Thermo-Acoustic Imaging in a Reverberating Cavity

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TAT/PAT in A Resonant Cavity

Thermo-acoustic tomography (TAT) is a hybrid form of medical imaging in which biological tissues are radiated with microwaves to illicit a thermal expansion which produces acoustic pressure waves. These waves are measured outside of the tissue. Thermo-Acoustic Tomography in a reverberating cavity is modelled by the following IVB/BVP problem:

$$\begin{cases} \frac{1}{c^2(\mathbf{x})}\frac{\partial^2}{\partial t^2}u = \Delta u & \mathbf{x}\in\Omega \\ \frac{\partial}{\partial \mathbf{n}}u = 0 & \mathbf{x}\in\Sigma \\ u(0,\mathbf{x}) = f, & \frac{\partial}{\partial t}u(0,\mathbf{x}) = 0 \end{cases} \qquad (1)$$

where, u is the pressure differential, c is the speed of sound, $\mathbf{n}$ is an outward normal unit vector, and $\Omega\subset\mathbb{R}^d$ is the region bounded by the reflecting surface, Σ. We assume that c is a twice differentiable function bounded away from zero,

$$0 < c_{min} \le c(\mathbf{x}) \le c_{max}.$$

$g(t,\mathbf{x})$ is the pressure measured on all or part of the reflecting surface. The measurement surface is denoted as Σ_1 and $\Sigma_2 = \Sigma\setminus\Sigma_1$ is the portion of the boundary where pressure is not measured.

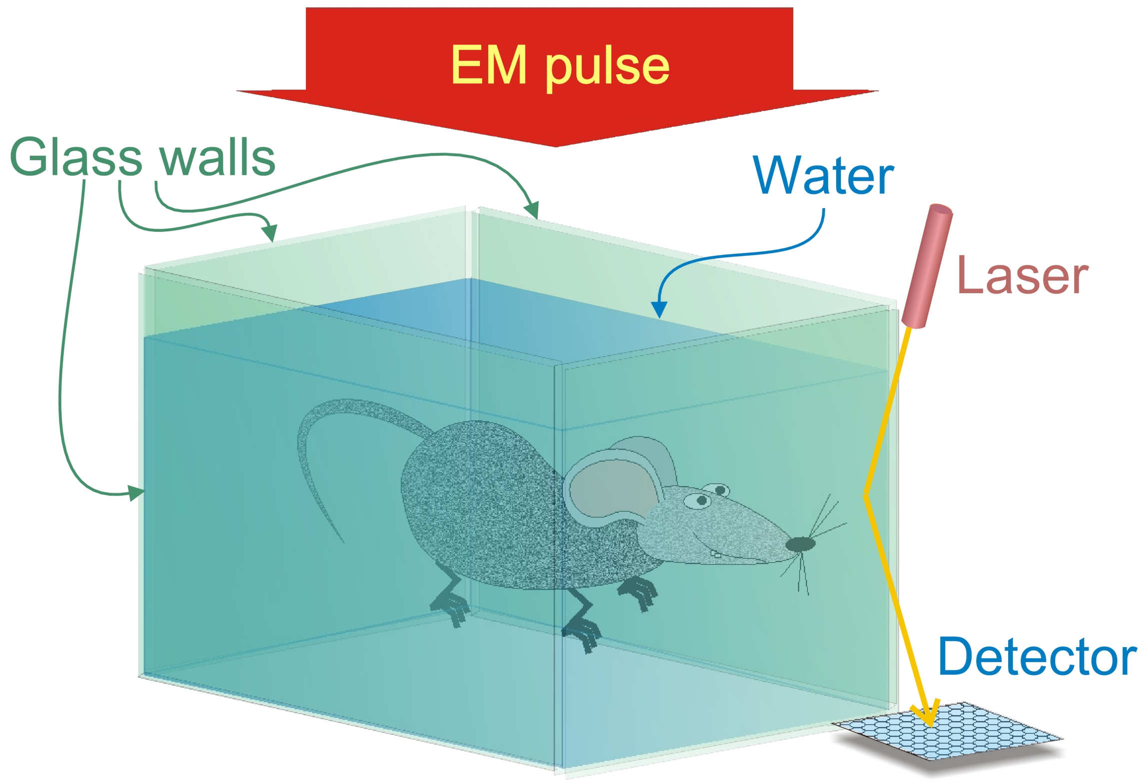


Figure : Depiction of TAT Apparatus with optical data acquisition scheme in a rectangular domain.

