PROF. BAYLY SOLUTIONS ①

Math 410 (Prof. Bayly) EXAM 1: Monday 1 August 2005

There are 4 problems on this exam. They are not all the same length or difficulty, nor the same number of points. You should read through the entire exam before deciding which problems you will work on earlier or later. You are not expected to complete everything, but you should do as much as you can.

You will not need a calculator on this exam. If your calculations become numerically awkward and time-consuming, you may describe the steps you would take if you had a calculator.

It is EXTREMELY important to show your work! Correct answers without documented support will have points deducted.

(1)(30 points) The linear system $A\vec{x} = \vec{b}$ has an infinite number of solutions with one free variable, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix} \quad , \quad \vec{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} .$$

(a)(5 points) Find the general solution and identify the free variable and null vector.

(b)(10 points) Express the squared length of the solution as a function of the free variable, and find the solution with the minimum length.

(c)(10 points) Calculate $K = AA^T$ and solve $K\vec{u} = \vec{b}$ for $\vec{u}$.

(d)(5 points) Calculate $A^T\vec{u}$. Is it the same as the min length solution you found in (b)?

ⓐ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow{L_{21}=2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 1 & -3 \end{array}\right)$ z free

$-y+z=-3 \qquad y = z+3$

$x+y+z=2 \qquad x = 2-(z+3)-z = -1-2z \qquad \vec{x} = \begin{pmatrix} -1-2z \\ z+3 \\ z \end{pmatrix}$

$\vec{x} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + z\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ kernel vector $= \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

ⓑ Length^2 $= L^2(z) = (1+2z)^2 + (z+3)^2 + z^2$

$= 1+4z+4z^2 + z^2+6z+9+z^2 = 6z^2+10z+10$

$$L'(z) = 12z + 10 \quad = 0 \text{ if } z = -5/6$$ 2

So min length solution is $\begin{pmatrix} 4/6 \\ 13/6 \\ -5/6 \end{pmatrix}$

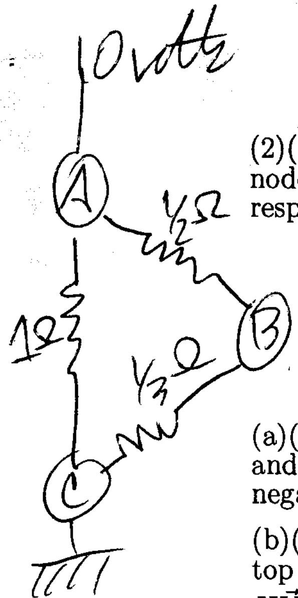
(b) $AA^T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$

so solve $\left(\begin{array}{cc|c} 3 & 6 & 2 \\ 6 & 14 & 1 \end{array}\right) \xrightarrow{l_{21}=2} \left(\begin{array}{cc|c} 3 & 6 & 2 \\ 0 & 2 & -3 \end{array}\right)$ $\quad v = -3/2$

$3u - 9 = 2 \qquad u = 11/3 \qquad$ so $\vec{u} = \begin{pmatrix} 11/3 \\ -3/2 \end{pmatrix}$

(c) $A^T \vec{u} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 11/3 \\ -3/2 \end{pmatrix} = \begin{pmatrix} 11/3 - 3 \\ 11/3 - 3/2 \\ 11/3 - 9/2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 13/6 \\ -5/6 \end{pmatrix}$

YES! $A^T \vec{u}$ is SAME as $\vec{x}_{me}$!



(2)(20 points) The electrical circuit on the accompanying page is described by the edge-node matrix, conductivity matrix (i.e. inverse of resistivity matrix), and voltage vector, respectively:

$$A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \vec{V} = \begin{pmatrix} V_A \\ V_B \\ V_C \end{pmatrix}.$$

(a)(10 points) The circuit equations involve the matrix $K = A^T C A$; calculate this matrix and verify that (i) it is symmetric, (ii) its diagonals are all positive, other entries all negative, and (iii) the sum of entries in each column are zero.

(b)(10 points) If we are given the voltages $V_A = 10$ and $V_C = 0$ (ground), we replace the top row of K with $(1, 0, 0)$ and the bottom row with $(0, 0, 1)$; call this new matrix H. Solve $H\vec{V} = (10, 0, 0)^T$ for $\vec{V}$.

