

MATH 440 (BAYLY) EXAM 3 SOLUTIONS ①

① $\vec{x} = \begin{pmatrix} z+2 \\ 1-z \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{x}_p} + z \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{\vec{x}_c} \quad \vec{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

② $\text{length}^2(\vec{x}) = (z+2)^2 + (1-z)^2 + z^2$
 $= (z^2 + 4z + 4) + (1 - 2z + z^2) + z^2$
 $= 3z^2 + 2z + 5$

③ $\frac{d(\text{length}^2(\vec{x}))}{dz} = 6z + 2 = 0 \text{ when } z = -\frac{1}{3}$

④ Resulting $\vec{x}_{ML} = \begin{pmatrix} 5/3 \\ 4/3 \\ -1/3 \end{pmatrix}$

Dot product $\vec{n}^T \vec{x}_{ML} = (1 \ -1 \ 1) \begin{pmatrix} 5/3 \\ 4/3 \\ -1/3 \end{pmatrix} = \frac{5}{3} - \frac{4}{3} - \frac{1}{3} = 0$ ✓
 $\Rightarrow$ ORTHOGONAL!

② $f(\vec{x}) = x^2 + 4xy + 3y^2 + 2xz + 8yz + 2z^2$

$$f(\vec{x}) = (x \ y \ z) \underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 2 \end{pmatrix}}_K \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (2)$$

(a)

(b) Row-reduce K . $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 2 \end{pmatrix} \xrightarrow[L_{31}=1]{L_{21}=2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$

$L_{32} = -2 \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$ This is U , but we're not done yet.

Note $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$, $U = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}}_D \underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T}$

$U = D L^T$

(c) SO $f(\vec{x}) = (\vec{x}^T L) D (L^T \vec{x})$

where $L^T \vec{x} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+z \\ y-2z \\ z \end{pmatrix}$

SO $f(\vec{x}) = (x+2y+z, y-2z, z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x+2y+z \\ y-2z \\ z \end{pmatrix}$
 $= 1(x+2y+z)^2 - 1(y-2z)^2 + 5z^2$

2c

Because ~~points~~ diagonal entries in D are BOTH positive AND negative, $f(\vec{x})$ is INDEFINITE.

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3 Data $(1,1), (2,1), (3,3), (4,4), (5,3)$

a) If $y = \alpha + \beta x$ then

$$\begin{array}{ll} 1 = \alpha + \beta & 4 = \alpha + 4\beta \\ 1 = \alpha + 2\beta & 3 = \alpha + 5\beta \\ 3 = \alpha + 3\beta & \end{array}$$

I.e.
$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \\ 3 \end{pmatrix}$$

NO SOLUTION!

Briefly substituting

1st 2 eqs $\Rightarrow \beta = 0$

while 2nd & 3rd $\Rightarrow \beta = 2$ INCONSISTENT

b) Use least squares $(A^T A) \vec{\alpha} = A^T \vec{b}$

Here $A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & 54 \end{pmatrix}$

$A^T \vec{b} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 43 \end{pmatrix}$

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3b) cont so $\begin{pmatrix} 5 & 10 \\ 10 & 54 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 12 \\ 43 \end{pmatrix}$

determine $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ least squares best approx.

4) $E = \begin{matrix} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} d \\ b \\ c \\ d \\ e \\ f \end{matrix} & \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{matrix}$

b) $\vec{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\vec{m}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\vec{m}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}$

c) Score difference vector is $\begin{pmatrix} 3 \\ 1 \\ -5 \\ -2 \\ 2 \\ 3 \end{pmatrix} = \vec{D}$

Note $\vec{m}_1^T \vec{D} = -9 \neq 0 \Rightarrow$ NO common
and $\vec{m}_2^T \vec{D} = -7 \neq 0$ also $\Rightarrow$
NO solution!

