NAME: Morning/Afternoon

Math 410 (Prof. Bayly) EXAM 3: Wednesday 8 November 2006

There are 5 problems on this exam. They are not all the same length or difficulty, nor the same number of points. You should read through the entire exam before deciding which problems you will work on earlier or later. You are not expected to complete everything, but you should do as much as you can. It is *extremely* important to show your work!

No calculators are allowed on this exam. If your calculations become numerically awkward and time-consuming, you should describe the steps you would take if you had a calculator.

Useful facts:

If the system $A\vec{x} = \vec{b}$ has no solutions, then the least-squares best approximate solution $\vec{x}_{LS}$ satisfies $A^T A\vec{x}_{LS} = A^T\vec{b}$.

If the system $A\vec{x} = \vec{b}$ has infinitely many solutions, then the minimum-length solution is $\vec{x}_{ML} = A^T\vec{u}$ where $AA^T\vec{u} = \vec{b}$.

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(1)(15 points) The general solution of a system of equations is

$$\vec{x} = \begin{pmatrix} z+2 \\ 1-z \\ z \end{pmatrix}.$$

(a)(3 points) Identify the particular $\vec{x}_p$ and complementary $\vec{x}_c$ parts of the solution, and the null vector $\vec{n}$.

(b)(3 points) Express the $length^2(\vec{x})$ as a function of z.

(c)(5 points) Find the value of z that minimizes $length^2(\vec{x})$.

(d)(4 points) Show that the resulting minimum length solution $\vec{x}_{ML}$ is perpendicular to $\vec{n}$.

(2)(25 points) Consider the quadratic function of three variables $\vec{x} = (x, y, z)^T$:

$$f(\vec{x}) = x^2 + 4xy + 3y^2 + 2xz + 8yz + 2z^2.$$

(a)(5 points) Find a symmetric matrix K for which $f(\vec{x}) = \vec{x}^T K\vec{x}$.

(b)(10 points) Find a lower-triangular matrix L and diagonal matrix D such that $LDL^T = K$.

(c)(10 points) Use L and D to express $f(\vec{x})$ as sums and/or differences of squared quantities. Is $f(\vec{x})$ positive definite, pos semidef, indefinite, neg semidef, or negative definite?

(3)(10 points) An experiment yields datapoints (1,1), (2,1), (3,3), (4, 4), and (5,3).

(a)(5 points) If the data is consistent with the hypothesis that $y = \alpha + \beta x$, then each datapoint gives an equation relating α, β. Write down these equations in matrix notation. Does this system have a solution?

(b)(5 points) What system of equations determines the Least Squares approximation? Do NOT try to solve it!

(4)(25 points) A league of 7 sports teams have played a certain number of games already. In game (a) A visits B and loses by 3, in game (b) B visits C and loses by 1 and in game (c) A visits C and wins by 5 goals. Meanwhile, in game (d) C visits D and wins by 2, game (e) C visits E and loses by 2, and in game (f) E visits D and loses by 3.

(a)(5 points) **In the space below** draw (large!) a network representing the games player so far, labeling the nodes with upper case letters, edges with lower-case, and score differences. Write down the edge-node incidence matrix for this situation, and the vector $\vec{D}$ of score differences.

(b)(5 points) Find the independent null vectors of E and E^T.

(c)(5) The "winning potentials" of the teams satisfy $E\vec{W} = \vec{D}$, or at least it would be nice if they did. Is there an exact solution to this system (giving reason)?

(d)(5 points) Find a system of equations (but do NOT try to solve them) that determine the least squares approximate solutions.

(e)(5 points) It turns out that the least-squares solution is not unique; there are two free variables corresponding to the separate "pieces" of the network. What does this mean for the real ranking problem? Why don't we use the min-length solution?

(5)(25 points) The matrix

$$A = \begin{pmatrix} 1 & 4 & 4 \\ 3 & -1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$$

has eigenvalues $\lambda = 1, -3, 5$.

(a)(5 points) Verify that the trace of the matrix (= sum of diagonal elements) equals the sum of the eigenvalues.

(b)(5 points) Verify that the determinant of the matrix equals the product of the eigenvalues.

(c)(5 points) Find the eigenvector of A belonging to $\lambda = 1$.

(d)(10 points) Find the eigenvalues and eigenvectors of $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

NETWORK FOR PROBLEM 4: