MATH 223, Linear Algebra, Fall 2004, Final Examination Questions

For your convenience, on the two sides of this sheet, we list all the problems on this exam. You may keep this sheet, but do not write anything on it during the exam. All the problems are of equal weight.

1. Let $A = \begin{pmatrix} 1 & 1 & 5 & -2 & 1 \\ 2 & 2 & 10 & -3 & 3 \\ 4 & 4 & 20 & -9 & 3 \end{pmatrix}$.

   (a) Find a basis for each of the row space, the column space, and the null space of A.

   (b) Find an invertible matrix Q such that QA is in reduced row-echelon form.

Solution: We row-reduce A, keeping track of what we're doing using elementary matrices. With $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we have $E_1A = \begin{pmatrix} 1 & 1 & 5 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 4 & 4 & 20 & -9 & 3 \end{pmatrix}$. With $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -4 & 0 & 1 \end{pmatrix}$, $E_2E_1A = \begin{pmatrix} 1 & 1 & 5 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$. With $E_3 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_3E_2E_1A = \begin{pmatrix} 1 & 1 & 5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$; with $E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $E_4E_3E_2E_1 = \begin{pmatrix} 1 & 1 & 5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, which is the RREF form of A. From this we can read off a basis for the row space — say $\{\begin{pmatrix} 1 & 1 & 5 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \end{pmatrix}\}$. A basis for the column space is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ -9 \end{pmatrix} \right\}$. If $\vec{v}$ is a 5-vector in the null space, then $x_1 = -r - 5s - 3t$, $x_2 = r$, $x_3 = s$, $x_4 = -t$ and $x_5 = t$ for some r, s, t in the reals. A basis for the null space is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$.
This does part (a).

For part (b), we have that $E_4E_3E_2E_1A$ is the RREF form. So we can take $Q = E_4E_3E_2E_1 = \begin{pmatrix} -3 & 2 & 0 \\ -2 & 1 & 0 \\ -6 & 1 & 1 \end{pmatrix}$. This is not the only possible answer, but it is very easy to check that it works.

2. For each of the following matrices A, find the characteristic polynomial χ_A and the minimal polynomial min_A. Find the eigenvalues, and a basis for each eigenspace. Decide in each case whether the matrix is diagonalizable over the reals. $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 0 \end{pmatrix}$; $A = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$; $A = \begin{pmatrix} 6 & 0 & 12 \\ 0 & 4 & 0 \\ -3 & 0 & -6 \end{pmatrix}$.

   Solution: Ah, the smorgasbord. For the first matrix A, $det(\lambda A - I) = det \begin{pmatrix} \lambda & -2 & -1 \\ -2 & \lambda - 3 & -2 \\ -1 & -2 & \lambda \end{pmatrix} = \lambda[(\lambda-3)\lambda - 4] + 2(-2\lambda - 2) - 1(4 + \lambda - 3) = \lambda^3 - 3\lambda^2 - 9\lambda - 5$. This is the characteristic polynomial $\chi_A(\lambda)$ (or $\Delta_A(\lambda)$). The only possible rational roots are ± 1, ± 5. 1 doesn't work, but -1 does. $\chi_A(\lambda) = (\lambda+1)(\lambda^2 - 4\lambda - 5) = (\lambda+1)^2(\lambda-5)$.

   For the root -1 of multiplicity two, $-I - A = \begin{pmatrix} -1 & -2 & -1 \\ -2 & -4 & -2 \\ -1 & -2 & -1 \end{pmatrix}$ which row-reduces to $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so we have a two-dimensional eigenspace; a basis for it is $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$. For the root 5 of multiplicity one, $5I - \lambda = \begin{pmatrix} 5 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 5 \end{pmatrix}$. This row-reduces to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$. A basis for the eigenspace is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$. Since we have a basis of eigenvectors of A, it is diagonalizable. (Of course we knew that from the start, as it is real and symmetric.) Its minimal polynomial has no repeated roots, so it is $(\lambda+1)(\lambda-5) = \lambda^2 - 4\lambda - 5$.

