# RANDOM FUNCTIONS AND POINT PROCESSES

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Contents



## 1. Introduction

We are interested in classifying or organizing all the conformally invariant measures on $H^m(\triangle)$. We can think of a probability measure on $H^m(\triangle)$ as a random function - pushing the measure forward under the evaluation map gives us a probability measure on $\mathbb{C}$ for every point in $\triangle$ - and we will use the terms measure and random function interchangeably. Obviously, there are a huge number of conformally invariant, or stationary, random functions on $\triangle$.

Every conformally invariant measure $\mu$ on $H^m(\triangle)$ also induces a point process on $\triangle$, by associating each function in $H^m(\triangle)$ with its zeros. Thus, one approach to studying the measures on $H^m(\triangle)$ is to study point processes on the disk. Point processes are characterized by their joint intensities. Roughly, the $k$-point intensity gives the expected number of points lying in $k$ disjoint neighborhoods in the disk; the one point intensity gives the expected number of points in any neighborhood. There are also a huge number of point processes on $\triangle$ - the compatibility requirements for the joint intensities of a point process are not strong. Since conformally invariant random functions induce stationary point processes, these are the ones we'd like to know something about. Conformal invariance places a restriction on the one-point intensity of point process - namely, the one-point intensity of a stationary point process must be a multiple of the Poincare metric on

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the disk. Even this is too weak, however, and attempting to classify the stationary point processes on $\triangle$ is a task of the same magnitude as the classification the conformally invariant measures.

Luckily, there are certain kinds of point processes whose joint intensities are more restricted. These are the determinantal and permanental processes, and their generalizations, the $\alpha$-determinantal processes. In this case, the joint intensities are by definition determined by a single function of two complex variables, the kernel. This restriction is sufficiently stringent that the stationary $\alpha$-determinantal processes on the disk form a two-parameter family.

One of the goals of this paper is to gather together facts and ideas which may be useful in illuminating the connection between random functions and point processes. In particular, we'd like to find a connection between the stationary $\alpha$-determinantal process of intensity $2\rho$ and a random measure on $H^\rho(\triangle)$.

## 2. Point Processes - Determinantal, Permanental, & Other

**Definition 2.1.** A **random element** is a measurable function from a probability space $(\Omega, \mathcal{E}, P)$ into a measurable space $(X, \mathcal{F})$, usually a topological vector space.

**Definition 2.2.** Let $S$ be an LCH space and $\mathcal{B}$ its Borel $\sigma$-algebra, and let $\mathfrak{N}(S)$ be the set of locally finite counting measures on $S$. Let $\mathcal{N}$ be the smallest $\sigma$-algebra on $\mathfrak{N}$ which makes the evaluation map for each $B \in \mathcal{B}$,

$$\Phi_B : \mathfrak{N} \longrightarrow \mathbb{Z} : n \longmapsto n(B),$$

a measurable function. Then we call $(\mathfrak{N}(S), \mathcal{N})$ the measurable space of **point sets** on $S$. A **point process** $\chi$ is a measurable map from a probability space $(\Omega, \mathcal{E}, P)$ into $(\mathfrak{N}(S), \mathcal{N})$.

Any point process can be represented as

$$\chi = \sum_{i=1}^{N} \delta_{X_i}$$

where $N$ is an integer-valued random variable on $\Omega$, and $X_i$ is an $S$-valued random element on $\Omega$ for each $i$.

**Definition 2.3.** Given a point process $\chi : \Omega \to \mathfrak{N}(S)$, let $\chi^{\wedge k} : \Omega \to \mathfrak{N}(S^k)$ be the point process assigning to each outcome $\omega$ every ordered k-tuple of points in $\chi(\omega)$. The **joint intensities** of $\chi$ with respect to the measure $\mu$ on $S$ are the functions $\rho_k : S^k \to [0, \infty)$ such that for any Borel set $B \subset S^k$,

$$E\left[\chi^{\wedge k}(B)\right] = \int_B \rho_k(x_1, \ldots, x_k) d\mu(x_1) \ldots d\mu(x_k).$$

Q: Is $\mathfrak{N}(S)^k$ with the product $\sigma$-algebra the same as $\mathfrak{N}(S^k)$?

When $B = A^k$ for some Borel $A \subset S$, then

$$E\left[\chi^{\wedge k}(B)\right] = E\left[\left(\begin{array}{c}\chi(A)\\ k\end{array}\right) k!\right],$$

so that integrating $\rho$ over $A^n$ tells us the expected number of of ordered $k$-tuples of points in $\chi$ lying in $A$. (The appearance of the binomial coefficient accounts for the fact that we are counting $k$-tuples, not points, and the $k!$ counts the different orderings of each $k$-tuple.) When $B = \prod_{i=1}^k A_i$ for *disjoint* Borel sets $A_i \subset S$, we have

$$E\left[\chi^{\wedge k}(B)\right] = E\left[\prod_{i=1}^{k} \chi(A_i)\right].$$

Indeed, if $S$ can be thought of as an open set in $\mathbb{R}^n$ and $\mu$ is Lebesgue measure, then we can write

$$\rho_k(x_1, \ldots, x_k) = \lim_{\epsilon \to 0} \frac{P\left(\chi \text{ has a zero in each } B_\epsilon(z_i)\right)}{Vol(B_\epsilon(0))^k}.$$

**Definition 2.4.** A **determinantal point process** with kernel $K$ is a point process $\chi : \Omega \to S$ with joint intensities

$$\rho_k(x_1, \ldots, x_k) = \det\left(K(x_i, x_j)\right)_{1 \le i,j \le k},$$

for every $k \geq 1$ and $x_1, \ldots, x_k \in S$.

