A Definition of Scalar Curvature on Simplicial Approximations to Riemannian Manifolds

Benjamin Pittman-Polletta

April 6, 2007

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Why Discrete Curvature?

- Lattice approximations to quantum gravity.
- Computational gravity.
- Computation of geometric flows.
- Curvature of data sets?

Curvature

Definition

- The **Riemann curvature tensor** is the 4-tensor

$$R(X, Y, Z, W) = g(R(X, Y)Z, W),$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Coefficients: R_{ijkl}.

- The **sectional curvature** is the function

$$K(X, Y) = \frac{R(X, Y, Y, X)}{|X|^2 |Y|^2 - g(X, Y)^2}.$$

Definition

- The **Ricci curvature** is the symmetric 2-tensor

$$R_{ij} = g^{kl} R_{kijl}.$$

- The **scalar curvature** is the contraction

$$R^2 = g^{ij} g^{kl} R_{kijl}.$$

Scalar Curvature

- In 2d, $R^2 = \frac{K}{2}$, and K is the **Gauss curvature**.
- In higher dimensions, $R^2(p)$ is the average of K over all planes in T_pM.
- Scalar curvature tells us how the volume of a ball changes:

$$V_g\left(B_t^g(p)\right) = \left[1 - \frac{R^2}{6(n+2)}t^2 + O(t^4)\right] V_E\left(B_t(0)\right).$$

Scalar Curvature

- The vacuum Einstein equations are derived from the Einstein-Hilbert action:

$$\int kR^2\sqrt{-\det g}\,dx^4 = \int kR^2\,dV_g.$$

- Ricci flow is also derived from an action involving R^2.

Simplicial Complexes

Definition

A **finite simplicial complex** K is a finite set of elements called **vertices** and a set of finite, nonempty subsets of vertices called **simplices** so that

- Any set containing a single vertex is a simplex.
- Any nonempty subset of a simplex is also a simplex.

- $j+1$ vertices have dimension j.
- The dimension of K is $\sup_{\sigma \subset K} \dim \sigma$.

Piecewise Flat Spaces

Definition

A **piecewise flat space** is a simplicial complex and a set of edge lengths

$$l : \{\sigma^1\} \to \mathbb{R}$$

defining a metric flat on each simplex in K.

The mesh of a piecewise flat space is $\eta = sup_{\{\sigma^1\}} l$.

Approximating Manifolds

To approximate a manfold M:

1. Embed a simplicial complex K in M with a smooth mapping f.
2. Take the geodesic distances between the vertices as the metric l.

In 2d, we can take each edge to be totally geodesic.

Motivation

In 2d, imagine a "secant approximation" of a surface.

1. At a "flat point", the sum of angles is 2π.
2. At a "positive curvature point", the sum of angles is less than 2π.
3. At a "negative curvature point", the sum of angles is more than 2π.

Scalar Curvature for Piecewise Linear Spaces of Dimension 2

Definition

Let M be a Riemannian 2-manifold, let T be a piecewise-flat approximation, and let $U \in M$ be measurable. Then the **angle-defect measure** is

$$R^2_\eta(U) := \sum_{v_i \in T \cap U} R^2_\eta(v_i),$$

where

$$R^2_\eta(v_i) := 4\pi - 2 \sum_{\triangle_j \ni v_i} \alpha^j_i,$$

where α^j_i is the angle at v_i of the Euclidean triangle $\triangle_j$.

Convergence Results in Dimension 2

- The angle-defect measure is supported at vertices, so it will not converge pointwise to scalar curvature.
- For triangulations with $\eta \longrightarrow 0$, it converges to the integral of scalar curvature.

Convergence Results in Dimension 2

Definition

Let U be a measurable subset of a Riemannian manifold M. Then the **scalar curvature measure** is

$$R^2(U) = \int_U R^2(x)dV.$$

Theorem

Let U be a measurable subset of a Riemannian 2-manifold. There is $c = c(\|R\|, \|\nabla R\|)$ so that

$$\left|R^2_\eta(U) - R^2(U)\right| \leq c\left[A(U)\eta + A(\partial_\eta U)\right].$$

where $A(U)$ is the Riemannian area of U, and $\partial_\eta U$ is the set of points less than η from ∂U.