   The other two matrices are basically 2×2. The characteristic polynomial of the second one is $[(\lambda-3)(\lambda-3)+1](\lambda-2) = \lambda^3 - 8\lambda^2 + 22\lambda - 20 = (\lambda-2)(\lambda-(3+i))(\lambda-(3-i))$. Since the eigenvalues aren't all real, this matrix is not diagonalizable over the reals, although it is over the complex numbers. As there are no repeated roots, the minimal polynomial is the same as the characteristic polynomial. The eigenvalues are 2,$3 \pm i$.

For $\lambda = 2$, a basis for the eigenspace is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$; for $\lambda = 3 + i$, a basis is $\left\{ \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \right\}$; and for $\lambda = 3 - i$, a basis is $\left\{ \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \right\}$.

For the third A, the characteristic polynomial is $(\lambda-4)[(\lambda-6)(\lambda+6)+36] = \lambda^3 - 4\lambda^2$. The only possibilities for min_A are $\lambda^3 - 4\lambda^2$ and $\lambda^2 - 4\lambda$ and the latter happens if and only if A is diagonalizable. It isn't, as we shall now see. For the root 0 of multiplicity two, we see that $0I - A$ row-reduces to $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so there is just a one-dimensional eigenspace — a basis for it is $\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$. For the eigenvalue 4, a basis for the eigenspace is just $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. So the matrix is not diagonalizable, and the minimal polynomial is $\lambda^3 - 4\lambda^2$.

3. Let V be the vector space $P_5(t)$ of polynomials over the reals of degree ≤ 5. Define $T : V \longrightarrow V$ by

$$T(p(t)) = t^2 p''(t) - 2tp'(t) + 2p(t).$$

(a) Show that T is linear.

(b) Show that each of 1, t, t^2, t^3, t^4 and t^5 is an eigenvector of T. In each case, give the corresponding eigenvalue.

(c) Find a basis for the kernel $ker(T)$ and the dimension $dim(Im(T))$ of the image of T.

Solution:

(a) For any polynomials $p_1(t)$ and $p_2(t)$ in $P_5(t)$, $T(p_1(t) + p_2(t)) = t^2(p_1(t) + p_2(t))'' - 2t(p_1(t) + p_2(t)) + 2(p_1(t) + p_2(t)) = t^2 p_1''(t) + t^2 p_2''(t) - 2tp_1'(t) - 2tp_2'(t) + 2p_1(t) + 2p_2(t) = T(p_1(t)) + T(p_2(t))$. Also, for any real r and $p(t) \in P_5(t)$, $T(rp(t)) = t^2(rp(t))'' - 2t(rp(t))' + 2(rp(t)) = r(t^2p''(t) - 2tp'(t) + 2p(t)) = rT(p(t))$. Hence T is linear.

(b) $T(1) = t^2(0) - 2t(0) + 2(1) = 2(1)$, so 1 is an eigenvector for the eigenvalue 2.

$T(t) = t^2(0) - 2t(1) + 2t = 0$, so t is an eigenvector for the eigenvalue 0.

$T(t^2) = t^2(2) - 2t(2t) + 2(t^2) = 0$, so t^2 is also an eigenvector for the eigenvalue 0.

$T(t^3) = t^2(6t) - 2t(3t^2) + 2(t^3) = 2(t^3)$, so t^3 is an eigenvector for the eigenvalue 2.

$T(t^4) = t^2(12t^2) - 2t(4t^3) + 2(t^4) = 6(t^4)$, so t^4 is an eigenvector for the eigenvalue 6.

Finally, $T(t^5) = t^2(20t^3) - 2t(5t^4) + 2(t^5) = 12(t^5)$, so t^5 is an eigenvector for the eigenvalue 12.