**Proposition 2.5.** *The one-point intensity $\rho_1$ of a stationary point process on $\triangle$ must be a multiple of the Poincare metric on $\triangle$.*

*Proof.* Let $dm(z) = dRe(z)dIm(z)$ be Lebesgue measure on $\mathbb{C}$. By stationarity, for any Borel set $B \subset \mathbb{C}^k$, and any Möbius transformation $M$, we must have

$$E\left[\chi(B)\right] = E\left[\chi(MB)\right],$$

which implies

$$\begin{aligned}
\int_B \rho_1(z)\pi^{-1}\left(1-|z|^2\right)^{-2} dm(z) &= \int_{MB} \rho_1(z)\pi^{-1}\left(1-|z|^2\right)^{-2} dm(z) \\
&= \int_B \rho_1(Mz)\pi^{-1}\left(1-|Mz|^2\right)^{-2}\left|\frac{\partial M}{\partial z}\right| dm(z),
\end{aligned}$$

where we used the change of variables formula in the last line. Now, let

$$M(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}.$$

Then,

$$\begin{aligned}
\left|\frac{\partial M}{\partial z}\right| &= \left|\frac{|\alpha|^2 - |\beta|^2}{\left(\bar{\beta} z + \bar{\alpha}\right)^2}\right| \\
&= \left|\bar{\beta} z + \bar{\alpha}\right|^{-2}.
\end{aligned}$$

Also,

$$\begin{aligned}
1 - |Mz|^2 =& \frac{|\bar{\beta} z + \bar{\alpha}|^2 - |\alpha z + \beta|^2}{|\bar{\beta} z + \bar{\alpha}|^2} \\
=& \frac{\left(|\beta|^2 - |\alpha|^2\right)|z|^2 + |\alpha|^2 - |\beta|^2}{|\bar{\beta} z + \bar{\alpha}|^2} \\
=& \frac{1 - |z|^2}{|\bar{\beta} z + \bar{\alpha}|^2}.
\end{aligned}$$

Thus,

$$\left(1-|Mz|^2\right)^{-2}\left|\frac{\partial M}{\partial z}\right| = \frac{\left|\bar\beta z+\bar\alpha\right|^4}{(1-|z|^2)^2}\left|\bar\beta z+\bar\alpha\right|^{-2} = \frac{\left|\bar\beta z+\bar\alpha\right|^2}{(1-|z|^2)^2}??$$

□

**Theorem 2.6.** *The only projective, stationary determinantal processes on $\triangle$ are those with kernel*

$$K_\rho(z,w) = \frac{\rho}{\pi}\left(1-z\bar w\right)^{-\rho-1}.$$

*Proof.* Let $\chi$ be a projective, stationary determinantal process on $\triangle$ with kernel $K$. Then, because $\chi$ is stationary,

$$\rho_1(z) = K(z,z) = \rho\lambda^2(z) = \frac{\rho}{\pi}\left(1-|z|^2\right)^{-2}.$$

Now,

$$\begin{aligned}\rho_k(z_1,\dots,z_k)dm(z_1)\dots dm(z_k) =& \frac{\rho_k(z_1,\dots,z_k)}{\lambda^2(z_1)\dots\lambda^2(z_k)}\lambda^2(z_1)dm(z_1)\dots\lambda^2(z_k)dm(z_k)\\ =& \frac{\rho^k\det\left(K(z_i,z_j)_{1\le i,j\le k}\right)}{K(z_1,z_1)\dots K(z_k,z_k)}d\mu(z_1)\dots d\mu(z_k).\end{aligned}$$

Notice also that

$$\frac{\rho^k\det\left(K(z_i,z_j)_{1\le i,j\le k}\right)}{K(z_1,z_1)\dots K(z_k,z_k)} = \det\left(\frac{\rho K(z_i,z_j)}{\sqrt{K(z_i,z_i)K(z_j,z_j)}}_{1\le i,j\le k}\right).$$

Thus, with respect to the Poincaré measure $\mu$, $\chi$ has kernel

$$\tilde K(z,w) = \frac{\rho K(z,w)}{\sqrt{K(z,z)K(w,w)}}.$$

Now,

$$\rho_2(z,w) = \det\tilde K(z,w) = \rho^2 - \frac{\rho^2|K(z,w)|^2}{K(z,z)K(w,w)},$$

since $K(w,z)=\overline{K(z,w)}$ due to the fact that $K$ is a projection. By stationarity, since we've expressed the joint intensities with respect to the Poincaré measure, we must have $\rho_2(Mz,Mw)=\rho_2(z,w)$, so

$$\frac{|K(z,w)|^2}{K(z,z)K(w,w)} = \frac{|K(Mz,Mw)|^2}{K(Mz,Mz)K(Mw,Mw)}$$

for any Möbius transformation $M$. This means that

$$\frac{|K(Mz,Mw)|^2}{K(Mz,Mz)K(Mw,Mw)} = C(z,w)$$

for every $M$, where $C$ is a function of $z$ and $w$ that doesn't depend on $M$. Now, if we write