(a)

$$A^T C A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -3 & 3 \\ -1 & 0 & 1 \\ -2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -2 & -1 \\ -2 & 5 & -3 \\ -1 & -3 & 4 \end{pmatrix}$$

YES! (i) symmetric, (ii) 3, 5, 4 all > 0

(iii) $3-2-1=0$, $-2+5-3=0$, $-1-3+4=0$ ✓

(b)

$$H = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ -2 & 5 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 5 & -3 & 20 \\ 0 & 0 & 1 & 0 \end{array}\right) \text{ Echelon form!}$$

Augmented matrix

$V_A = 10$, $V_C = 0$ $\quad 5V_B = 20 \Rightarrow V_B = 4$ volts.

(3)(20 points) The roof structure on the accompanying page supports a weight of 6000 newtons at the top vertex B, and is itself supported by unknown forces F at the left corner A and G at the right corner C. The "scaled tensions" in the beams satisfy $A\vec{\tau} = \vec{f}$, where

$$A = \begin{pmatrix} 0 & -3 & -1 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \vec{\tau} = \begin{pmatrix} \tau_a \\ \tau_b \\ \tau_c \end{pmatrix} \quad , \quad \vec{f} = \begin{pmatrix} 0 \\ F \\ 0 \\ -6000 \\ 0 \\ G \end{pmatrix}.$$

We are NOT interested in the actual values of $\vec{\tau}$, but we ARE interested in whether a solution exists.

(a)(10 points) Verify that the following are kernel vectors of A^T:

$$\vec{s}_1 = (1,0,1,0,1,0)^T \quad , \quad \vec{s}_2 = (0,1,0,1,0,1)^T \quad , \quad \vec{s}_3 = (0,1,1,0,1,-2)^T.$$

(b)(5 points) Use these vectors to find conditions under which $A\vec{\tau} = \vec{f}$ will have a solution.

(c)(5 points) Find the values of F, G for which the solvability conditions in (b) are satisfied. Do they seem reasonable for a 6000-newton roof?

(a) We just check $A^T\vec{s} = \vec{0}$ for each $\vec{s}$.
For efficiency we'll put the $\vec{s}$ vectors in a matrix S

$$A^TS = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 0 \\ -3 & -1 & 0 & 0 & 3 & 1 \\ -1 & -1 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -1+1 & 0 & -1+1 \\ -3+3 & -1+1 & -1+3-2 \\ -1+1 & -1+1 & -1+1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Each column is one of the $A^T\vec{s}$'s.

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(b) We need $\vec{s}^T \vec{f} = 0$ for a solution to exist, for each $\vec{s}$. Lumping them all together

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ F \\ 0 \\ -6000 \\ 0 \\ G \end{pmatrix} = \begin{pmatrix} 0+0+0+0+0+0 \\ F-6000+G \\ F-2G \end{pmatrix}$$

Need these to all $=0$ to have a solution to exist.

So (c) we need

$$\left.\begin{aligned} F+G &= 6000 \\ F &= 2G \end{aligned}\right\} \Rightarrow 3G = 6000$$

$G = 2000$ newtons

$F = 2 \times 2000 = 4000$ newtons

It totally makes sense to support a 6000-newton roof with 2000 newtons on one wall, and 4000 newtons on the other.

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(4)(30 points) Four soccer teams play four games over three days, and we want to rank them. In this problem you are encouraged to NOT solve any linear systems if you can answer the questions otherwise.

In case you've forgotten, we represent a game by an arrow from the "visitor" to the "home" team, and label with the score difference (home) - (visitor).

(a)(5 points) On the first day A visits B and loses by 1 goal, and D visits C and also loses by 1 goal. Draw a digraph that expresses this situation, and say why we cannot yet rank all the teams with respect to each other.

(b)(5 points) On the second day, B visits C and wins by 2. Draw the new digraph and rank the teams, if possible.

(c)(5 points) On the third day, A visits D and loses by 4. Draw the digraph, and say why there is no perfectly consistent ranking.

(d)(5 points) Write down the edge-node matrix A for the digraph in (c), and find (without doing any calculation) a kernel vector $\vec{s}$ of A^T.

(e)(10 points) Find a linear system of equations whose solution would give a reasonable approximate ranking. You do NOT have to solve them, but you should find the matrix and right-hand side vector explicitly.

