RESEARCH STATEMENT

BENJAMIN PITTMAN-POLLETTA

My research is in Lie theory, the study of groups with a compatible manifold structure, and has connections to representation theory and combinatorics. I study loop groups, the simplest class of infinite-dimensional Lie groups. The elements of a loop group are smooth maps from the circle into a finite-dimensional Lie group. Loop groups are related to the study of solitons and integrable systems in PDE, and to quantum field theory (QFT). They are spaces of fields for two-dimensional versions of the standard model of particle physics. Quantizing these fields requires constructing suitable probability measures on loop groups.

Choosing the correct coordinates is the key to many problems in mathematics. One goal of my research is finding the right coordinates for studying the invariant measure on a loop group. Coordinates on a compact, semisimple Lie group K in which the invariant measure can be written as a product appear in the work of Lu [35] and Soibelman [49]. These coordinates are connected to the geometry and topology of the flag manifold of K. In [39], my advisor Doug Pickrell used the framework of Kac-Moody Lie algebras, in which loop groups are generalizations of semisimple Lie groups, to prove the existence of an analogous coordinate system on the group of loops into $SU(2)$. In [41], Pickrell and I prove that such a coordinate system exists on the group of loops into any compact, simple group K. Besides conjecturally factoring the invariant measure on the loop group of K, these coordinates provide a new tool for probing the analytic properties of this group. They allow us to factor the determinants of Toeplitz operators, providing a nonabelian extension of a formula of Szegö and Widom [53], and the determinants of Hankel operators, providing a number of new integral formulas. These results are discussed in Section 1.

These coordinates are also connected to the study of affine flag manifolds, and possibly to recent work on affine generalizations of the Schubert calculus [34, 32, 33]. One connection is through the affine Weyl group W_{aff}, a reflection group associated to K. The coordinates appearing in [41] depend in part on data coming from W_{aff}, and lead to a number of combinatorial questions about this group. Explicit reduced words for certain translations in W_{aff} will appear in my dissertation. Currently, I am investigating variations of the factorization appearing in [41], which should lead to new integral and determinant formulas. Using the work of Lam & Pylyavskyy [33] on infinite reduced words in W_{aff}, I hope to classify all possible variations of the factorization result in [41], and to find factorizations which are optimal for various purposes. I hope this perspective will illuminate both factorizations and measures on loop groups, and the combinatorics of W_{aff}. This work is discussed in Section 2.

In an attempt to understand and generalize formulas for the determinants of Hankel operators appearing in [39], I studied higher-order Hankel operators [42]. These representation-theoretic generalizations of Hankel operators were discovered by Janson & Peetre [26]. By establishing a connection between these operators and the Lie algebra of holomorphic vector fields on the unit disk, I was able to re-derive results of Jansen & Peetre, giving concrete formulas for the higher-order Hankel operators as differential operators [42]. These formulas also generated new binomial coefficient identities. These results are discussed in Section 3.

My background is broad, including applied mathematics and neuroscience. In the future, I hope to make interdisciplinary connections with biology and sociology. Much recent activity has focused on applying techniques from geometry and topology to biology through data analysis and statistics. My interests in this direction are described in Section 4.

1. Factorization and Coordinates on Loop Groups

The group of smooth maps from the circle (loops) into a Lie group G is denoted LG. It is a smooth infinite-dimensional manifold modeled on its Lie algebra, the topological vector space $L\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. The product of two loops is the pointwise product of their images in G: $(gh)(\theta) = g(\theta)h(\theta)$.

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Save a handful of exceptions, all the compact simple groups K are matrix groups, so we can think of elements of LK as matrices with entries depending smoothly on the coordinate $z = e^{i\theta}$ for S^1.