Relating Angles to Curvature

1. As we shrink a geodesic triangle to a vertex it becomes Euclidean.
2. Take a geodesic triangle with side length tl_1, tl_2 and tl_3, expand the angle at v_1 around $t = 0$.
3. Result:

$$\alpha_1 = \alpha_E + \frac{K}{3} A_E + O(t^3).$$

Proving Convergence

1. Summing twice the above result around a vertex we have

$$R^2_\eta(v) = \frac{1}{3}R^2(v)A_E\left(\cup_{\triangle_j \ni v}\triangle_j\right) + O(\eta^3).$$

2. We must replace the Euclidean area on the right hand side with Riemannian area.
3. The product of curvature with area on the right hand side can be used to estimate the integral.

The Angle Difference Formula

Proposition

Let $\triangle$ be a geodesic triangle on a Riemannian manifold. Let R and ∇R be bounded, Let d be the diameter of $\triangle$. Then,

$$\alpha_i = \alpha_E + \frac{K}{3} A_E + O(d^3),$$

where α_i is the angle of $\triangle$ at v_i, α_E is the angle of $\hat{\triangle}$, and A_E is the area of $\hat{\triangle}$.

Constant-Curvature Law of Cosines

- The law of cosines relates angles to side lengths:

$$\cos \alpha_1 = \frac{\cos(\sqrt{K} l_1) - \cos(\sqrt{K} l_2) \cos(\sqrt{K} l_3)}{\sin(\sqrt{K} l_2) \sin(\sqrt{K} l_3)}.$$

- We Taylor expand this law, in terms of

$$s_1 = \sqrt{K} l_1, \qquad s_2 = \sqrt{K} l_2, \qquad s_3 = \sqrt{K} l_3.$$

Taylor Expanding

Numerator:

$$\left[1 - \frac{(s_1)^2}{2} + \frac{(s_1)^4}{24} + O((s_1)^6)\right] - \left[1 - \frac{(s_2)^2}{2} + \frac{(s_2)^4}{24} + O((s_2)^6)\right]\left[1 - \frac{(s_3)^2}{2} + \frac{(s_3)^4}{24} + O((s_3)^6)\right]$$

Taylor Expanding

Denominator:

$$\left[s_2 - \frac{(s_2)^3}{6} + O((s_2)^5)\right]\left[s_3 - \frac{(s_3)^3}{6} + O((s_3)^5)\right]$$

Taylor Expanding

Altogether:

$$\begin{aligned}\cos(\alpha_1) = &-\frac{1}{s_2 s_3} \\ &\times \left[\frac{1}{2}\left[(s_1)^2 - (s_2)^2 - (s_3)^2\right] + \frac{1}{4}(s_2 s_3)^2 \right. \\ &\quad \left. - \frac{1}{24}\left[(s_1)^4 - (s_2)^4 - (s_3)^4\right] + O(K^3 d^6)\right] \\ &\times \left[1 + \frac{1}{6}\left[(s_2)^2 + (s_3)^2\right] + O(Kd^4)\right] + O(d^3)\end{aligned}$$

Taylor Expanding

Continuing:

$$\cos(\alpha_1) = \frac{(s_2)^2 + (s_3)^2 - (s_1)^2}{2s_2s_3}$$
$$+ \frac{s_2s_3}{6}\left(\frac{(s_1)^2 + (s_2)^2 + (s_3)^2 - 2\{(s_1s_2)^2 + (s_1s_3)^2 + (s_2s_3)^2\}}{4(s_2s_3)^2}\right)$$
$$+ O(d^3)$$

Eureka!

The Euclidean law of cosines!

$$\cos(\alpha_E) = \frac{(l_2)^2 + (l_3)^2 - (l_1)^2}{2 l_2 l_3},$$

$$\cos^2(\alpha_E) = \frac{(l_1)^4 + (l_2)^4 + (l_3)^4 + 2(l_2)^2(l_3)^2 - 2(l_1)^2\left[(l_2)^2 + (l_3)^2\right]}{4(l_2 l_3)^2}.$$

Simplifying

$$\begin{aligned}\cos(\alpha_1) &= \cos(\alpha_E) + \frac{s_2 s_3}{6}\left[cos^2(\alpha_E) - 1\right] + O(d^3) \\ &= \cos(\alpha_E) - \frac{s_2 s_3}{6} sin^2(\alpha_E) + O(d^3).\end{aligned}$$

Angles or Cosines?

