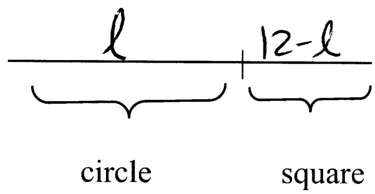
OPTIMIZATION – PART 1 NAME ______________________

1. A wire of length 12 inches can be bent into a circle, a square, or cut to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure(s) is to be a minimum? A maximum?



Circum. of circle : $l = 2\pi r$ $\rightarrow r = \dfrac{l}{2\pi}$.

Perimeter of square : $12 - l = 4s$ $\rightarrow s = \dfrac{12-l}{4}$

$A = A_{circle} + A_{square} = \pi r^2 + s^2 = \pi\left(\dfrac{l}{2\pi}\right)^2 + \left(\dfrac{12-l}{4}\right)^2$

$$A'(l) = 2\pi\left(\frac{l}{2\pi}\right)\cdot\frac{1}{2\pi} + 2\left(\frac{12-l}{4}\right)\cdot\left(-\frac{1}{4}\right)$$

$$= \frac{l}{2\pi} - \frac{24}{16} + \frac{2l}{16}$$

$$= \frac{4l}{8\pi} + \frac{\pi l}{8\pi} - \frac{3}{2} = \left(\frac{4+\pi}{8\pi}\right)l - \frac{3}{2}.$$

Crit. pts.

A' always defined

$A' = 0 \rightarrow \left(\frac{4+\pi}{8\pi}\right)l - \frac{3}{2} = 0$

$\rightarrow \left(\frac{4+\pi}{8\pi}\right)l = \frac{3}{2}$

$\rightarrow l = \frac{3}{2}\left(\frac{8\pi}{4+\pi}\right) = \frac{12\pi}{4+\pi}$.

2nd deriv. test

$A''(l) = \frac{4+\pi}{8\pi} > 0$.

$\boxed{\text{So minimum area when } \frac{12\pi}{4+\pi} \text{ in. of wire used for circle}}$

Max area: check end pts.

$A(0) = \pi\left(\frac{0}{2\pi}\right)^2 + \left(\frac{12-0}{4}\right)^2 = 9$.

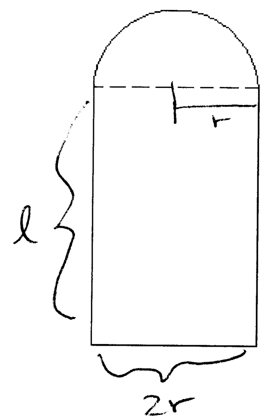
$A(12) = \pi\left(\frac{12}{2\pi}\right)^2 + \left(\frac{12-12}{4}\right)^2$

$= \pi \cdot \frac{144}{4\pi^2} = \frac{36}{\pi}$.

Since $\frac{36}{\pi} > 9$,

$\boxed{\text{max. area when 12 in. of wire used for circle}}$

2. A window consisting of a rectangle topped by a semicircle is to have an outer perimeter *P*. Find the radius of the semicircle if the area of the window is to be a maximum.



$P = P_{rect} + P_{semicircle}$

$= 2l + 2r + \pi r$

Pick r as indep. variable

$\rightarrow$ write $l = \dfrac{P - 2r - \pi r}{2} = \dfrac{P - (\pi+2)r}{2}$

Area $A = A_{rect} + A_{semicircle}$

$$= \frac{\pi}{2}r^2 + 2rl$$

$$= \frac{\pi}{2}r^2 + 2r\left(\frac{P - (\pi+2)r}{2}\right)$$

$$= \frac{\pi}{2}r^2 + Pr - (\pi+2)r^2$$

$$= \left(-\frac{\pi}{2} - 2\right)r^2 + Pr.$$

$\rightarrow A'(r) = 2\left(-\frac{\pi}{2} - 2\right)r + P = (-\pi - 4)r + P$.