The theory of Kac-Moody Lie algebras allows us to think of the loop group LG as a generalization of the Lie group G. Roughly speaking, the structure theory of LG is like the structure theory of G, except that it also incorporates the Fourier transform on S^1. For example, for constant loops, the notion of triangular factorization in LG corresponds to triangular factorization in G. For a nonconstant loop g, a triangular factorization $g = ldu$ is a product of an antiholomorphic element l which is a power series in z^{-1}, a constant diagonal element d, and a holomorphic element u which is a power series in z.

Coordinates on a simple, compact group K in which the invariant measure μ_K for K is a product measure were written down by Lu [35] and Soibelman [49]. This coordinate system is related to the following factorization. Every element $k \in K$ having a triangular factorization can be written as a unique product of very simple elements k_i' and a unitary diagonal element t:

$$k = k_n' \dots k_2' k_1' t =: k' t. \tag{1}$$

Each element k_i' lives in a subgroup S_i of K isomorphic to $SU(2)$, and has real diagonal. The order of the factors k_i' of k' is determined by a longest reduced word in the Weyl group of K. This factorization arises in the study of the flag manifold K/T, where T is the diagonal subgroup of K, and is incipient in the work of Bott & Samelson [6]. Parametrizing S_i/S^1 with a single complex coordinate ζ_i, and T with coordinates $\{\chi_i\}_{i=1}^r$, we have for constants c_j

$$\mu_K = \prod_{j=1}^n \frac{c_j}{1+|\zeta_j|^2} d\zeta_j d\bar{\zeta}_j \prod_{i=1}^r d\chi_i.$$

To find coordinates diagonalizing the invariant measure on LK, it makes sense to generalize (1) to LK. An analog of (1) for $LSU(2)$ was shown to exist by Pickrell [40]. Every loop $k \in LSU(2)$ having a triangular factorization can be written as a unique product of infinitely many simple elements k_i' and k_i'', along with a smooth function λ from S^1 into the diagonal subgroup of $SU(2)$:

$$k \quad = \quad \left((k_0')^* \, (k_1')^* \dots (k_n')^* \dots \right) \quad \lambda \quad \left(\dots k_n'' \dots k_2'' k_1'' \right) \quad =: \quad (k')^* \, \lambda k''. \tag{2}$$

The elements k_i' and k_i'' depend on complex coefficients η_i and ζ_i, and are analogous to Fourier modes:

$$k_i' = \frac{1}{\sqrt{1+|\eta_i|^2}} \begin{pmatrix} 1 & -\bar{\eta}_i z^i \\ \eta_i z^{-i} & 1 \end{pmatrix}, \qquad k_i'' = \frac{1}{\sqrt{1+|\zeta_i|^2}} \begin{pmatrix} 1 & \zeta_i z^{-i} \\ -\bar{\zeta}_i z^i & 1 \end{pmatrix}. \tag{3}$$

As with the Fourier transform, the sequences of coefficients η and ζ must be rapidly decreasing for the product (2) to converge to a smooth loop. The products k' and k'' are determined by associating one factor to each generator in an infinite reduced word in the Weyl group W_{aff} of $LSU(2)$. There are two such products since in this case W_{aff} is the infinite dihedral group, which has only two infinite reduced words. One of these words corresponds to the product k', and the other corresponds to k''.

Crucial to showing that the product $(k')^* \lambda k''$ has a triangular factorization is the fact that the antiholomorphic components of k' and k'' have the form

$$l' = \begin{pmatrix} 1 & \\ \sum_{i=0}^\infty y_i z^{-i} & 1 \end{pmatrix}, \qquad l'' = \begin{pmatrix} 1 & \sum_{j=1}^\infty x_i z^{-i} \\ & 1 \end{pmatrix},$$

for rapidly-decreasing complex sequences x and y. Pickrell also characterized k' and k'' by their entries. Namely, the top row of k' and the bottom row of k'' must be holomorphic and nonvanishing in the open unit disk Δ.