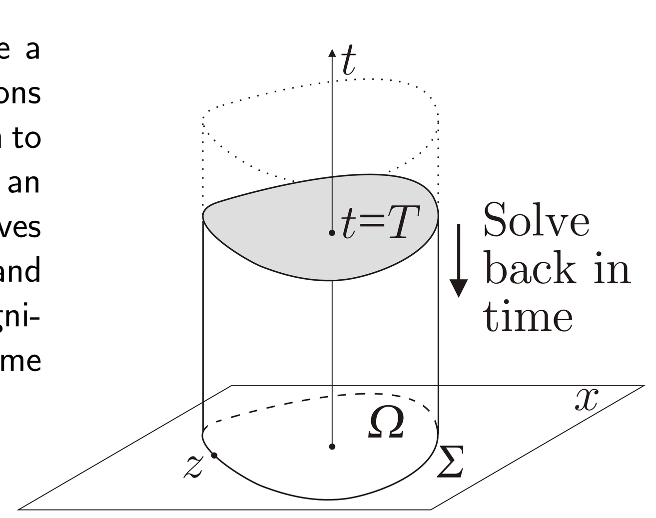
Inverse Problem The inverse problem is to reconstruct the initial acoustic pressure within the region of interest from measured boundary data.

Failure of Traditional Time Reversal

Practically all existing theory on TAT/PAT is based on the assumption that acoustic waves propagate in free space, and that reflections from the detectors and walls of the water tank can either be neglected or gated out. Almost all reconstruction techniques developed for the free-space problem (including time reversal) rely upon the decay of energy within the region of interest. However, energy is conserved in the model of TAT in a resonant cavity (1) which implies that traditional time reversal is not applicable to this situation.

$$E \equiv \int_\Omega \left(\frac{1}{c^2(\mathbf{x})}\left|\frac{\partial}{\partial t}u(t,\mathbf{x})\right|^2 + |\nabla u(t,\mathbf{x})|^2 \right) d\mathbf{x}$$

The traditional time reversal algorithm is to solve a Dirichlet problem back in time from initial conditions $u(T,\mathbf{x}) = \frac{\partial}{\partial t}u(T,bx) = 0$. Applying this algorithm to data measured from a resonant cavity introduces an error proportional to the energy of the acoustic waves at the time T, which will propagate toward $t=0$ and create artefacts roughly of the same order of magnitude as $f(\mathbf{x})$. We developed a modified form of time reversal that overcomes this difficulty.



Gradual Time Reversal

We propose a modified form of the time reversal algorithm that does not rely upon the decay of energy within the region of interest. This method consists in solving back in time on the interval $[0,T]$ the initial/boundary value from for the wave equation with Dirichlet boundary data multiplied by a smooth cut-off function

Gradual Time Reversal

Reconstruct $u(0,\mathbf{x})$ by solving the following for v:

$$\begin{cases} \frac{1}{c^2(\mathbf{x})}\frac{\partial^2}{\partial t^2}v = \Delta v & \mathbf{x}\in\Omega,\quad t\in[0,T] \\ v(t,\mathbf{x}) = g(t,\mathbf{x})\eta(\varepsilon t) & \mathbf{x}\in\Sigma_1,\quad t\in[0,T] \\ \frac{\partial}{\partial \mathbf{n}}v(t,\mathbf{x}) = 0 & \mathbf{x}\in\Sigma_2,\quad t\in[0,T] \\ v(T,\mathbf{x}) = 0, & \frac{\partial}{\partial t}v(T,\mathbf{x}) = 0. \end{cases} \qquad (2)$$

- η is a smooth cut-off function
- $\eta = 1$ in a neighbourhood of zero
- $\eta(T) = 0$ along with all of its derivatives.