(c) It would be easy at this stage to write down the 6×6 matrix $[T]_{\mathcal{B}}$, where $\mathcal{B}$ is the standard basis for $P_5(t)$ (we just did the work; the matrix is diagonal). But now we can see directly that a basis for $ker(T)$ is just $\{t, t^2\}$. Since the dimensions of the kernel and the image must add up to six (the dimension of the space $P_5(t)$), $dim(Im(T)) = 4$. (In fact, a basis is $\{1, t^3, t^4, t^5\}$.)

4. Suppose that V is a finite-dimensional vector space, and $T : V \longrightarrow V$ is a linear operator on V such that $T^2 = T$, but T is not the zero operator or the identity operator.

   (a) Give an example of such a T. (You may let $V = \mathcal{R}^2$.)

   (b) Show that the minimal polynomial of T is $t^2 - t$. Show that T is diagonalizable.

   (c) Letting W_0 be the eigenspace corresponding to 0, and W_1 the eigenspace corresponding to 1, show that $V = W_0 \oplus W_1$; show also that $ker(T) = W_0$ and $Im(T) = W_1$.

   (d) In case $dim(V) = 3$, list all possible diagonal matrices which are $[T]_B$ for some ordered basis B of V.

   Solution: This is very similar to a problem on one of the assignments. Anyway, here goes.

   (a) If $T = T_A$, where $A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, it is easily seen that $A^2 = A$, so $T^2 = T$.

   (b) Since $T^2 - T = 0$, the minimal polynomial divides $t^2 - t$; it has to be either $t^2 - t$ itself, or t or $t - 1$. But if it were t, we'd have $T = 0$; if it were $t - 1$, we'd have $T - I = 0$, so $T = I$. So the minimal polynomial is indeed $t^2 - t$ itself. Since this polynomial has no repeated roots, T is diagonal. (And it diagonalizes with just 1's and 0's on the diagonal.)

   (c) $W_0 = \{\vec{v} \in V : T\vec{v} = \vec{0}\} = ker(T)$ and $W_1 = \{\vec{v} \in V : T\vec{v} = \vec{v}\}$. Clearly any $\vec{v} \in W_1$ is in $Im(T)$, but if $\vec{v} \in Im(T)$, then $\vec{v} = T\vec{w}$

for some $\vec{w} \in V$ and so $T\vec{v} = T^2\vec{w} = T\vec{w} = \vec{v}$ and $\vec{v} \in W_1$. So $W_1 = Im(T)$. (We have used $im(T)$ in this class.)

Given any $\vec{v} \in V$, let $\vec{w}_1 = T\vec{v}$ and $\vec{w}_0 = \vec{v} - T\vec{v}$; obviously $\vec{v} = \vec{w}_0 + \vec{w}_1$. But $T\vec{w}_0 = T(\vec{v} - T\vec{v}) = T\vec{v} - T^2\vec{v} = T\vec{v} - T\vec{v} = \vec{v}$, so $\vec{w}_0 \in W_0$; also $T(\vec{w}_1) = T^2\vec{v} = T\vec{v} = \vec{w}_1$, so $\vec{w}_1 \in W_1$. This shows that $V = W_0 + W_1$; the sum is direct since if $\vec{v} \in W_0 \cap W_1$, then $T\vec{v} = \vec{0}$ and also $T\vec{v} = \vec{v}$. So the only thing in the intersection is $\vec{0}$, and thus $V = W_0 \oplus W_1$.

(d) As noted above, the only eigenvalues are 0 and 1. The possible matrices are $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

5. (a) Find

$$det \begin{pmatrix} a & -1 & 0 & 0 \\ -1 & a & -1 & 0 \\ 0 & -1 & a & -1 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

(b) For which (complex) values of a is this matrix singular?

Solution:

(a) Expanding along the first row (or column) tells us the given determinant is $a \cdot det \begin{pmatrix} a & -1 & 0 \\ -1 & a & -1 \\ 0 & -1 & a \end{pmatrix} + 1 \cdot det \begin{pmatrix} -1 & -1 & 0 \\ 0 & a & -1 \\ 0 & -1 & a \end{pmatrix}$, and this equals $a[a \cdot det \begin{pmatrix} a & -1 \\ -1 & a \end{pmatrix} + 1 \cdot det \begin{pmatrix} -1 & -1 \\ 0 & a \end{pmatrix}] - 1 \cdot det \begin{pmatrix} a & -1 \\ -1 & a \end{pmatrix}$. This equals $a[a(a^2-1)-a]-(a^2-1) = a^4-3a^2+1$.