In the joint paper [41], we proved that for an arbitrary simple, compact group K, every loop $k \in LK$ having a triangular factorization also has a factorization of the form (2). Now, however, $\lambda(z)$ is a smooth map from S^1 into the diagonal subgroup T of K. The "Fourier modes" k_i' and k_i'' are elements of the form (3), embedded into subgroups of K isomorphic to $SU(2)$. For example, in the $LSU(3)$ case each factor k_i' or k_i'' has one of the three shapes

$$\begin{pmatrix} * & * & \\ * & * & \\ & & 1 \end{pmatrix}, \qquad \begin{pmatrix} * & & * \\ & 1 & \\ * & & * \end{pmatrix}, \qquad \begin{pmatrix} 1 & & \\ & * & * \\ & * & * \end{pmatrix}.$$

The products k' and k'' are again associated to infinite reduced words in W_{aff}. These words are chosen so that the antiholomorphic components of the products k' and k'' take values in the spaces of lower- and upper-triangular unipotent matrices, respectively, and so that the shapes of the factors k_i' and k_i'' repeat periodically with i. This is discussed further in the next section.

In [41], we obtain representation-theoretic characterizations of the elements k' and k'', such as the following theorem.

Theorem 1.1. *Infinite products of the form $k'' = \ldots k_n'' \ldots k_2'' k_1''$ are characterized by their action on the lowest-weight vector ϕ in a highest-weight representation π of K: The vector $\pi\left((k'')^{-1}\right)\phi$ is holomorphic and nonvanishing in Δ.*

I proved this directly for k'' having a finite Laurent series. The hard part is showing that an element satisfying the conditions of the Theorem has a triangular factorization of the right form. While, there is no general algorithm for finding the triangular factorization of a loop, the conditions of Theorem 1.1 allowed me to construct a triangular factorization for k'' by combining pointwise and loop triangular factorization.

I used a similar argument to prove the following in [41].

Theorem 1.2. *A loop $k \in LK$ has a factorization $k = (k')^* \lambda k''$ if and only if it has a triangular factorization $k = ldu$.*

The hard part is going from ldu to $(k')^* \lambda k''$. Intuitively, we must go from maps l and u having values in the complex group $K^{\mathbb{C}}$ to loops k' and k'' having values in the unitary group K. This suggests the use of something like the Gram-Schmidt algorithm, which can be applied to the columns of a nonsingular matrix to get a unitary matrix. The proof makes this precise.

1.1. **Formulas for Toeplitz & Hankel Determinants.**

A Toeplitz matrix is constant along diagonals, and a Hankel matrix is constant along antidiagonals. This idea is generalized to operators on Hilbert spaces by considering so-called compressions of multiplication operators. Compression here means we restrict the domain and the range of the multiplication operator. Let P_+ be the operator which takes the holomorphic part of a Laurent series, and let P_- be the operator taking the antiholomorphic part, so that

$$P_+\left(\sum_{i\in\mathbb{Z}} f_i z^i\right) = \sum_{i=0}^{\infty} f_i z^i, \qquad P_-\left(\sum_{i\in\mathbb{Z}} f_i z^i\right) = \sum_{i=1}^{\infty} f_{-i} z^{-i}.$$

Then the Toeplitz operator associated to a multiplication operator M is $A(M) = P_+ \circ M \circ P_+$, and the Hankel operator associated to M is $B(M) = P_+ \circ M \circ P_-$. In the basis $\{z^n\}_{n\in\mathbb{Z}}$ for $L^2\left(S^1\right)$, the matrix for $A(M)$ is constant on diagonals, while the matrix for $B(M)$ is constant on antidiagonals. There are many identities relating the Toeplitz and Hankel operators of a given multiplication operator.