$$M(z) = \frac{\alpha z+\beta}{\bar\beta z+\bar\alpha}$$

then we can parameterize the space of Möbius transformations by $\alpha$ and $\beta$, and $C$ is thus constant in these parameters. We will apply a differential operator, in the derivatives of $\alpha$ and $\beta$, to the (logarithm of the) previous expression to obtain a differential equation for $K$. First, define

$$\begin{aligned} L :=& \left(\frac{\partial}{\partial \beta} - \bar{w}\frac{\partial}{\partial \alpha}\right)\left(\frac{\partial}{\partial \bar{\beta}} - z\frac{\partial}{\partial \bar{\alpha}}\right) \\ =& \frac{\partial^2}{\partial \beta \partial \bar{\beta}} - z\frac{\partial^2}{\partial \beta \partial \bar{\alpha}} - \bar{w}\frac{\partial^2}{\partial \alpha \partial \bar{\beta}} + z\bar{w}\frac{\partial^2}{\partial \alpha \partial \bar{\alpha}} \\ =:& L_1 + L_2 + L_3 + L_4. \end{aligned}$$

Then, note that if $g(z, w)$ is a function analytic in its first argument and anti-analytic in its second argument, then

$$L_i g(M(z), M(w)) = \partial_1 \bar{\partial}_2 g(M(z), M(w)) L_i M(z) L_i M(w),$$

where $\partial_1$ is the derivative with respect to the first argument, and $\partial_2$ is the derivative with respect to the section argument. Then,

$$LM(z) = \frac{z - z^2\bar{w} - z + z^2\bar{w}}{-\left(\bar{\beta}z + \bar{\alpha}\right)^2} = 0,$$

while

$$LM(w) =$$

$\square$

**Proposition 2.7.** *The only projective, stationary $\alpha$-determinantal processes on $\triangle$ are those with kernel*

$$K_\rho(z, w) = \frac{\rho}{\pi}\left(1 - z\bar{w}\right)^{-\rho-1}.$$

*Proof.* As above, stationarity implies that

$$\rho_1(z) = K(z, z) = \frac{\rho}{\pi}\left(1 - |z|^2\right)^{-2}$$

and that

$$\frac{\rho_2(z, w)}{\lambda^2(z)\lambda^2(w)} = \frac{\rho_2(Mz, Mw)}{\lambda^2(Mz)\lambda^2(Mw)}$$

for any $M \in PSU(1, 1)$. But,

$$\begin{aligned} \rho_2(z, w) =& \det{}_\alpha K(z, w) \\ =& \alpha^2 K(z, z)K(w, w) - \alpha K(z, w)K(w, z) \\ =& \alpha^2\rho^2\lambda^2(z)\lambda^2(w) - \alpha\left|K(z, w)\right|^2. \end{aligned}$$

Thus,

$$\alpha^2\rho^2 - \frac{\alpha\left|K(z, w)\right|^2}{\lambda^2(z)\lambda^2(w)} = \alpha^2\rho^2 - \frac{\alpha\left|K(Mz, Mw)\right|^2}{\lambda^2(Mz)\lambda^2(Mw)},$$

so as above

$$\frac{|K(Mz, Mw)|^2}{K(Mz, Mz)K(Mw, Mw)} = C(z, w).$$

The rest of the argument proceeds as above. $\square$

**Proposition 2.8.** *There is a one-parameter family of conformally-invariant Poisson processes on $\triangle$, with joint intensities*

$$\rho_k^s(z_1, \ldots, z_k) = s^k \prod_1^k \frac{1}{(1 - |z_i|^2)^2}.$$

*Proof.* The conformally-invariant Poisson processes on $\triangle$ are defined by the property that

$$P\left(\cap_1^n \{\chi_s(B_i) = n_i\}\right) = \prod_1^n \frac{s^{n_i}}{n_i!} \mu(B_i)_i^n \exp\left(-s\mu(B_i)\right),$$

for disjoint Borel subsets $\{B_i\}_1^n$, $B_i \subset \mathbb{C}$. Thus, we have that

$$\begin{aligned}
E\left[\prod_1^n \chi_s(B_i)\right] &= \sum_{n_1,\ldots,n_n=0}^{\infty} \prod_1^n n_i \frac{s^{n_i}}{n_i!} \mu(B_i)^{n_i} \exp\left(-s\mu(B_i)\right) \\
&= \exp\left(-s \sum_1^n \mu(B_i)\right) \prod_1^n \sum_{m_i=0}^{\infty} \frac{s^{n_i}}{(n_i - 1)!} \mu(B_i)^{n_i} \\
&= \exp\left(-s \sum_1^n \mu(B_i)\right) \left[1 + s^n \prod_1^n \mu(B_i) \sum_{m_i=0}^{n} \frac{(s\mu(B_i))^{n_i - 1}}{(n_i - 1)!}\right] \\
&= \exp\left(-s \sum_1^n \mu(B_i)\right) \left[1 + s^n \prod_1^n \mu(B_i) \exp\left(s\mu(B_i)\right)\right] \\
&= \exp\left(-s \sum_1^n \mu(B_i)\right) + s^n \prod_1^n \mu(B_i).
\end{aligned}$$

□

## 3. Zeroes of the i.i.d. Gaussian power series.

Peres & Virág ([1]) have shown that the zeros of the power series with i.i.d. Gaussian coefficients are a conformally invariant determinantal process in the plane. I will repeat their proof here. However, first we will need a number of preliminary lemmas. The most important of these is the following.

