A GEOMETRIC INTERPRETATION AND EXPLICIT FORM FOR HIGHER-ORDER HANKEL OPERATORS

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1. Introduction

A classical Hankel operator has a matrix of the form

$$B_0(x) = \begin{pmatrix} \vdots & & & \\ & & & \\ x_3 & & \iddots & \\ x_2 & x_3 & & \\ x_1 & x_2 & x_3 & \dots \end{pmatrix},$$

where $x = \sum_{j=1}^{\infty} x_j z^j$ is the symbol of the operator. The map $x \longmapsto B_0(x)$ is conformally equivariant in a sense explained below. In this paper we derive explicit expressions for certain well-known higher-order generalizations of this map, which are also conformally equivariant. Indeed, we will prove

Theorem 1.1. *The higher-order Hankel operator of order $s+1$ with symbol $x(z) = \sum_{j=s+1}^{\infty} x_j z^{s+j}$ has the formula*

$$B_s(x) = \mathcal{P}_+ L_s(x),$$

where $\mathcal{P}_+$ is a projection and $L_s(x)$ is the differential operator

$$L_s(x) = \sum_{j=0}^{s} \frac{1}{s!} \begin{pmatrix} s \\ j \end{pmatrix} \begin{pmatrix} s+j \\ j \end{pmatrix} x^{(s-j)} \left(\frac{\partial}{\partial z} \right)^j.$$

While these expressions are basically known, our method appears to have some novelty. In §2, we fix our notation and present an outline of the proof. In §3 and §4, we present key lemmas. In §5, we prove Theorem 1.1. In §6, we present a pair of combinatorial identities resulting from Theorem 1.1, and in §7 we connect our results to prior work.

2. An Outline of the Proof

The group

$$G = PSU(1,1) = \left\{ g = \pm \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}$$

acts on the Riemann sphere $\hat{\mathbb{C}}$ by linear fractional transformations,

$$g : z \longmapsto \frac{\bar{b} + \bar{a} z}{a + bz}.$$

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The decomposition

$$\hat{\mathbb{C}} = \Delta^* \sqcup S^1 \sqcup \Delta, \tag{1}$$

where Δ denotes the unit disk, is stable under this action. Restricting this action to Δ identifies G with the group of conformal automorphisms of Δ.

For each half-integer m, the action of G on $\hat{\mathbb{C}}$ lifts to an action of $SU(1,1)$ on κ^m, the mth (tensor) power of the canonical bundle. This induces an action of $SU(1,1)$ on the space of holomorphic sections of $\kappa^m|_\Delta$, the holomorphic differentials of degree m on Δ. We denote such a differential, and the space of all such differentials, by $f(z)(dz)^m$ and $H^m = H^m(\Delta)$, respectively. The action of $g \in SU(1,1)$ on H^m is

$$g : f(z)(dz)^m \longmapsto f\left(\frac{-\bar{b}+az}{\bar{a}-bz}\right)(\bar{a}-bz)^{-2m}(dz)^m. \tag{2}$$

The action of $SU(1,1)$ on H^m is essentially unitary for $m > 0$; we denote the dense Hilbert space by $H^m_{L^2} = H^m_{L^2}(\triangle)$.

We denote the space of sections of $\kappa^{1/2}|_{S^1}$ by $\Omega^{1/2}(S^1)$. The action of $SU(1,1)$ on this space is also given by (2), with $m = 1/2$. There is an $SU(1,1)$-invariant Hermitian inner product $\langle\theta,\eta\rangle = \int_{S^1}\theta\bar{\eta}$. (If $\theta = f(z)(dz)^{1/2}$ and $\eta = g(z)(dz)^{1/2}$, then $\theta\bar{\eta} = f\bar{g}|dz|$ is a one density on S^1.) We will write $\Omega^{1/2}_{L^2}(S^1)$ when we are thinking of this space as a Hilbert space.

The polarization (1) corresponds to an $SU(1,1)$-stable polarization

$$\Omega^{1/2}_{L^2}(S^1) = H^{1/2}_{L^2} \oplus H^{1/2}_{L^2}(\Delta^*).$$

We denote by $\mathcal{P}_+$ the projection onto $H^{1/2}_{L^2}$, and by $\mathcal{P}_-$ the projection onto $H^{1/2}_{L^2}(\triangle^*)$. If $\theta = f(z)(dz)^{1/2} \in H^{1/2}_{L^2}$ and $\eta = g(z)(dz)^{1/2} \in H^{1/2}_{L^2}(\Delta^*)$, then $\theta\eta = fg dz$ is a one density on S^1 that can be integrated to a nontrivial constant, so $\left(H^{1/2}_{L^2}\right)^* = H^{1/2}_{L^2}(\Delta^*)$.

The above polarization induces a diagonal action of $SU(1,1)$ on Hilbert-Schmidt operators sending $H^{1/2}_{L^2}(\triangle^*)$ to $H^{1/2}_{L^2}$, via the identifications

$$\mathcal{L}_2\left(H^{1/2}_{L^2}(\triangle^*), H^{1/2}_{L^2}\right) = H^{1/2}_{L^2} \otimes \left(H^{1/2}_{L^2}(\triangle^*)\right)^* = H^{1/2}_{L^2} \otimes H^{1/2}_{L^2}.$$

It is a fact from representation theory that

$$H^{1/2}_{L^2} \otimes H^{1/2}_{L^2} = H^1_{L^2} \oplus H^2_{L^2} \oplus H^3_{L^2} \oplus \dots$$

so for each $m \in \mathbb{N}$ there is an intertwining map $H^m_{L^2} \longrightarrow \mathcal{L}_2\left(H^{1/2}_{L^2}(\triangle^*), H^{1/2}_{L^2}\right)$.