If K has a representation on a vector space V, we can think of a loop $k \in LK$ as a multiplication operator acting on LV. Let $A(k)$ be the Toeplitz operator associated to k. The coordinates appearing in [41] allow us to give the following nonabelian generalization of a determinant formula of Szegö and Widom [53, 41]. If we parametrize $C^\infty\left(S^1, T\right)$ by the complex coordinates $\{\chi_l\}_{l\in\mathbb{Z}}$ with $\chi_l = -\bar{\chi}_{-l}$, then

$$\det|A(k)|^2 = \prod_{i\geq 0}\left(1+|\eta_i|^2\right)^{-p'(i)} \prod_{l\geq 1} e^{-(r+1)l|\chi_l|^2} \prod_{j\geq 1}\left(1+|\zeta_j|^2\right)^{-p''(j)}. \tag{4}$$

The integers $p'(i)$ and $p''(i)$ depend on choices of infinite reduced words in W_{aff}; for $LSU(2)$, $p'(i) = p''(i) = i$.

It is not surprising that (4) also tells us something about Hankel operators. For example, given a triangular factorization $k'' = l''d''u''$ for a loop $k'' \in LSU(2)$ as in (2),

$$\det|A(k'')|^2 = \det\left(1+|B(l'')|^2\right)^{-1} = \det\left(1+|B\left(M_x\right)|^2\right)^{-1},$$

where M_x denotes multiplication by the function x [39]. Thus, the sequence ζ is a coordinate on the space of power series with rapidly decreasing coefficients, in which the above determinants are products. When x is polynomial, this allows these determinants to be integrated.

1.2. **Future Work.** For a finite dimensional Lie group K with Weyl group W, there is a canonical longest reduced word in W leading to a simplest set of coordinates on K. Are there canonical choices of infinite reduced words leading to simplest, or otherwise optimal, coordinates for LK? Given a loop k, is there a factorization that best captures its behavior? To answer these questions, I am studying how different infinite reduced words in W_{aff} lead to variations of (2). This is discussed further below.

It remains to be shown that the invariant measure on LK is a product measure in the $\{\eta, \chi, \zeta\}$ coordinates. The product measure μ in these coordinates which is a candidate for the invariant measure is well-defined on loops having a finite Laurent series. It must be extended to a suitable completion of LK, and its multiplication invariance must be shown.

Lu [35] showed that her coordinates diagonalize a symplectic structure. Do the $\{\eta, \chi, \zeta\}$ coordinates diagonalize the analogous symplectic form on LK? Lu's proof relies on a property that the symplectic form on LK lacks, so a new approach is needed in the loop case.

A far-out but exciting possibility is to generalize the ideas contained in the factorization (2) to infinite-dimensional Lie groups having less structure than loop groups, like $\text{Diff}(S^1)$. Measures analogous to μ on $\text{Diff}(S^1)$, called Malliavin measures, are conjecturally connected to the stochastic Loewner evolution [31].

2. Reduced Decompositions in W_{aff} and Alcove Walks

If K has rank r, the associated affine Weyl group, denoted W_{aff}, is the automorphism group of a set $\mathcal{L}$ of affine hyperplanes in $\mathbb{R}^r$ (see Figure 1 for G_2). A finite subset $\{L_\alpha\}$ of the hyperplanes in $\mathcal{L}$ pass through the origin. All the others belong to regularly spaced families $\{L_{k,\alpha}\}_{k\in\mathbb{Z}}$ of hyperplanes parallel to each L_α.

Removing $\mathcal{L}$ from $\mathbb{R}^r$ leaves a set of connected components called alcoves, which can be identified with finite elements of W_{aff} by choosing a distinguished alcove A_0. A word in W_{aff} is a sequence of alcoves $A_0, A_1, \ldots, A_n$ such that A_i and A_{i+1} are adjacent. This word is reduced if and only if there is no shorter word from A_0 to A_n, and an infinite reduced word is an infinite word for which every initial subword is reduced.