### 3.1. **The form of the joint intensity.**

**Proposition 3.1.** *Let $f$ be a Gaussian analytic function on $\triangle$ so that $E[f(z)] = 0$ for all $z \in \triangle$. Then the joint intensities for the zeros of $f$ exist and satisfy*

$$\rho_k(z_1, \ldots, z_k) = \frac{E\left[|f'(z_1) \ldots f'(z_k)|^2 \middle| f(z_1) = \ldots f(z_k) = 0\right]}{\pi^k \det\left(E\left[f(z_i)\overline{f(z_j)}\right]_{1 \le i,j \le k}\right)}.$$

This, in turn, relies on Hammersley's formula, which we present without proof.

**Theorem 3.2** (Hammersley's Formula)**.** *Let $f_n(z) = a_n z^n + \ldots + a_0$ be a random polynomial so that $(a_0, \ldots, a_n)$ has a distribution that is absolutely continuous with respect to Lebesgue measure on $\mathbb{C}^{n+1}$. Then the joint intensity of the zeros of $f_n$ exists, and can be written*

$$\rho_k(z_1, \ldots, z_k) = \lim_{\epsilon \to 0} \left(\pi\epsilon^2\right)^{-k} \int_{\cap_1^k \{f_n(z_i) \in B_\epsilon(0)\}} |f'(z_1) \ldots f'(z_k)|^2 da_0 \ldots da_n.$$

Our strategy is the following. First, we show that Proposition 3.1 holds for products of disjoint Borel subsets of $\triangle$. That this is true for polynomials with i.i.d. Gaussian coefficients is a direct consequence of Hammersley's formula. For i.i.d. Gaussian power series, we use approximation by polynomials. Next, we show that Gaussian analytic functions have no double zeros; in this case, the given form of the joint intensity also suffices for arbitrary Borel subsets of $\mathbb{C}^k$.

*Proof.* First, we assume that $\Lambda_1, \ldots, \Lambda_k$ are disjoint Borel subsets of $\triangle$. Then, let $N_f(\Lambda)$ be the number of zeros of $f$ in $\Lambda$. Now, by Hammersley's formula, if $f$ is a polynomial whose coefficients have joint density absolutely continuous with respect to Lebesgue measure, then

$$E\left[\prod_{i=1}^k N_f(\Lambda_i)\right] = \int_{\prod_1^k \Lambda_i} \lim_{\epsilon \to 0} \left(\pi\epsilon^2\right)^{-k} \times \int_{\cap_1^k \{f_n(z_i) \in B_\epsilon(0)\}} |f'(z_1) \ldots f'(z_k)|^2 \prod_1^n da_i \prod_1^k dz_i.$$

Note that the integrand above is

$$\lim_{\epsilon \to 0} \frac{E\left[|f'(z_1) \ldots f'(z_k)|^2 \middle| f(z_i) \in B_\epsilon(0), i = 1, \ldots, k\right]}{\pi^k \epsilon^{2k}}.$$

For a polynomial with coefficients which are i.i.d. standard complex Gaussians,

$$\left(\pi\epsilon^2\right)^{-k} \int_{\cap_1^k \{f_n(z_i) \in B_\epsilon(0)\}} |f'(z_1) \ldots f'(z_k)|^2 \prod_1^n da_i = \pi^{-(n+k)} \int_{\cap_1^k \{f_n(z_i) \in B_\epsilon(0)\}} \frac{|f'(z_1) \ldots f'(z_k)|^2}{\epsilon^{2k} \exp\left(\sum_1^n |z_i|^2\right)} \prod_1^n dz_i.$$

Now,

$$\exp\left(\sum_1^n |z_i|^2\right) = \exp\left(Tr\left((z_i \bar{z}_j)_{1 \le i,j \le n}\right)\right) = \det\left(\exp\left((z_i \bar{z}_j)_{1 \le i,j \le n}\right)\right).$$

Note that

$$\left((z_i \bar{z}_j)_{1 \le i,j \le n}\right)^m = \left(\sum_1^n |z_i|^2\right)^{m-1} \left((z_i \bar{z}_j)_{1 \le i,j \le n}\right),$$

so that

$$\exp\left((z_i \bar{z}_j)_{1 \le i,j \le n}\right) = \frac{\exp\left(\sum_1^n |z_i|^2\right)}{\sum_1^n |z_i|^2} \left((z_i \bar{z}_j)_{1 \le i,j \le n}\right),$$

and

$$\det\left(\exp\left((z_i\bar{z}_j)_{1\le i,j\le n}\right)\right) = \left(\frac{\exp\left(\sum_1^n |z_i|^2\right)}{\sum_1^n |z_i|^2}\right)^n \det\left((z_i\bar{z}_j)_{1\le i,j\le n}\right).$$

This gives the handy formula

$$\det\left((z_i\bar{z}_j)_{1\le i,j\le n}\right) = \left(\sum_1^n |z_i|^2\right)^n \exp\left((1-n)\sum_1^n |z_i|^2\right),$$

but doesn't get us closer to what we want.

Anyway, assuming we have the proposition for polynomials with i.i.d. Gaussian coefficients, □

3.2. **The main theorem.**

**Theorem 3.3.** *The joint intensity of zeros for the i.i.d. Gaussian power series*

$$f_\triangle(z) = \sum_{k=0}^\infty a_k z^k$$

*in the unit disk exists, and satisfies*

$$p(z_1,\ldots,z_n) = \pi^{-n}\det\left[\frac{1}{(1-z_i\bar{z}_j)^2}\right]_{ij} = \det\left[K_\triangle(z_i,z_j)\right]_{1\le i,j\le n}$$

*where $K_\triangle(z,w) = \pi^{-1}(1-z\bar{w})^{-2}$ is the Bergman kernel.*

Before proving this theorem, we will need a few more preliminary results. None of these, however, is deserving of its own section.