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d) Least squares solution would satisfy

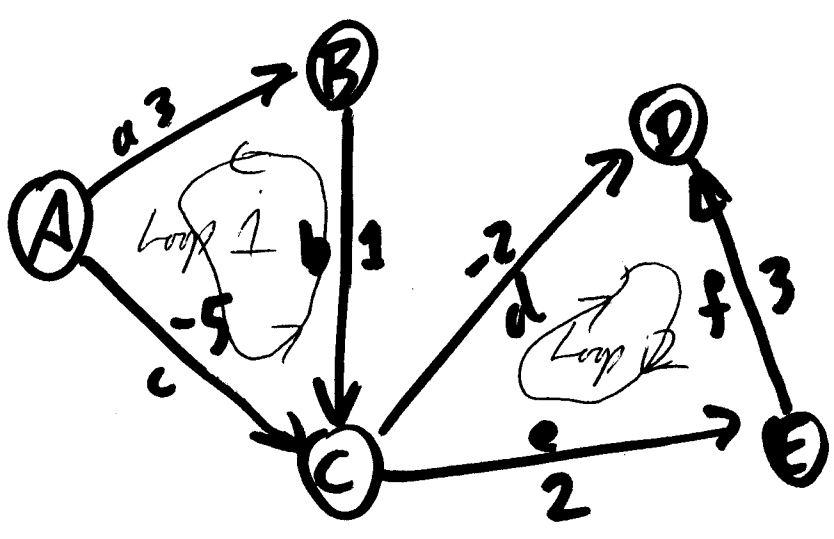
$$E^T E \vec{W}_{LS} = E^T \vec{D}$$

Here $E^T E = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$

$$E^T \vec{D} = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -5 \\ -2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -4 \\ 1 \\ -1 \end{pmatrix}$$

NETWORK FOR PROBLEM 4:

a)



(6)

(4e) We don't use MIN LENGTH edition when there's a single free variable, because all the information about ranking can be obtained from the general form. ~~The fact we~~

~~That~~ When there are 2 free variables it is VERY important to keep them both free, as it allows the possibility of one league being much better than the other. If we used MIN LENGTH we would be making a big mistake in a lot of circumstances!

(5) $A = \begin{pmatrix} 1 & 4 & 4 \\ 3 & -1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$ They give us $\lambda's = 1, -3, 5$

(a) $\text{Trace}(A) = 1 - 1 + 3 = 3$ ✓
 $\text{Sum of evals} = 1 - 3 + 5 = 3$ ✓

(b) $\text{Det}(A) = 1 \det \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} - 3 \det \begin{pmatrix} 4 & 4 \\ 2 & 3 \end{pmatrix}$
 $= 1(-3) - 3(12 - 8) = -3 - 12 = -15$ ✓

$\text{Product of evals} = 1 \times (-3) \times 5 = -15$ ✓

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$$\lambda = 1 \quad (A - \lambda I) \vec{x} = \vec{0} \Rightarrow \begin{pmatrix} 0 & 4 & 4 & | & 0 \\ 3 & -2 & 0 & | & 0 \\ 0 & 2 & 2 & | & 0 \end{pmatrix}$$

Swap rows 1 & 2

$$\begin{pmatrix} 3 & -2 & 0 & | & 0 \\ 0 & 4 & 4 & | & 0 \\ 0 & 2 & 2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & -2 & 0 & | & 0 \\ 0 & 4 & 4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

z free $4y + 4z = 0 \Rightarrow y = -z$

$$3x - 2y = 0 \Rightarrow x = \frac{2}{3}y = -\frac{2}{3}z$$

So $\vec{x} = \begin{pmatrix} -2/3 z \\ -z \\ z \end{pmatrix} = z \begin{pmatrix} -2/3 \\ -1 \\ 1 \end{pmatrix}$ $\vec{z} = \begin{pmatrix} -2/3 \\ -1 \\ 1 \end{pmatrix}$

(d) $J - \lambda I = \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$ $\det = \lambda^2 + 1$
 Roots $\lambda = \pm i$

For $\lambda = i$ $(J - \lambda I) \vec{x} = \vec{0} \Rightarrow \begin{pmatrix} -i & -1 & | & 0 \\ 1 & -i & | & 0 \end{pmatrix}$

$\hookrightarrow \begin{pmatrix} -i & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$ y free, $-ix - y = 0 \Rightarrow -ix = y \Rightarrow x = iy$

so $\vec{x} = y \begin{pmatrix} i \\ 1 \end{pmatrix}$ $\vec{z} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ is vector belonging to $\lambda = i$
 BY CONJUGATION $\vec{z} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ goes with $\lambda = -i$