We associate an infinite product of simple loops to an infinite reduced word by adding a factor for every hyperplane $L_{k,\alpha}$ crossed by the word. The shape of the factor is determined by α, while the exponent of z is determined by k. In [41], we associate one infinite product to each of two infinite reduced words. The infinite reduced word determining k' must start at A_0 and cross every hyperplane $L_{k,\alpha}$ for which $k \le 0$, while the infinite reduced word determining the infinite product k'' must start at A_0 and cross every hyperplane $L_{k,\alpha}$ for which $k \ge 1$. The former is easily obtained from the latter.

The word determining k'' proceeds arbitrarily deeply into the interior of the cone C_0 spanned by A_0. To construct it, we use the fact that W_{aff} contains as a subgroup a lattice $\check{T}$ which acts by translations. Picking a translation $t \in \check{T}$ which lies in the interior of C_0 and repeating a reduced word from A_0 to $A_0 + t$ takes us arbitrarily far into C_0.

Explicit reduced words can be constructed for the following simple choice of translation. The edges of the cone C_0 are rays spanned by vectors $\Theta_1, \ldots, \Theta_r$, so that $h^* = \sum_{i=1}^r \Theta_i$ lies in C_0. Furthermore, $2h^*$ is always an element of $\check{T}$. A large number of reduced words for the translation $2h^*$ can be constructed by associating a reduced word w_i to each vector Θ_i. This word is not a translation, but it translates the center of A_0 by Θ_i. The following will appear in my dissertation.

Theorem 2.1. *Any concatenation of the reduced words w_i, in which each w_i appears twice, is a reduced word for $2h^*$.*

2.1. **Current and Future Work.** More generally, one might ask how many reduced words there are for $2h^*$, or for any (finite) element $w \in W_{\text{aff}}$. For translations, the words w_i could provide an answer this question. Any element $t \in \check{T}$ can be decomposed as a sum $t = \sum_{i=1}^r a_i\Theta_i$, and it should be possible to obtain reduced words for t by concatenating a_i copies of each word w_i. Any reduced word for t can be obtained from such a concatenation, and preliminary calculations suggest that the total number of reduced words for t should be on the order of the number of such concatenations. Making this precise could provide a method for enumerating all the reduced words for t. Lam's [32] recent generalization of Stanley symmetric functions to affine Weyl groups provides another avenue of attack. Stanley [50] created these functions while studying reduced words in finite Weyl groups, and used them to show that the number of reduced words for

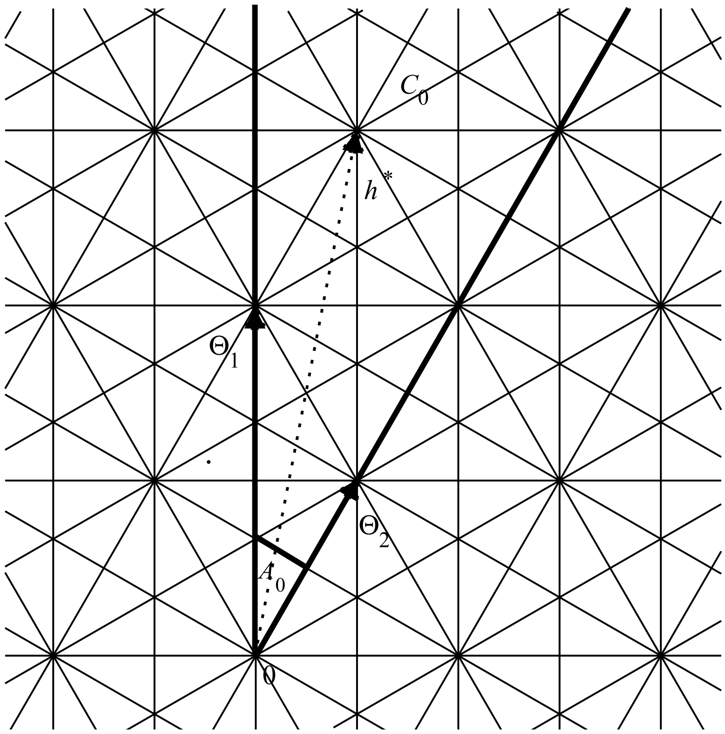

FIGURE 1. Alcove diagram for G_2.