We plug this into another expansion,

$$\cos^{-1}(x) = \cos^{-1}(a) - \frac{1}{\sqrt{1-a^2}}(x-a) + O(x-a)^2.$$

The result is as desired:

$$\alpha_1 = \alpha_E + \frac{K}{3}A_E + O(d^3).$$

A Nonconstant Curvature Law of Cosines

Proposition

Let $\triangle$ be a geodesic triangle on a Riemannian 2-manifold. Let $\|R\|$ and $\|\nabla R\|$ be bounded. Let d be the diameter of $\triangle$. Assume $\triangle$ contains no conjugate points. Then,

$$\cos(\alpha_i) = \frac{\cos(\sqrt{K}l_i) - \cos(\sqrt{K}l_j)\cos(\sqrt{K}l_k)}{\sin(\sqrt{K}l_j)\sin(\sqrt{K}l_k)} + O\left(d^3\right),$$

where α_i is the angle at v_i, l_m is the length of the side opposite v_m, and K is the sectional curvature at v_i.

A Constant Curvature Approximation

1. A metric g can be Taylor expanded:

$$g = \delta_{ij} - \frac{1}{3} R_{iklj}(0) x^k x^l + \nabla_m R_{iklj}(0) x^k x^l x^m + O(|x|^4).$$

2. We can approximate g up to second order by a constant curvature metric; lengths should agree to high order.
3. We take a manifold with constant sectional curvature equal to $K(v_1)$, and line its normal coordinates up with ours.

A Constant Curvature Approximation

1. We know

$$\alpha_1^K = \alpha_1, \qquad s_2^K = s_2, \qquad s_3^K = s_3.$$

2. By the law of cosines already have

$$\cos(\alpha_1) = \frac{\cos(\sqrt{K} l_1^K) - \cos(\sqrt{K} l_2)\cos(\sqrt{K} l_3)}{\sin(\sqrt{K} l_2)\sin(\sqrt{K} l_3)}.$$

3. We must compute

$$|l_1 - l_1^K|.$$

Comparing Curves in the Tangent Space

We must compare lengths in the tangent space.

$$\hat{g} = \exp_g^* g, \qquad \hat{g}_K = \exp_{g_K}^* g_K,$$
$$\hat{\gamma}_1 = \exp_g^{-1} \gamma_1(t), \qquad \hat{\gamma}_1^K = \exp_{g_K}^{-1} \gamma_1^K(t).$$

Comparing Curves in the Tangent Space

- Both curves are geodesics for their respective metrics, so

$$l_1 \leq \int_{\hat{\gamma}_1^K} \sqrt{\hat{g}(\dot{\hat{\gamma}}_1^K(t), \dot{\hat{\gamma}}_1^K(t))}dt,$$

$$l_1^K \leq \int_{\hat{\gamma}_1} \sqrt{\hat{g}_K(\dot{\hat{\gamma}}_1(t), \dot{\hat{\gamma}}_1(t))}dt.$$

Comparing Curves in the Tangent Space

- If $l_1 > l_1^K$, then

$$
\begin{aligned}
|l_1 - l_1^K| &= l_1 - l_1^K \\
&= \int_{[0,1]} \left[\sqrt{\hat{g}(\dot{\hat{\gamma}}_1(t), \dot{\hat{\gamma}}_1(t))} - \sqrt{\hat{g}_K(\dot{\hat{\gamma}}_1^K(t), \dot{\hat{\gamma}}_1^K(t))} \right] dt \\
&\leq \int_{[0,1]} \left[\sqrt{\hat{g}(\dot{\hat{\gamma}}_1^K(t), \dot{\hat{\gamma}}_1^K(t))} - \sqrt{\hat{g}_K(\dot{\hat{\gamma}}_1^K(t), \dot{\hat{\gamma}}_1^K(t))} \right] dt.
\end{aligned}
$$

- If $l_1 < l_1^K$, then

$$
|l_1 - l_1^K| \leq \int_{[0,1]} \left[\sqrt{\hat{g}_K(\dot{\hat{\gamma}}_1(t), \dot{\hat{\gamma}}_1(t))} - \sqrt{\hat{g}(\dot{\hat{\gamma}}_1(t), \dot{\hat{\gamma}}_1(t))} \right] dt.
$$

Comparing Curves in the Tangent Space

Thus we have the bound

$$|l_1 - l_1^K| \leq \max\{\int_{[0,1]} \left|\sqrt{g(\dot\gamma_1(t), \dot\gamma_1(t))} - \sqrt{g_K(\dot\gamma_1(t), \dot\gamma_1(t))}\right| dt, \int_{[0,1]} \left|\sqrt{g(\dot\gamma_1^K(t), \dot\gamma_1^K(t))} - \sqrt{g_K(\dot\gamma_1^K(t), \dot\gamma_1^K(t))}\right| dt\}$$