Crit pts:

A' always defined

$A' = 0 \rightarrow (-\pi-4)r + P = 0$

$\rightarrow (-\pi-4)r = -P$

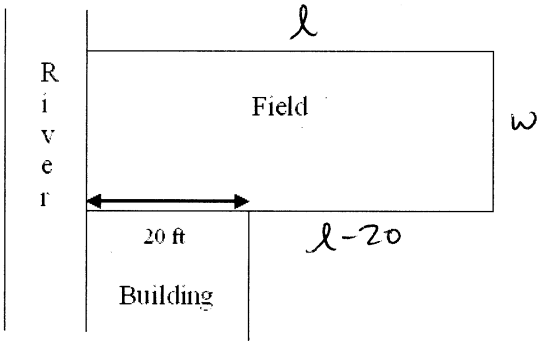
$\rightarrow r = \frac{-P}{-\pi-4} = \frac{P}{\pi+4}$

2nd deriv. test

$A''(r) = -\pi - 4 < 0$.

So $\boxed{r = \frac{P}{\pi+4} \text{ maximizes area}}$

3. A rectangular field as shown is to be bounded by a fence. Find the dimensions of the field with maximum area that can be enclosed with 1000 feet of fencing. You can assume that fencing is not needed along the river and building.



$1000 = P = \ell + w + (\ell - 20) = 2\ell + w - 20$

$A = \ell \cdot w$

Pick ℓ as indep. variable

$\rightarrow$ Write $w = 1000 - 2\ell + 20 = 1020 - 2\ell$

So Area $A = \ell w = \ell(1020 - 2\ell) = 1020\ell - 2\ell^2$

$\rightarrow A'(\ell) = 1020 - 4\ell$

crit pts:

A' always defined

$A' = 0 \rightarrow 1020 - 4\ell = 0$

$\rightarrow 1020 = 4\ell$

$\rightarrow \ell = 255$

2nd deriv. test

$A''(\ell) = -4 < 0.$

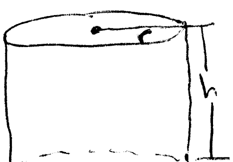
So $\boxed{\ell = 255 \text{ ft yields a maximum area}}$

$w = 1020 - 2\ell = 1020 - 2(255)$

$\rightarrow w = 1020 - 510$

$\rightarrow \boxed{w = 510 \text{ ft}}$

4. A company manufactures cylindrical barrels to store nuclear waste. The top and bottom of the barrels are to be made with material that costs \$10 per square foot and the rest is made with material that costs \$8 per square foot. If each barrel is to hold 5 cubic feet, find the dimensions of the barrel that will minimize the total cost.



Area $A = 2\pi r^2 + 2\pi r h$

$\rightarrow$ Cost $C = 10(2\pi r^2) + 8(2\pi r h)$

$= 20\pi r^2 + 16\pi r h$

$5 = \text{Volume} = \pi r^2 h$

$h = \frac{5}{\pi r^2}.$

Pick r as indep. variable

$\rightarrow C = 20\pi r^2 + 16\cancel{\pi}\cancel{r}\left(\frac{5}{\cancel{\pi} r^{\cancel{2}}}\right)$

$= 20\pi r^2 + \frac{80}{r}.$

$\rightarrow C'(r) = 40\pi r - \frac{80}{r^2}.$

crit pts:

C' undef. at ~~$r=0$~~

($r = 0$ means no cylinder, so not in domain)

$C' = 0 \rightarrow 40\pi r - \frac{80}{r^2} = 0$

$\rightarrow 40\pi r = \frac{80}{r^2}$

$\rightarrow \pi r^3 = 2$

$\rightarrow r^3 = \frac{2}{\pi}$

$\rightarrow r = \sqrt[3]{\frac{2}{\pi}}$

2nd deriv. test

$C''(r) = 40\pi + \frac{160}{r^3} > 0.$

So $r = \sqrt[3]{2/\pi}$ minimizes cost.

AND $h = \frac{5}{\pi(2/\pi)^{2/3}}$