the longest permutation in S_n is

$$\frac{C_2^n!}{1^{n-1}3^{n-2}\dots(2n-3)^1}.$$

For a higher-rank group K, choosing two infinite words is not the only way to produce a factorization result. Instead of choosing one infinite reduced word which crosses all $L_{k,\alpha}$ with $k \geq 1$, we could choose several infinite reduced words, each of which crosses hyperplanes $L_{k,\alpha}$ for all $k \geq 1$ but only some α. These infinite reduced words can be constructed from translations lying in the walls of C_0. The resulting variations of the factorization (2) should provide new integral and determinant formulas, as well as different and possibly useful expressions for the measure μ.

Intriguingly, infinite reduced words resulting from repeated translations play a prominent role in recent work of Lam & Pylyavskyy [33]. I plan to use their study of infinite reduced words in W_{aff} to provide a classification of possible variations of (2), and to answer the question of how to choose optimal or canonical factorizations for different purposes, either in general or given a particular loop k. From another perspective, associating a measure to each infinite reduced word in W_{aff} through a factorization of the type (2) might shed light on the structure of W_{aff}.

Finally, there are a number of connections between Weyl groups, random walks, random matrices, and Brownian motion [20] [21] [4]. The number of walks on the lattice $\sum_{i=1}^{r} \mathbb{Z}\Theta_i$ having a given length and endpoints was calculated by Grabiner [21]. Reduced words for translations are in a sense refinements of these walks, and it would be interesting to introduce randomness and study the scaling limits of these words. This may present opportunities for computational experimentation and undergraduate research.

3. Higher-Order Hankel Operators

In studying Toeplitz and Hankel operators, it is natural to identify the space of power series $H_+ := P_+\left(L^2\left(S^1\right)\right)$ with a space of holomorphic functions on the open unit disk Δ, and the space $H_- := P_-\left(L^2\left(S^1\right)\right)$ with a space of holomorphic functions on the open unit disk around infinity, Δ^*. The automorphism group of Δ is the group $\mathcal{M}$ of Möbius transformations. It is thus not surprising that $\mathcal{M}$ has well-known representations on the spaces H_+ and H_-, as well as on the space of Hilbert-Schmidt operators from H_- to H_+, denoted $\mathcal{L}_2(H_-, H_+)$.

Janson and Peetre [26] introduced the representation theory of $\mathcal{M}$ into the study of Hankel operators, by studying the decomposition of $\mathcal{L}_2(H_-, H_+)$ into $\mathcal{M}$-invariant subspaces. There are countably many such invariant subspaces, each of which is canonically isomorphic to a space of holomorphic functions, H^m. The subspace H^1 can be identified with the space of Hankel operators. Each function v in H^1 is the "symbol" for a Hankel operator $B_1(v) = B(M_v)$, where M_v is multiplication by v. Janson and Peetre named the operators in the unique $\mathcal{M}$-invariant subspace isomorphic to H^m Hankel operators of order m.

In [42], I set out to find a formula for the operator $B_m(v)$, when v is a function in H^m. The main novel insight was that H^2 can be identified with the space H^{-1} of holomorphic vector fields on the disk, and there is the following relationship between H^m and the universal enveloping algebra of H^{-1}.

Theorem 3.1. *H^m can be embedded in the symmetric tensor product of $n-1$ copies of H^{-1}.*

Vector fields are differential operators of order one. As a composition of $m-1$ vector fields, $B_m(v)$ is a differential operator of order at most $m-1$.