- Taylor expanding, the first two terms cancel by construction.
- Since tangent vectors are also bounded, we get

$$|l_1 - l_1^K| = O(d^4).$$

- We plug this in to the cosine law:

$$\cos(\alpha_1) = \frac{\cos(\sqrt{K}l_1) - \cos(\sqrt{K}l_2)\cos(\sqrt{K}l_3)}{\sin(\sqrt{K}l_2)\sin(\sqrt{K}l_3)} + O(d^3).$$

Approximating the Integral

Proposition

Let U be measurable. Let R and ∇R be bounded on U. If T is a geodesic triangulation of U with mesh η, then

$$\left| R^2(U) - \frac{1}{3} \sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A(\triangle_j) \right| \leq c(\|R\|, \|\nabla R\|) \left[A(U)\eta + A(\partial_\eta U) \right],$$

where v_i and $\triangle_j$ are vertices and triangles of T, A is the Riemannian area, and $\partial_\eta U$ is the set of points less than η from ∂U.

Start With an Expansion

- We Taylor expanding R^2 around v_1 and integrate over $\triangle$:

$$R^2(\triangle) - R^2(v_1)A(\triangle) \leq \|\nabla R^2\| A(\triangle) d + O(d^2).$$

- We sum over triangles:

$$R^2\left(\bigcup_{\triangle_j \in U} \triangle_j\right) - \sum_{\triangle_j \in U} R^2(v_1^j)A(\triangle_j) \leq \|\nabla R\| A(U)\eta + O(\eta^2)$$

More Estimates

- Counting triangles or vertices?

$$\left| \frac{1}{3} \sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A(\triangle_j) - \sum_{\triangle_j \in U} R^2(v_1^j) A(\triangle_j) \right| = O(\eta).$$

- Difference between U and its triangulation:

$$R^2(U) - R^2 \left(\bigcup_{\triangle_j \in U} \triangle_j \right) \leq \|R\| A\left(\partial_\eta U\right).$$

Our Estimates' Powers Combined

Combining:

$$\left| R^2(U) - \frac{1}{3} \sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A(\triangle_j) \right|$$
$$\leq c(\|R\|, \|\nabla R\|) \left[A(U)\eta + A(\partial_\eta U) \right]$$

Out of Many Estimates, One Result

Theorem

$$\left|R^2_\eta(U) - R^2(U)\right| \leq c\left[A(U)\eta + A(\partial_\eta U)\right].$$

The Triangle Inequality

$$\begin{aligned}
&\left|R^2_\eta(U) - R^2(U)\right| \\
&\leq \left|R^2_\eta(U) - \frac{1}{3}\sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A_E(\triangle_j)\right| \\
&\quad + \left|\frac{1}{3}\sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A_E(\triangle_j) - \frac{1}{3}\sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A(\triangle_j)\right| \\
&\quad + \left|\frac{1}{3}\sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A(\triangle_j) - R^2(U)\right|
\end{aligned}$$

We've estimated the first and last term:

- $$\left| R^2_\eta(U) - \frac{1}{3} \sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A_E(\triangle_j) \right| = O(\eta^3).$$
- $$\left| \frac{1}{3} \sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A(\triangle_j) - R^2(U) \right| \leq c(\|R\|, \|\nabla R\|) \left[A(U)\eta + A(\partial_\eta U) \right].$$

The middle term is a difference between Euclidean and Riemannian areas.

$$\left| \frac{1}{3} \sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A(\triangle_j) - \frac{1}{3} \sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A_E(\triangle_j) \right|$$

$$\leq \frac{1}{3} \sum_{v_i \in U} \left| R^2(v_i) \right| \sum_{\triangle_j \ni v_i} \left| A(\triangle_j) - A_E(\triangle_j) \right| .$$

A Final Expansion

We use a volume form expansion,

$$dV_g = \left[1 - \frac{1}{6} R_{kl} x^k x^l + O(|x|^3)\right] dV_E,$$

to show

$$\frac{1}{3} \sum_{v_i \in U} |R^2(v_i)| \sum_{\triangle_j \ni v_i} |A(\triangle_j) - A_E(\triangle_j)| = O(\eta^2).$$

Our Result!

Thus, to leading order,

$$\left|R^2_\eta(U) - R^2(U)\right| \leq c(\|R\|, \|\nabla R\|)\left[A(U)\eta + A(\partial_\eta U)\right].$$

Concluding Remarks

- In higher dimensions, look at angle defect around simplices of codimension 2.
- Proof in higher dimensions is *not local*.
- We hope to find a local proof in dimension 3.
- Possible use: approximating Ricci flow.