**Proposition 3.4.** *Let $M$ be a Möbius transformation fixing the unit disk. Then the random functions $f_\triangle(z)$ and*

$$f_\triangle^M(z) = \sqrt{M'(z)}f_\triangle(M(z))$$

*are identically distributed.*

*Proof.* Since $f_\triangle(z)$ is Gaussian, its distributions are determined by its covariance structure. We have

$$\begin{aligned} f_\triangle(z)\overline{f_\triangle(w)} &= \left(\sum_{k=0}^\infty a_k z^k\right)\overline{\left(\sum_{k=0}^\infty a_k w^k\right)} \\ &= \sum_{k=0}^\infty |a_k|^2 (z\bar{w})^k + \sum_{m\ne n} a_m\bar{a}_n z^m \bar{z}^n. \end{aligned}$$

Then,

$$E\left[f_\triangle(z)\overline{f_\triangle(w)}\right] = \sum_{k=0}^\infty E\left[|a_k|^2\right](z\bar{w})^k + \sum_{m\ne n} E\left[a_m\bar{a}_n\right] z^m\bar{z}^n.$$

Because the $a_i$ are independent,

$$E\left[a_m\bar{a}_n\right] = E\left[a_m\right]E\left[\bar{a}_n\right] = 0.$$

Also, $E\left[|a_k|^2\right]$ is just the variance of $a_k$, which is 1. Thus,

$$E\left[f_\triangle(z)\overline{f_\triangle(w)}\right] = \sum_{k=0}^{\infty}(z\bar{w})^k = \frac{1}{1-z\bar{w}} =: S_\triangle(z,w).$$

Likewise,

$$E\left[f_\triangle^M(z)\overline{f_\triangle^M(w)}\right] = \frac{\sqrt{M'(z)\overline{M'(w)}}}{1-M(z)\overline{M(w)}}.$$

Let

$$M(z) = \frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}, \qquad |\alpha|^2-|\beta|^2 = 1.$$

Then,

$$1-M(z)\overline{M(w)} = \frac{1-z\bar{w}}{(\bar{\beta}z+\bar{\alpha})(\beta\bar{w}+\alpha)},$$

and

$$M'(z) = \left(\bar{\beta}z+\bar{\alpha}\right)^{-2},$$

so that

$$S_\triangle(z,w) = \left(M'(z)\overline{M'(w)}\right)^{1/2} S_\triangle(M(z),M(w)),$$

and thus $f_\triangle$ and $f_\triangle^M$ are identically distributed. □

**Lemma 3.5.** *The distribution of the random function*

$$T_{z_1}(z)\dots T_{z_n}(z)f_\triangle(z)$$

*is the same as the conditional distribution of $f_\triangle(z)$ given $f(z_1) = \dots = f(z_n) = 0$.*

*Proof.* For $n = 1$ and $z_1 = 0$, we have $T_0(z) = z$ and so

$$T_0(z)f_\triangle(z) = zf_\triangle(z) = \sum_{k=0}^{\infty}a_k z^{k+1} = \sum_{k=1}^{\infty}a_{k-1}z^k.$$

Now, if $f_\triangle(z) = a_0 = 0$, then

$$f_\triangle(z) = \sum_{k=1}^{\infty}a_k z^k,$$

but since the $a_i$ are iid, the distribution of this function is the same as the distribution of $zf_\triangle(z)$. To see that this holds for arbitrary $z_1$, recall that the random function

$$\tilde{f}(z) = \sqrt{T'_{z_1}(z)}f_\triangle\left(T_{z_1}(z)\right)$$

has the same distribution as $f_\triangle$. If $z_1$ is a zero of $f$, then 0 is a zero of $\tilde{f}$. Thus, by the same reasoning as above, $T_{z_1}(z)\tilde{f}(z)$ has the same distribution as $f_\triangle$ given $z_1$ is a zero of $f_\triangle$; but $T_{z_1}\tilde{f}(z)$ and $T_{z_1}f(z)$ are identically distributed. For arbitrary $n$, we use induction. By assumption, $(f_\triangle|f_\triangle(z_1) = \dots = f_\triangle(z_k) = 0)$ and $T_{z_1}(z)\dots T_{z_k}(z)f_\triangle(z)$ are identically distributed. We must condition the first on $f_\triangle(z_{k+1}) = 0$, but this is the same as conditioning the second on the same event. But conditioning is a projection in $L^2(\Omega,P)$ for any Gaussian process on $(\Omega,P)$, and so it commutes with pointwise multiplication. Thus, we can apply the base case to show that $(f_\triangle(z)|f_\triangle(z_{k+1})=0)$ is equal in distribution to $T_{z_{k+1}}(z)f_\triangle(z)$, so obtaining our result. $\square$

**Proposition 3.6.** *If $Z_1,\dots,Z_n$ are mean 0 jointly complex Gaussian random variables with covariance matrix $M_{ij}=E\left[Z_i\bar{Z}_j\right]$, then $E\left[|Z_1\dots Z_n|^2\right]=$ $perm M$.*

*Proof.* $\square$

The following, and final, preliminary result is presented without proof.