Using the action of $\mathcal{M}$, it is possible to pick out distinguished elements v_m of H^m, as well as their images V_m in $\mathcal{L}_2(H_-, H_+)$. Given that $B_m(v_m)$ is a differential operator of order at most $m-1$, the linear equation $B_m(v_m) = V_m$ determines the coefficients of B_m up to a multiplicative constant. This equation can be solved using some beautiful identities for binomial coefficients. Further properties of B_m lead to novel binomial coefficient identities.

3.1. **Future Work.** It remains an open question whether there are coordinates for the spaces H^m in which the determinants of higher-order Hankel operators can be written as products. The relation between H^m and the universal enveloping algebra $\mathcal{U}$ of H^{-1} suggests that the formula obtained for B_n may be related to a coproduct in a quantum group associated to $\mathcal{U}$. I hope to pursue these questions further in the future.

4. Biology and Data Analysis

Biology was my first love in the sciences. I majored in neuroscience as an undergraduate, and intended to study mathematical neuroscience in graduate school. Trouble reconciling my interest in biology with my taste for geometry and algebra led me to physics, where these disciplines have brought qualitative insights into the analysis of complicated nonlinear and high-dimensional systems. A recent workshop at the Mathematical Biosciences Institute, "Mathematical Developments Arising from Biology," made it clear to me that there is no shortage of applications of geometry and algebra to biological systems. I am particularly interested in applications of geometry and topology to statistics, data analysis, and the study of networks.

4.1. **Statistics on Manifolds.** In the study of biological shape, data of interest lives on a Riemannian manifold or its quotient. Certain representations of anatomical structures are Lie group elements [15], while in object recognition and medical imaging studies, data lie in shape spaces [29]. These are spaces of k-tuples of points in Euclidean or projective space, representing k landmarks on an object. To study object shape, we discard location, size, and orientation information, and deal with k-tuples up to equivalence under a transformation group. Thus, shape spaces are quotients of Riemannian manifolds by isometric Lie group actions.

Recently, this has motivated a number of generalizations of statistical techniques to Riemannian manifolds and their quotients, where curvature introduces new wrinkles. For example, the Fréchet means of a probability measure μ on a Riemannian manifold (M, g) are points minimizing the function $\int_M d_g(x, y) d\mu(y)$, and they many not be unique. The large-sample statistics of data on shape spaces have been studied [3]. Principal geodesic analysis, which produces a geodesic best approximating a data set, has been studied on Lie groups [16], Riemannian manifolds [24], and quotients of Riemannian manifolds by isometric Lie group actions [22]. Multi-factorial statistical analysis has also been developed for quotients of Riemannian manifolds [23]. Infinite-dimensional shape spaces have also been employed [30].

Last year, I participated in a study group with Rabi Bhattacharya and Jorge Ramirez, constructing a smooth structure on a projective shape space [36], with an eye towards doing statistics on this space. I hope to continue this research, and to pursue open questions about the uniqueness of Fréchet means and principal geodesic components, as well as the generalization to principal geodesic submanifolds of higher dimension.

4.2. **Geometric and Topological Structures for Data Analysis.** The analysis of large, high-dimensional data sets is a problem of great practical and theoretical interest in all areas of modern science. The study of dynamical systems suggests that the evolution of otherwise unpredictable systems may be constrained to lie on or near low-dimensional manifolds, and that identifying these manifolds can provide a satisfying, qualitative understanding of such systems. The high dimensionality of many biological systems makes it difficult to identify such underlying geometric structures analytically. Using experimental data to identify these structures represents an attractive alternative.