(b) The matrix is singular if and only if its determinant is zero. Letting $b = a^2$, we see that $a^4 - 3a^2 + 1 = 0$ if and only if $b^2 - 3b + 1 = 0$, which occurs just if $b = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$. This happens for four values of a (all real, in fact) $\sqrt{\frac{3}{2} + \frac{\sqrt{5}}{2}}, -\sqrt{\frac{3}{2} + \frac{\sqrt{5}}{2}}, \sqrt{\frac{3}{2} - \frac{\sqrt{5}}{2}}, -\sqrt{\frac{3}{2} - \frac{\sqrt{5}}{2}}$. (In fact, these values can be simplified as $\frac{3}{2} + \frac{\sqrt{5}}{2} = [\frac{1}{2}(1 + \sqrt{5})]^2$ and so on. But this is not necessary.)

6. In this problem we use the standard Hermitian inner product on $\mathcal{C}^3$. We let $U \subseteq \mathcal{C}^3$ be the subspace spanned by $\{\vec{u}_1, \vec{u}_2\}$, where $\vec{u}_1 = \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}$

and $\vec{u}_2 = \begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix}$.

(a) Find an orthonormal basis for U.

(b) If $\vec{v} = \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}$, find the vector $\vec{u}$ in U such that the norm $||\vec{v} - \vec{u}||$ is as small as possible. What is this norm?

Solution:

(a) We start by replacing $\vec{u}_2$ by $\vec{u}_2 - \frac{<\vec{u}_2,\vec{u}_1>}{<\vec{u}_1,\vec{u}_1>}\vec{u}_1$. $< \vec{u}_2, \vec{u}_1 >= i(1) + 0(-i) + 2(+i) = 3i$ and $< \vec{u}_1, \vec{u}_1 >= 1(1) + i(-i) + 1(1) = 3$, so our new second vector is $\begin{pmatrix} i \\ 0 \\ 2 \end{pmatrix} - i \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Normalizing gives us our orthonormal basis $\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

(b) $\vec{u}$ is the projection $Proj_U \vec{v}$; we use the orthogonal basis $\{\vec{w}_1, \vec{w}_2\} = \left\{ \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ to avoid worrying about square roots. The projection is

$$\frac{< \vec{v}, \vec{w}_1 >}{< \vec{w}_1, \vec{w}_1 >}\vec{w}_1 + \frac{< \vec{v}, \vec{w}_2 >}{< \vec{w}_2, \vec{w}_2 >}\vec{w}_2.$$

Now $< \vec{v}, \vec{w}_1 >= i(-i) = 1$, $< \vec{v}, \vec{w}_2 >= i(1) = i$ and $< \vec{w}_1, \vec{w}_1 >= 3$, $< \vec{w}_2, \vec{w}_2 >= 2$. So $\vec{u} = \frac{1}{6} \begin{pmatrix} 6 \\ 5i \\ i \end{pmatrix}$. $\vec{v} - \vec{u} = \frac{1}{6} \begin{pmatrix} -6 \\ i \\ -i \end{pmatrix}$, and this has norm $\frac{\sqrt{38}}{6}$

7. Let $V = \mathcal{R}^n$ be given the standard inner product — i.e., the dot product.

(a) State the Cauchy-Schwarz inequality for this inner product space.

(b) If $n \geq 2$ and $x_1, \ldots, x_n \in \mathcal{R}$, show that

$$(\sum_{1 \leq j < k \leq n} x_j x_k)^2 \leq \frac{n-1}{2} \sum_{\ell=1}^{n} x_\ell^2.$$

[Hint: Apply your statement in part (a) with one vector having all its entries 1.]

