - \vec{u}||$ is as small as possible. What is this norm?

7. Let $V = \mathcal{R}^n$ be given the standard inner product — i.e., the dot product.

   (a) State the Cauchy-Schwarz inequality for this inner product space.

   (b) If $n \geq 2$ and $x_1, \ldots, x_n \in \mathcal{R}$, show that
$$\sum_{1 \leq j < k \leq n} x_j x_k \leq \frac{n-1}{2} \sum_{\ell=1}^{n} x_\ell^2.$$

   [Hint: Apply your statement in part (a) with one vector having all its entries 1.]

8. (a) Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.

   (b) Let $E = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathcal{R}^3 : x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 = 1\}$. Find an orthonormal basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathcal{R}^3$ and real numbers λ_1,λ_2 and λ_3 such that $E = \{y_1\vec{v}_1 + y_2\vec{v}_2 + y_3\vec{v}_3 : \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1\}$.