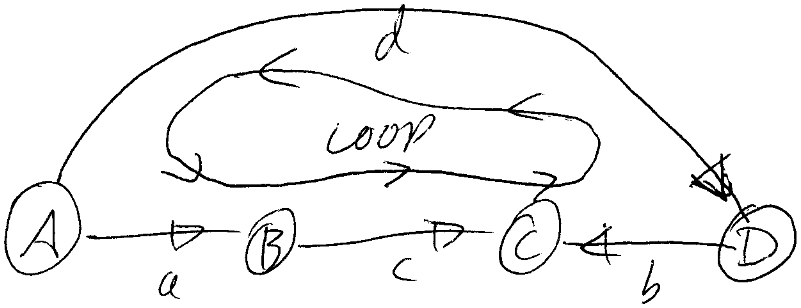
(a) $A \xrightarrow[a]{1} B \qquad C \xleftarrow[b]{1} D$ Can't rank teams because there is no basis for comparing group A, B with group C, D.

(b) Now $A \xrightarrow[a]{1} B \xrightarrow[c]{-2} C \xleftarrow[b]{1} D$

B is 2 better than C and 1 better than A

Therefore A is 1 better than C, Also C is 1 better than D $\Rightarrow$ B best, then A, then C, then D.

(c)



D beating A by 4 is INCONSISTENT with D being ranked below A in part (b)

(d) Observe there is a LOOP in the network above: that goes along

a in + direction

b in − "

c in + "

d in − "

so $\vec{s} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

Edge-node matrix is

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

~~check:~~

NOT NEEDED BUT INTERESTING

check $A^T\vec{s} = \begin{pmatrix} -1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ✓

(e) Since $A\vec{Q} = \vec{D} = \begin{pmatrix}1\\-2\\1\\4\end{pmatrix}$

is inconsistent,

we have to use LEAST SQUARES $A^TA\vec{Q} = A^T\vec{D}$

here $$A^TA = \begin{pmatrix}-1&0&0&-1\\1&0&-1&0\\0&1&1&0\\0&-1&0&1\end{pmatrix}\begin{pmatrix}-1&1&0&0\\0&0&1&-1\\0&-1&1&0\\-1&0&0&1\end{pmatrix} = \begin{pmatrix}2&-1&0&-1\\-1&2&-1&0\\0&-1&2&-1\\-1&0&-1&2\end{pmatrix}$$

and $$A^T\vec{D} = \begin{pmatrix}-1&0&0&-1\\1&0&-1&0\\0&1&1&0\\0&-1&0&1\end{pmatrix}\begin{pmatrix}1\\-2\\1\\4\end{pmatrix} = \begin{pmatrix}-5\\0\\1\\4\end{pmatrix}$$

SO $\vec{Q} = \begin{pmatrix}w-7/2\\w-2\\w-1/2\\w\end{pmatrix} = \begin{array}{c}Q_A\\Q_B\\Q_C\\Q_D\end{array}$

SO D Best then C, then B, then A

Lets solve it, though not asked.

$$\left(\begin{array}{cccc|c}2&-1&0&-1&-5\\-1&2&-1&0&0\\0&-1&2&-1&1\\-1&0&-1&2&4\end{array}\right) \xrightarrow[L_{41}=-1/2]{L_{21}=-1/2} \left(\begin{array}{cccc|c}2&-1&0&-1&-5\\0&3/2&-1&-1/2&-5/2\\0&-1&2&-1&1\\0&-1/2&-1&+3/2&+3/2\end{array}\right) \xrightarrow[L_{42}=-1/3]{L_{32}=-2/3} \left(\begin{array}{cccc|c}2&-1&0&-1&-5\\0&3/2&-1&-1/2&-5/2\\0&0&4/3&-\frac{4}{3}&-2/3\\0&0&-4/3&\frac{4}{3}&2/3\end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c}2&-1&0&-1&-5\\0&3/2&-1&-1/2&-5/2\\0&0&\frac{4}{3}&-\frac{4}{3}&-2/3\\0&0&0&0&0\end{array}\right) \leftarrow \text{OK!}$$

w free $= Q_D$ $\quad \frac{3}{2}y - (w-\frac12) - \frac{w}{2} = -\frac{5}{2}$

$z = w - \frac12 \quad \Rightarrow \frac{3}{2}y = \frac{3}{2}w - 3 \quad y = w-2$

$2x - (w-2) - w = -5 \quad 2x = 2w - 7 \quad x = w - \frac72$