**Theorem 3.7** (Borchardt's Determinant Identity)**.**

$$per\left((x_i+y_j)^{-1}_{1\le i,j\le k}\right)\det\left((x_i+y_j)^{-1}_{1\le i,j\le k}\right)=\det\left((x_i+y_j)^{-2}_{1\le i,j\le k}\right).$$

Now we are ready to prove the main theorem.

*Theorem 3.3.* For ease of notation, we define

$$\Upsilon(z)=\prod_{i=1}^n T_{z_i}(z),$$

and note the following.

$$\Upsilon'(z)=\sum_{j=1}^n T'_{z_j}(z)\prod_{i\neq j}T_{z_i}(z_j),$$

and since $T_{z_j}(z_j)=0$,

$$\begin{aligned}\Upsilon'(z_j)=&T'_{z_j}(z_j)\prod_{i\neq j}T_{z_i}(z),\\ =&(1-z_j\bar{z}_j)^{-1}\prod_{i\neq j}\frac{z_j-z_i}{1-\bar{z}_i z_j}\\ =&\prod_{i=1}^n(1-\bar{z}_i z_j)^{-1}\prod_{i\neq j}(z_j-z_i).\end{aligned}$$

Then, by Lemma **??**, the conditional joint distribution of $f_\triangle(z_j)$ given $f_\triangle(z_1)=\ldots=f_\triangle(z_n)=0$ is the same as the unconditional distribution of $\Upsilon(z_j)f_\triangle(z_j)$. Since $\Upsilon(z_j)=0$, the derivative of $\Upsilon(z)f_\triangle(z)$ at $z_j$ is $\Upsilon'(z_j)f_\triangle(z_j)$, and so the random variables $\Upsilon'(z_j)f_\triangle(z_j)$ have the same joint distribution as the variables $f_\triangle(z_j)$, once the latter are conditioned on $z_j$ being a zero for $f_\triangle$. Thus,

$$E\left[\left|f'_\triangle(z_1)\dots f'_\triangle(z_n)\right|^2\left|f_\triangle(z_1)=\ldots=f_\triangle(z_n)=0\right.\right]=\prod_1^n\left|\Upsilon'(z_i)\right|^2 E\left[\left|f_\triangle(z_1)\dots f_\triangle(z_n)\right|^2\right].$$

As we've shown,

$$\prod_{j=1}^n\left|\Upsilon'(z_j)\right|^2=\prod_{i\neq j}|z_j-z_i|^2\prod_{i,j=1}^n|1-\bar{z}_i z_j|^{-2},$$

and by Cauchy's determinant formula, we have

$$det\left(\frac{1}{1-z_i\bar{z}_j}_{1\leq i,j\leq n}\right)=\prod_{i<j}|z_j-z_i|^2\prod_{i,j=1}^{n}(1-z_i\bar{z}_j)^{-1}.$$

But then, since $|z_i-z_j|^2=|z_j-z_i|^2$ for all $i,j$, we have

$$\prod_{i\neq j}|z_j-z_i|^2=\left(\prod_{i<j}|z_j-z_i|^2\right)^2,$$

and so

$$\prod_1^n\left|\Upsilon'(z_i)\right|^2=\left|det\left(\frac{1}{1-z_i\bar{z}_j}_{1\leq i,j\leq n}\right)\right|^2.$$

□

## 4. Appendix

4.1. **Probability Spaces and Random Variables.** Probability theory is a mathematical attempt to describe phenomena which involve chance or randomness. While it developed independently from measure theory, modern probability theory relies heavily on measure theoretic concepts. In particular, the set of all outcomes of a random process is associated with a space, and the events made up of subsets of these outcomes are associated with a $\sigma$-algebra on that space. A measure is then used to assign a probability of occurrence to each of these events.

**Definition 4.1.** A **sample space** or **probability space** is a measure space $(\Omega, B, P)$ with the property that $P(\Omega) = 1$. The measure $P$ is called a probability measure, the elements of $\Omega$ are called outcomes, and the elements of $B$ are called events. For $E \in B$, $P(E)$ is said to be the probability of the event $E$.

For any $E \in B$, $P(E)$ is a number between zero and one. We can think of it as the proportion of the time we can expect $E$ to occur.

Often, it is useful to associate to each outcome in our sample space a numerical or other value. For example, we may like to associate a hand of blackjack with its total value; we could also assign a hand of blackjack a value of 1 if it contains a face card, and a value of 0 otherwise.

The set of values we assign to our outcomes is called the **state space** of our random process. It is also equipped with a $\sigma$-algebra. The map that takes outcomes to elements of our state space is called a random variable. The most common state space is $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, so we define a random variable as real-valued.

**Definition 4.2.** A random variable $X$ is a measurable function from a sample space $(\Omega, B, P)$ to $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

A real-valued random variable assigns a number to each outcome. The probability that the random variable takes on a certain value $r$ can be calculated as the measure of the event

$$\{\omega : X(\omega) = r\},$$

or, equivalently, the measure of the level set $X^{-1}(r)$. In this way, every random variable defines a probability measure on the real numbers, called its **distribution**.

**Definition 4.3.** Let $(\Omega, B, P)$ be a sample space, and let $X$ be a random variable on $\Omega$. For $E \in \mathcal{B}_{\mathbb{R}}$, let $P_X(E) = P(X^{-1}(E))$. Then $P_X$ is a probability measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, called the distribution of $X$. The function $F_X(t) = P_X((-\infty, t])$ is called the **distribution function** of $X$.

