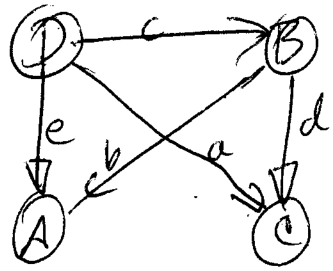
MATH 410 (BAYLY) SELECTED SOLUTIONS 1

2.6.5



I'll let you do the calculations!

(a)

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

(b) RANK $= 3$

(c) Dim RANGE(A) = rank $= 3$ = Dim CORANGE(A)

Dim KERNEL$(A) = n - r = 4 - 3 = 1$

Dim COKERNEL$(A) = m - r = 5 - 3 = 2$

(d) Basis for KERNEL is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ ☆ see problem 2.6.11

We know from 2.6.11 that this vector is ALWAYS a kernel vector, and is the ONLY kernel vector if the graph is IN ONE PIECE

Basis for COKERNEL consists of the INDEPENDENT LOOPS.

The loop $A \to B \to D$ can be written as the vector

$$\begin{array}{c c} a & \\ b & \\ c & \\ d & \\ e & \end{array}\begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ +1 \end{pmatrix} \leftarrow \vec{s}_1$$

These TWO vectors SPAN the KERNEL of A^T, i.e. the COKERNEL of A

The loop $B \to C \to D$ can be written by the vector

$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \vec{s}_2$$

(e) The SOLVABILITY CONDITION for $A\vec{x} = \vec{b}$ is for $\vec{s}_1^{\,T}\vec{b} = 0$, $\vec{s}_2^{\,T}\vec{b} = 0$

(f) Explicitly, if $\vec{b} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{pmatrix}$, we need

$$\vec{s}_1^{\,T}\vec{b} = -\beta - \gamma + \varepsilon = 0 \quad \text{, i.e. } \varepsilon = \beta + \gamma$$

$$\vec{s}_2^{\,T}\vec{b} = -\alpha + \gamma + \delta = 0 \quad \text{i.e. } \delta = \alpha - \gamma$$

<u>SO</u> any $\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \alpha - \gamma \\ \beta + \gamma \end{pmatrix}$ will work! $\begin{pmatrix} 1 \\ 2 \\ 3 \\ -2 \\ 5 \end{pmatrix}$ for example.

Row-reduction (with swaps)

$$
\underset{(A \mid \vec{b})}{\left(\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & -1 & 1 & 0 & -2 \\ 1 & 0 & 0 & -1 & 5 \end{array}\right)}
\longrightarrow
\underset{U \mid \vec{c}}{\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}}
\begin{array}{l} \\ \\ \\ \\ \leftarrow \text{ok!} \\ \leftarrow \text{ok!} \end{array}
$$

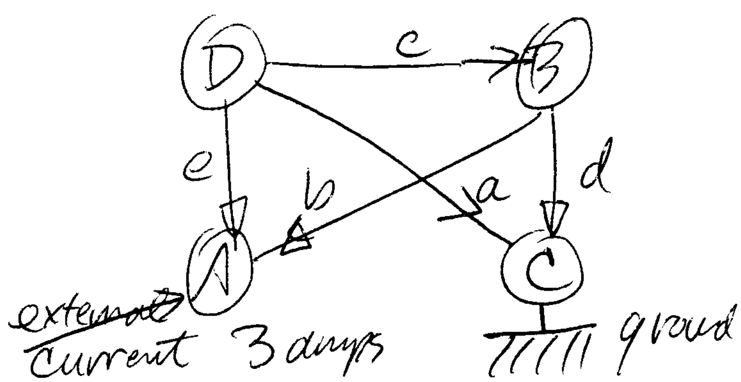
$\Rightarrow \quad z = w+1 \qquad y = w+3 \qquad x = w+5$

General solution $\vec{x} = \begin{pmatrix} 5 \\ 3 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

6.2.2 (sketch solution)

Let's use A, B, C, D for nodes ~~...~~ as in 2.6.5, edges a, b, c, d, e

so that $A = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$



If $\vec{V} = \begin{pmatrix} V_A \\ V_B \\ V_C \\ V_D \end{pmatrix}$ is the vector of voltages (to be solved for)

then the vector of currents is

$$\vec{I} = \begin{pmatrix} I_a \\ I_b \\ I_c \\ I_d \\ I_e \end{pmatrix} = R^{-1} A \vec{V} = A \vec{V}$$ since all resistances are 1 OHM

KIRCHOFF'S CURRENT LAW says $A^T \vec{I} = \vec{0}$ for the internal nodes, which are B & D.
So we just use the rows of A^T that correspond to nodes B & D, which are

$$
\begin{array}{c|ccccc|l}
 & a & b & c & d & e & \\
\hline
A & 0 & 1 & 0 & 0 & 1 & \text{row } A \\
B & 0 & -1 & 1 & -1 & 0 & \text{row } B \\
C & -1 & 0 & 0 & 1 & 0 & \\
D & 1 & 0 & -1 & 0 & -1 & \text{row } D
\end{array}
$$

KIRCHOFF'S CURRENT LAW also applies to node A where the external current 3 amps must be canceled by total NEGATIVE current of 3 amps along edges b, e

FOR THIS WE USE ROW A of A^T

AT node C, current is allowed to escape to ground so no KCL. But being ground means $V_C = 0$

So we have

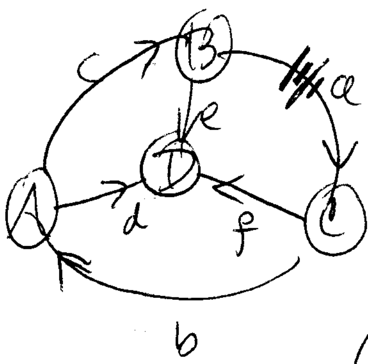
$$
\left.
\begin{aligned}
(0\ 1\ 0\ 0\ 1)\, A\vec{V} &= -3 \\
(0\ -1\ 1\ -1\ 0)\, A\vec{V} &= 0 \\
(1\ 0\ -1\ 0\ -1)\, A\vec{V} &= 0 \\
\text{and} \quad V_C &= 0
\end{aligned}
\right\} \text{4 equations for } \vec{V}
$$

I.e.

$$
\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & -1 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\begin{pmatrix} V_A \\ V_B \\ V_C \\ V_D \end{pmatrix}
=
\begin{pmatrix} -3 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

<u>This</u> you can solve!

6.2.6 Beginning only! It's easier to draw



Each edge has a $3\,\Omega$ resistor; edge a has a 10 volt battery. They want us to find current I_d (as well as all other currents for me!)

There are NO grounds or external nodes; ONLY the battery ~~adds~~ subtracts 10 volts ~~to~~ from $I_a R_a$

so that $I_a = R_a^{-1}\left(V_C - V_D + 10\right)$

[I am probably ~~making~~ violating ANOTHER ± sign convention!]

Everything stays the same, now: $A^T \vec{I} = \vec{0}$ at every node

$$\Rightarrow \quad A^T R^{-1} A \vec{V} = -A^T \begin{pmatrix} R_a^{-1} 10 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This simplifies since all R's are $= 3$, ~~to~~

$$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} V_A \\ V_B \\ V_C \\ V_D \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ -10 \\ 0 \end{pmatrix}$$

which you can now go ahead and solve!