Solution:

(a) Generally, Cauchy-Schwartz says that $| < \vec{v}, \vec{w} > | \leq ||\vec{v}|| \cdot ||\vec{w}||$ for any vectors $\vec{v}$ and $\vec{w}$ in an inner product space. In this case, say that the vector $\vec{v} = \begin{pmatrix} x_1 \\ \vdots \\ v_n \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. Then $< \vec{v}, \vec{w} >= \vec{v} \cdot \vec{w} = \sum_{j=1}^n x_j y_j$, $||\vec{v}|| = (\sum_{j=1}^n x_j^2)^{\frac{1}{2}}$ and $||\vec{w}|| = (\sum_{j=1}^n y_j^2)^{\frac{1}{2}}$, so the inequality becomes

$$|\sum_{j=1}^n x_j y_j| \leq (\sum_{j=1}^n x_j^2)^{\frac{1}{2}} (\sum_{j=1}^n y_j^2)^{\frac{1}{2}}.$$

(b) If $\vec{v}$ is any vector in $\mathcal{R}^n$, and every $y_j = 1$, the above equation simplifies; if we then square both sides, we get

$$(\sum_{j=1}^n x_j)^2 \leq (\sum_{j=1}^n x_j^2) \cdot n.$$

The left-hand side of this is $x_1^2 + 2x_1x_2 + x_2^2 + 2x_1x_3 + 2x_2x_3 + x_3^2 + \cdots$, i.e., it's $\sum_{j=1}^n x_j^2 + 2\sum_{1 \leq j < k \leq n} x_j x_k$. If we subtract $\sum_{j=1}^n x_j^2$ from both sides of the inequality, and then divide both sides by 2, we get

$$\sum_{1 \leq j < k \leq n} x_j x_k \leq \frac{n-1}{2} \sum_{j=1}^n x_j^2,$$

as advertized.

8. (a) Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^T A P = D$.

(b) Let $E = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathcal{R}^3 : x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 = 1 \}$. Find an orthonormal basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathcal{R}^3$ and real numbers λ_1, λ_2 and λ_3 such that $E = \{y_1\vec{v}_1 + y_2\vec{v}_2 + y_3\vec{v}_3 : \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1\}$.

Solution:

(a) $det(\lambda I - A) = det \begin{pmatrix} \lambda - 2 & 1 & 0 \\ 1 & \lambda - 2 & 1 \\ 0 & 1 & \lambda - 2 \end{pmatrix} =$ $(\lambda - 2)[(\lambda - 2)^2 - 1] - 1(\lambda - 2) = \lambda^3 - 6\lambda^2 + 10\lambda - 4$. The roots of this characteristic polynomial are 2, $2 \pm \sqrt{2}$. For $\lambda_1 = 2$, a normalized eigenvector is easily seen to be $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. For $\lambda_2 = 2+\sqrt{2}$, a normalized eigenvector is $\frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$, and for $2-\sqrt{2}$, $\frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$. We can take $P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, and then $P^T A P = D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2-\sqrt{2} \end{pmatrix}$.

(b) This is almost the same problem, slightly disguised. The surface E can also be defined by $2x_1^2+2x_2^2+2x_3^2-2x_1x_2-2x_2x_3 = 2$. With $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, the equation for E becomes $\vec{x}^T A\vec{x} = 2$. Using the change-of-basis matrix P we just computed, and letting $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = P^T\vec{x}$, the equation for E becomes $\vec{y}^T D\vec{y} = 2$ with D as computed above. This is true for $\vec{y}$ if and only if $2y_1^2 + (2+\sqrt{2})y_2^2 + (2-\sqrt{2})y_3^2 = 2$. We can choose $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ to be the eigenvectors we computed above, but we must divide the eigenvalues in half — for this part of the problem, let $\lambda_1 = 1$, $\lambda_2 = 1+\frac{\sqrt{2}}{2}$ and $\lambda_3 = 1-\frac{\sqrt{2}}{2}$.