Note that $P_X$ is the push-forward of $P$ by $X$. A family $\{X_\alpha\}_{\alpha \in A}$ is called identically distributed if $P_{X_\alpha} = P_{X_\beta}$ for all $\alpha, \beta \in A$.

A collection $X_1, ..., X_n$ of random variables on $(\Omega, B, P)$ defines a map $X$ from $(\Omega, B, P)$ to $\mathbb{R}^n$ by

$$X(\omega) = (X_1(\omega), \ldots, X_n(\omega)).$$

This map, in turn, defines a probability measure on $\mathbb{R}^n$.

**Definition 4.4.** Let $X = (X_1, \ldots, X_n)$ be a collection of random variables on $(\Omega, B, P)$. The **joint distribution** of $X_1, \ldots, X_n$ is the push-forward of $P$ by $(X_1, \ldots, X_n)$. It is the unique probability measure on $\mathbb{R}^n$ which is defined as follows for a Borel rectangle $\prod_{i=1}^n B_i$, where $B_i \in \mathcal{B}_{\mathbb{R}}$ for each $i$:

$$P_{(X_1,\ldots,X_n)}\left(\prod_{i=1}^n B_i\right) = P\left(X^{-1}\left(\prod_{i=1}^n B_i\right)\right) = P\left(\bigcap_{i=1}^n X_i^{-1}(B_i)\right).$$

### 4.2. Moments and Expectations.

**Definition 4.5.** The $n$th moment of a random variable $X$ on $(\Omega, B, P)$ about $c \in \mathbb{R}$ is

$$\begin{aligned}\mu_n^c &= \int_\Omega (X(\omega) - c)^n dP(\omega) \\ &= \int_{\mathbb{R}} (t-c)^n dP_X(t).\end{aligned}$$

The mean or **expectation** of $X$ is $E(X) = \mu_1^0$, the integral of $X$ with respect to $P$, and $\mu_n^{E(X)}$ is called the $n$th **central moment** of $X$.

Besides the expectation, various central moements of a random variable have special names and significance.

**Definition 4.6.** The **variance** of a random variable $X$ on $(\Omega, B, P)$ is

$$\sigma^2(X) = \mu_2^{E(X)} = \int_\Omega (X - E(X))^2 dP.$$

If $X \notin L^2(\Omega, P)$, then $\sigma^2(X) = \infty$. If $X \in L^2(\Omega, P)$, however,

$$\sigma^2(X) = E\left(X^2\right) - E(X)^2.$$

The variance is a measure of the spread of a random variable's distribution.

The concepts of mean and variance can be defined for any probability measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, and distributions allow us to define these concepts for arbitrary random variables as well.

**Definition 4.7.** Let $P$ ($P_X$) be a probability measure on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ (distribution). Then the mean of $P$ ($P_X$) is given by

$$\bar{P} = \int_\mathbb{R} t dP(t) \qquad \left(\bar{X} = \int_\mathbb{R} t dP_X(t)\right),$$

and the variance of $P$ ($P_X$) is given by

$$\sigma_P^2 = \int_\mathbb{R} (t - \bar{P})^2 dP(t) \qquad \left(\sigma_X^2 = \int_\mathbb{R} (t - \bar{X})^2 dP_X(t)\right),$$

if these integrals exist.

4.3. **Independence.** One probabilistic concept that has no measure-theoretic analogue is that of independence. Two events $E, F \in B$ are independent if they do not overlap too much - that is, if they overlap in proportion to their respective measures. More precisely:

**Definition 4.8.** Let $(\Omega, B, P)$ be a probability space, and let $\{E_\alpha\}_{\alpha \in A} \subset B$. Then the events $\{E_\alpha\}$ are **independent** if for all $\{\alpha_1, \ldots, \alpha_n\} \subset A$

$$P(E_{\alpha_1} \cap \ldots \cap E_{\alpha_n}) = \prod_{i=1}^n P(E_{\alpha_i}).$$

The meaning of this is as follows. Say $E$ and $F$ are independent events. If we restrict ourselves to event $B$, and measure the fraction of the outcomes that are in $A$, our answer is the same as if we had not restricted ourselves to $B$. We can use this concept to define independence for random variables.

**Definition 4.9.** A collection $\{X_\alpha\}_{\alpha \in A}$ of random variables on $\Omega$ is independent if the events $\{X_\alpha^{-1}(B_\alpha)\}$ are independent for any $\{B_\alpha\} \subset \mathcal{B}(\mathbb{R})$.

If we let $B_\alpha = r \in \mathbb{R}$ for all $\alpha$, we can see that the level sets of the $X_\alpha$ must be independent; that is, the level sets of independent random variables do not overlap "too much". This is a partial explanation for the first part of the following proposition.

**Proposition 4.10.** *Suppose that $\{X_j\}_1^n$ are independent random variables.*
(1) *If $X_j \in L^1(\Omega, P)$ for all $j$, then $\prod_1^n X_j \in L^1(\Omega, P)$.*
(2) *$E\left(\sum_1^n X_j\right) = \sum_1^n E(X_j)$.*
(3) *$E\left(\prod_1^n X_j\right) = \prod_1^n E(X_j)$.*

To prove this proposition, we will need a different characterization of independence.