Convergence of Gradual Time Reversal

Represent v as

$$v = u\eta + w.$$

The error in approximating the actual initial pressure, $u(0,\mathbf{x})$, by the result of gradual time reversal, $v(0,\mathbf{x})$, is the function $w(0,\mathbf{x})$. Later, it will be convenient to represent u by an expansion in the series of eigenfunctions of the weighted Neumann Laplacian and w by an expansion in the series of the eigenfunctions of the weighted Laplacian with boundary conditions given in (2); that is,

$$u(t,\mathbf{x}) = \sum_n f_n\varphi(\mathbf{x})\cos(\nu_n t), \qquad \text{and} \qquad w(t,\mathbf{x}) = \sum_k w_k(t)\psi_k(\mathbf{x}).$$

Theorem If the weighted Laplacian with boundary conditions defined by (2) has eigenvalues that are distinct from the eigenvalues of the weighted Neumann Laplacian, then the result $v(0,\mathbf{x})$ of gradual time reversal converges weakly to $f(\mathbf{x})$ in $H^1(\Omega)$ as $\varepsilon\to 0$ (or, equivalently, as the measurement time $T\to\infty$).

Proof Sketch Substituting the form specified for v into the first line of (2) and rearranging produces

$$\frac{1}{c^2(\mathbf{x})}\frac{\partial^2}{\partial t^2}w(t,\mathbf{x}) - \Delta w(t,\mathbf{x}) = \frac{1}{c^2(\mathbf{x})}F_\varepsilon(t,\mathbf{x}), \quad F_\varepsilon(t,\mathbf{x}) = -\varepsilon\left(2\eta'(\varepsilon t)\frac{\partial}{\partial t}u(t,\mathbf{x}) + \varepsilon\eta''(\varepsilon t)\,u(t,\mathbf{x})\right).$$

Due to the orthogonality of the eigenfunctions $\psi_k(\mathbf{x})$ (in the sense of the weighted inner product) each generalized Fourier coefficient of w satisfies the following ODE

$$w_k''(t) + \lambda_k^2 w_k(t) = F_k(t), \qquad F_k(t) = \langle F_\varepsilon, \psi_k\rangle_{c^{-2}}.$$

The causal Green's function of each equation is $\frac{\sin(\lambda_k t)}{\lambda_k}$ which provides an integral representation for $w_k(t)$.

$$\lambda_k w_k(0) = \sum_n f_n I_{n,k}(\varepsilon)\langle\psi_k,\varphi_n\rangle_{c^{-2}}$$

$$I_{n,k}(\varepsilon) = \varepsilon\int_0^{1/\varepsilon}\left[2\eta'(\varepsilon t)\nu_n\sin(\nu_n t) - \varepsilon\eta''(\varepsilon t)\cos(\nu_n t)\right]\sin(\lambda_k t)\,dt$$

Each coefficient of the error term may be bounded

$$|\lambda_k w_k(0)| \le 2\varepsilon^M B(M)\,||f||_{H^1}\sum_n\frac{1}{|\nu_n-\lambda_k|^M}.$$

Remark If the weighted Laplacian with boundary conditions defined by (2) has eigenvalues that are not distinct from the eigenvalues of the weighted Neumann Laplacian, then the result $v(0,\mathbf{x})$ of gradual time reversal can posses error that is non-decreasing with ε. The generalized Fourier Coefficients of the error function w will converge to

$$w_k(0) = -\sum_{n \text{ s.t. } \nu_n=\lambda_k} f_n\langle\psi_k,\varphi_n\rangle_{c^{-2}}\psi_k(\mathbf{x}).$$