The new field of topological data analysis applies ideas from topology to study the connectivity of point cloud data [8], through tools such as persistent homology [14]. Homology is an algebraic invariant which allows one to count the number of holes or hollows, of a given dimension, in a manifold. To apply homology to a data set, we place a ball of radius r at each data point, and approximate the resulting solid with a simplicial complex. The homology of this complex can then be computed. Varying r allows us to examine the connectivity of the data at different scales, and to distinguish important topological features, which persist over large r intervals, from sampling artifacts. This technique has been applied to images from natural scenes [9], genome data from cancer patients, and neuronal recording data [48]. In [48], topological data analysis was applied to neural recordings of spontaneous and driven activation in the visual cortex of macaque monkeys. It was discovered that, in both cases, the topological structure of neural activity mirrored the topological structure of natural image data, lying either on a two-dimensional surface deformable to a sphere, or on a one-dimensional closed loop. While spontaneous neural activity was mostly distributed over the surface, when the monkeys viewed natural images, their neural activity collapsed onto a loop. Understanding the significance of such findings requires a way of putting topologically relevant "coordinates" on data.

Geometric flows, such as the Ricci and mean curvature flows, have recently been used to study the topology of manifolds with great success. These flows change the metric of a Riemannian manifold, in many cases making it more regular, and illuminating the topology of the underlying space. Putting Riemannian structures, such as curvature, on simplicial complexes has a long history in physics, where it is related to the quantization of gravity. Putting a metric on a simplicial complex amounts to giving a length to each edge. Each simplex then inherits a Euclidean metric structure determined by its edge lengths, resulting in a piecewise linear manifold. For my oral comprehensive exam, I studied Cheeger, Müller & Schrader's [11] scalar curvature measure for two-dimensional piecewise linear manifolds. My supervisor for this project, David Glickenstein, has studied conformal structures and geometric flows on piecewise linear two- and three-manifolds using circle packings [17, 18, 2]. I am interested in applying these flows to data analysis, for example to put topologically relevant coordinates on data sets, by deforming the associated simplicial complexes to canonical shapes.

4.3. **Dynamics, Topology, and Harmonic Analysis on Networks.** The ubiquity of networks in biology, from neural networks to food webs to metabolic and gene regulatory networks, makes the study of dynamics on networks a crucial problem in mathematical biology. Active research is being conducted on the ways that network topology interacts with network dynamics [5, 38, 51, 54, 1, 5], with the dual goals of predicting dynamics from architecture, and deducing architecture from dynamics.

Predicting dynamics from topology is easiest for networks with highly regular structures, such as networks whose connectivity admits a group of symmetries. [19, 7]. Networks having a modular structure, characterized by the repetition of identical motifs, are common in neuroscience. I am interested in hierarchical modular networks, in which modules at one level become nodes in a module of the same form at a higher level [44]. Such networks have many properties characteristic of real world networks, and metabolic networks have been shown to have this form [43]. In recent research, signatures of hierarchical modular networks, and tools for extracting modules from such networks, were developed [45]. There is a possibility that such hierarchical modularity exists in the brain, for example in association cortex, and it would be interesting to study its effects on network dynamics.

On the other hand, attempts have been made to quantify the information processing properties of neural networks which lead to phenomena such as consciousness. Network complexity [52] is a measure of statistical independence at small scales combined with statistical dependence at large scales, relative to the size of the network. It is designed to captures the ability of the brain to process stimuli in distinct sensory modalities independently, while integrating these sensory modalities at the level of perception. I am interested in finding ways to compute this measure for neural data, and in using similar measures to quantify "progress" in evolutionary systems.

Diffusion geometry [12] is a tool which can be used to study either networks or data. The idea is simple: Given a graph and its Laplacian L, the diffusion e^{-L} and its eigenvectors ϕ_i, seen as functions on the vertices, provide information about the geometry of the graph. Ignoring all but the first n eigenvectors, for n small, and using the coordinates $(\phi_1(v), \ldots, \phi_n(v))$ for each vertex v provide a low-dimensional embedding of the data with interesting properties [37]. The diffusion L can be used to scale and smooth functions on the graph, providing a framework for harmonic and multiresolution analysis on graphs and networks [13]. These techniques can be used to understand the functional modules of networks, and I am interested in analyzing time series data from neuronal populations with this technology.

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Research Statement