**Definition 4.11.** Let $(\Omega, B, P)$ be a probability space. The collection of events $\{E_\alpha\}_{\alpha \in A} \subset B$ is said to be independent if

$$P(E_{\alpha_1} \cap \ldots \cap E_{\alpha_n}) = \prod_{i=1}^n P(E_{\alpha_i})$$

for all $n \in \mathbb{N}$ and all distinct $(\alpha_1, \ldots, \alpha_n) \in A^n$.

**Definition 4.12.** Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of random variables on the probability space $(\Omega, B, P)$. Then $\{X_\alpha\}_{\alpha \in A}$ is independent if for each $B \in \mathcal{B}_\mathbb{R}$ the set of events $\{X_\alpha^{-1}(B)\}_{\alpha \in A}$ are independent.

There is an equivalent characterization of independence for a collection of random variables that is often easier to work with.

**Proposition 4.13.** *Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of random variables on the probability space $(\Omega, B, P)$. Then $\{X_\alpha\}_{\alpha \in A}$ is independent if for each finite subset $(\alpha_1, \ldots, \alpha_n) \subset A$ and for each $n$, the joint distribution*

$$P_{(X_{\alpha_1}, \ldots, X_{\alpha_n})} = \prod_{i=1}^{n} P_{X_{\alpha_i}}.$$

*In other words, the finite joint distributions of $\{X_\alpha\}_{\alpha \in A}$ factor.*

4.3.1. *The Normal Distribution.* One of the most important distribution functions is the gaussian or normal distribution.

**Definition 4.14.** A random variable $X$ is said to be normally distributed if it has distribution

$$P_X(E) = \int_E g(x; \bar{X}, \sigma_X) dx = \int_E \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{(x-\bar{X})^2}{2\sigma_X^2}\right] dx,$$

and corresponding distribution function

$$F_X(t) = \int_{-\infty}^{t} g(x; \bar{X}, \sigma_X) dx.$$

$g(x; \bar{X}, \sigma_X)$ is called the gaussian with mean $\bar{X}$ and variance $\sigma_X^2$. The normal distribution with mean zero and variance one is called the standard normal distribution.

We will often abbreviate $g(x; 0, \sigma_X)$ by $g(x; \sigma_X)$. The importance of the normal distribution is in how it relates to other distributions. If a finite collection of $n$ independent, identically distributed random variables is averaged together in a suitable way, the distribution of the average approaches the standard normal distribution as $n$ grows large. This may explain why many natural random processes have a normal distribution; they are the result of many other random processes, occurring on smaller scales.

**Theorem 4.15** (**Central Limit Theorem**). *Let $\{X_i\}_{i=1}^{\infty}$ be a family of independent, identically distributed random variables with mean $\bar{X}$ and variance $\sigma_X^2$. Define the random variable*

$$Z_n = \sum_{i=1}^{n} \frac{X_i - \bar{X}}{\sigma_X \sqrt{n}}.$$

*Then for all $t$,*

$$\lim_{n \to \infty} F_{Z_n}(t) = \lim_{n \to \infty} P(Z_n \leq t) = \int_{-\infty}^{t} g(x; 0, 1) dx.$$

4.3.2. *Stochastic Processes.* A stochastic process is a mathematical model of a random process that evolves in time. It consists of a family of random variables indexed by time.

**Definition 4.16.** Let a probability space $(\Omega, \Sigma, P)$ and a set $T$ be given. Then a stochastic process on $\Omega \times T$ is a family $\{X_t\}_{t \in T}$ of real-valued random variables on $\Omega$.

The random variable $X_t$ assigns a numerical value to each event in $B$ at time $t$; its distribution $P_{X_t}$ gives the probability that the system takes on certain real values at time $t$. The probability that the system takes on values in the Borel sets $B_1, \ldots, B_n$ at times $t_1, \ldots, t_n$ can be calculated as

$$P\left(\bigcap_{i=1}^{n} X_{t_i}^{-1}(B_i)\right).$$

This defines a unique probability measure on $\mathbb{R}^n$.

**Definition 4.17.** Let $\{X_t\}_{t\in T}$ be a stochastic process. The finite distributions of the process are the probability measures defined on Borel rectangles $\prod_{i=1}^{n} B_i$ by

$$P\left(\bigcap_{i=1}^{n} X_{t_i}^{-1}(B_i)\right),$$

for all $(t_1, \ldots, t_n) \in T^n$, and for all $n$.

A stochastic process is completely characterized by its finite distributions.

Given $t, s \in T$, the difference $X_t - X_s$ of random variables is itself a random variable, called an increment of the stochastic process.

**Definition 4.18.** A stochastic process $\{X_t\}_{t\in T}$ on $(\Omega, B, P)$ has independent increments if for all $s, t, u, v \in T$ with $s < t \leq u < v$, the random variables $X_t - X_s$ and $X_v - X_u$ are independent.

If a stochastic process has independent increments, its finite distributions are easy to write down.

**Proposition 4.19.** *If a stochastic process $\{X_t\}_{t\in T}$ on $(\Omega, B, P)$ has independent increments, then its finite distributions are*

$$P_{(X_{t_1},\ldots,X_{t_n})} = \prod_{i=1}^{n} P_{X_{t_i} - X_{t_{i-1}}}.$$

4.4. **Independence.**

4.5. **Conditional Expectations.**

4.6. **Central Limit Theorems.** Lindeberg's CLT for Triangular Arrays

## References

[1] Peres, Y. & Virág, B. "Zeros of the i.i.d. Gaussian power series."

Random Functions and Point Processes