Example: Disk Domain

Theorem If the domain Ω is a disk, the measurement surface Σ_1 is the boundary of the disk, and $c(\mathbf{x}) = c_0$ is a constant speed of sound then the reconstruction produced by gradual time reversal converges to the actual initial pressure strongly in $H^1(\Omega)$. The following bound is placed on the H^1 norm of the error in the reconstruction:

$$||w(0,\mathbf{x}))||_{H^1}^2 \le C(M)\varepsilon^M\left(\frac{1}{\lambda_{1,0}^2}+1\right)||f||_{H^1}^2 \xrightarrow[\varepsilon\to 0]{} 0$$

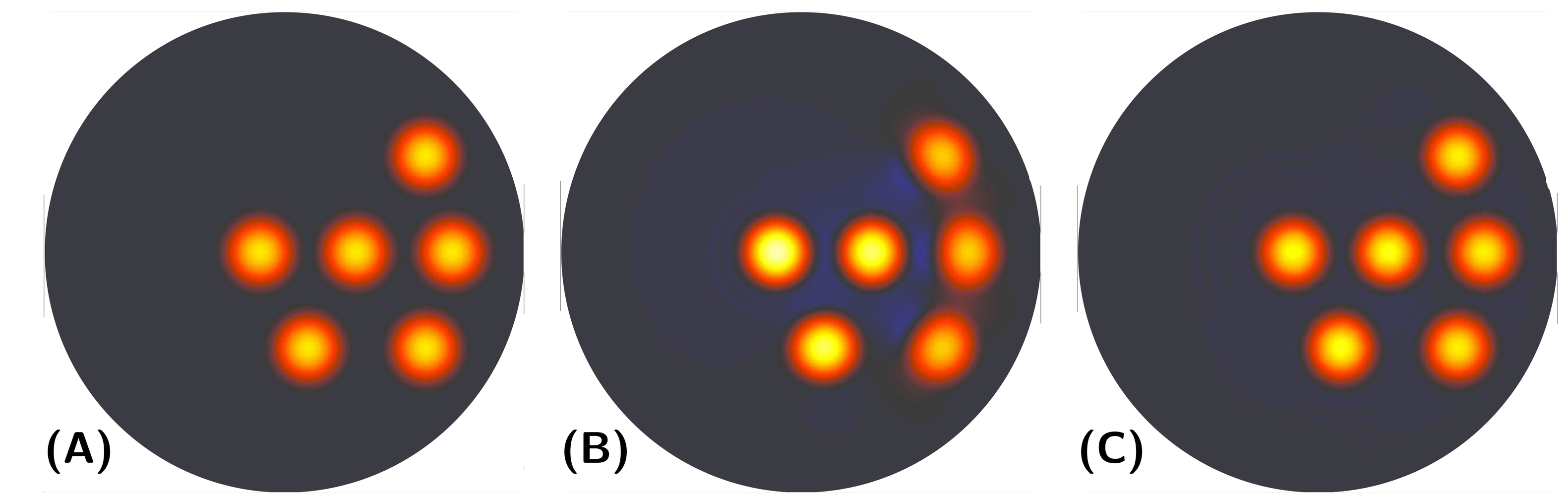


Figure : (A) Phantom used to generate boundary data, (B) reconstruction from 5.3 seconds of boundary data, and (C) reconstruction computed from 10.6 seconds of boundary data.

Example: Rectangular Domain

Consider a rectangular domain with $c(\mathbf{x}) = c_0$. Specific measurement surfaces of interest are the following:

- If the domain is $\Omega = [0,A]\times[0,B]$ where A and B are rational numbers and $\Sigma_1 = \partial\Omega$ then the Dirichlet and Neumann Laplacians will have infinitely many repeated eigenvalues. Reconstructions computed using the gradual time reversal algorithm posses error terms predicted by the analysis of the method that do not decrease.
- If the domain is $\Omega = [0,A]\times[0,B]$ where A and B are incommensurable numbers and $\Sigma_1 = \partial\Omega$ then there are no coinciding eigenvalues between the Dirichlet and Neumann Laplacians. Thus, the gradual time reversal method converges for each generalized Fourier coefficient.
- If the domain is $\Omega = [0,A]\times[0,B]$ then the measurement surface $\Sigma_1\subset\partial\Omega$ can be carefully chosen so that the eigenvalues of the Laplacian with boundary conditions prescribed by (2) are distinct from the eigenvalues of the Neumann Laplacian. Thus, the gradual time reversal method converges for each generalized Fourier coefficient. The following figure depicts reconstructions from this type of situation and clearly displays the non-uniform convergence of coefficients.

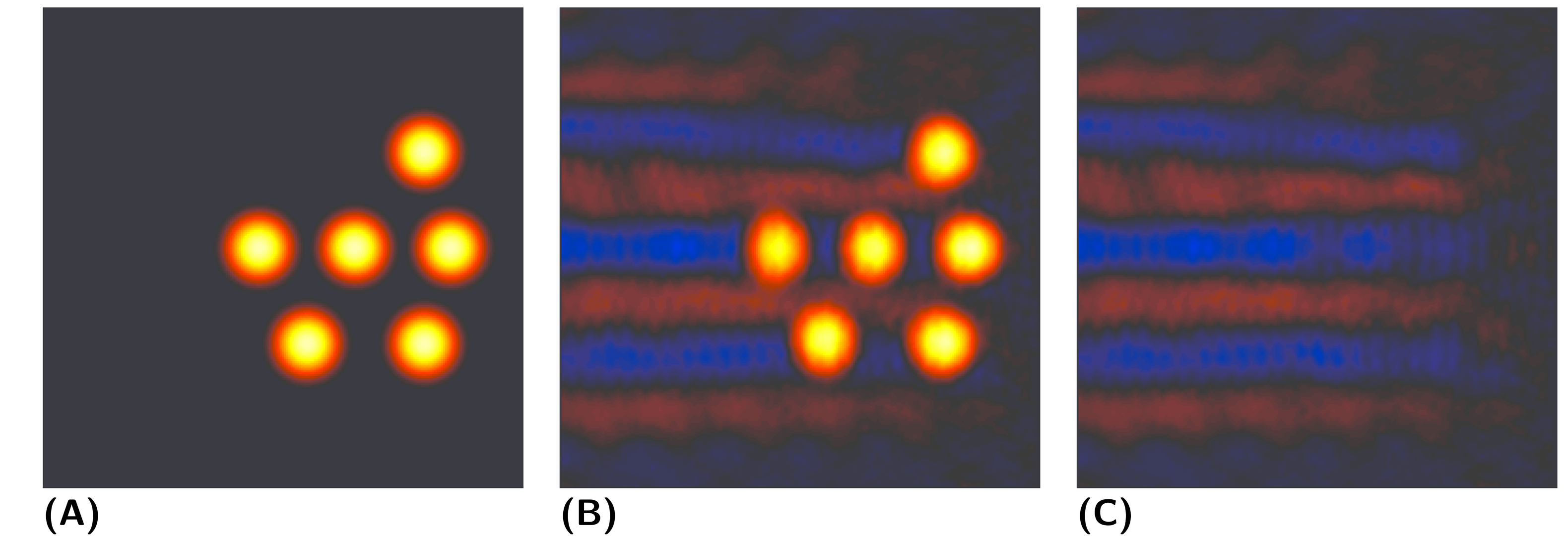


Figure : Reconstruction in $\Omega = [0,1]\times[0,1]$ with Σ_1 being the vertical line at $x=1$. (A) Phantom used to generate boundary data, (B) reconstruction from 100 seconds of boundary data (from one side), and (C) error in the reconstruction.

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References

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