Let $x \in H^0/\mathbb{C}$. Then x acts on $\Omega^{1/2}(S^1)$ by multiplication. With respect to the Hardy polarization, we can write the multiplication operator M_x as

$$M_x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

$B = B_0(x) = \mathcal{P}_+ M_x \mathcal{P}_- \in \mathcal{L}_2\left(H^{1/2}_{L^2}(\triangle^*), H^{1/2}_{L^2}\right)$ is the Hankel operator associated to x. The action of G on $H^0/\mathbb{C}$ intertwines with the diagonal action of $SU(1,1)$ on $\mathcal{L}_2\left(H^{1/2}_{L^2}(\triangle^*), H^{1/2}_{L^2}\right)$, and $B_0(x)$ is Hilbert-Schmidt precisely when $\theta = x'dz \in H^1_{L^2}$. Thus B_0 is an $SU(1,1)$-equivariant map from $H^1_{L^2}$ to $\mathcal{L}_2\left(H^{1/2}_{L^2}(\triangle^*), H^{1/2}_{L^2}\right)$. This motivates the following.

Definition 2.1. The Hankel operator of order m with symbol $\theta \in H^m_{L^2}$ is the image of θ under the intertwining map

$$H^m_{L^2} \longrightarrow \mathcal{L}_2\left(H^{1/2}_{L^2}(\triangle^*), H^{1/2}_{L^2}\right) = H^{1/2}_{L^2} \otimes H^{1/2}_{L^2}.$$

The space of Hankel operators of order m is the image of $H^m_{L^2}$ under this map.

Our conceptual explanation of these operators relies on two facts. First, for $m > 0$, the action of $SU(1,1)$ on H^m is essentially irreducible. But for $m \leq 0$, say $m = -s$ for $s \in \mathbb{N}$, there is a short exact sequence

$$0 \longrightarrow \mathbb{C}^{2s+1} = span\left\{(dz)^{-s}, \ldots, z^{2s}(dz)^{-s}\right\} \longrightarrow H^{-s} \xrightarrow{I_s} H^{s+1} \longrightarrow 0.$$

$H^{-s}/\mathbb{C}^{2s+1}$ can be given a Hilbert space structure via the intertwining map I_s, and we refer to this Hilbert space as $H^{-s}_{L^2}$. We study the maps $B_s : H^{-s}_{L^2} \longrightarrow H^{1/2}_{L^2} \otimes H^{1/2}_{L^2}$.

Second, in the case $s = 1$, H^{-1} is a Lie algebra of complex vector fields on the unit disk (note $(dz)^{-1} = \frac{\partial}{\partial z}$) . These vector fields act on $\Omega^{1/2}(S^1)$ as differential operators of order 1 via the Lie derivative. Compressing this action gives us a Hankel operator of order two. For $\theta = f(z)(dz)^{1/2} \in H^{1/2}_{L^2}(\triangle^*)$, $x(z)\frac{\partial}{\partial z} \in H^{-1}_{L^2}$, a formal calculation leads to

$$B_1(x)\theta := \mathcal{P}_+ \frac{1}{2} L_{x(z)\frac{\partial}{\partial z}} f(z)(dz)^{1/2} = \mathcal{P}_+ \left(\frac{1}{2}x'f + xf'\right)(dz)^{1/2}.$$

This operator has the matrix representation

$$B_1(x) = \begin{pmatrix} \vdots & & & & & \\ & & & \iddots & & \\ 4x_5 & & & & & \\ 3x_4 & 2x_5 & & & & \\ 2x_3 & x_4 & 0 & & & \\ x_2 & 0 & -x_4 & -2x_5 & & \\ 0 & -x_2 & -2x_3 & -3x_4 & -4x_5 & \ldots \end{pmatrix}.$$

The key is that the above action of H^{-1} determines an action of H^{-s} on $\Omega^{1/2}(S^1)$, as differential operators of order $\leq s$. Each representation of H^{-1} corresponds to a representation of the universal enveloping algebra $\mathcal{U}\left(H^{-1}\right)$. The elements of $\mathcal{U}\left(H^{-1}\right)$ act as linear differential operators on $\Omega^{1/2}(S^1)$. Let $S^s\left(H^{-1}\right)$ be the space of symmetric tensors of order $\leq s$ over H^{-1}. $S\left(H^{-1}\right) = \cup_{s=0}^{\infty} S^s\left(H^{-1}\right)$ is isomorphic as a filtered vector space to $\mathcal{U}\left(H^{-1}\right)$, so elements of $S^s\left(H^{-1}\right)$ can be mapped to linear differential operators of order $\leq s$ on $\Omega^{1/2}(S^1)$. In §3, we show the existence of a G-equivariant embedding of H^{-s} into $S^s\left(H^{-1}\right)$. Thus the image of H^{-s} under B_s contains linear differential operators of order $\leq s$.

B_s is unique up to multiplication by a constant, and must map the lowest-weight vector in $H^{-s}/\mathbb{C}^{2s+1}$, namely $z^{2s+1}\left(\frac{\partial}{\partial z}\right)^s$, to the lowest-weight vector

l_s of weight $2(s+1)$ in $H^{1/2}_{L^2} \otimes H^{1/2}_{L^2}$. We find the form of l_s in §4, and use it to find B_s in §5.

Finally, a technical point. The elements of H^{-1} may not extend to holomorphic vector fields on S^1, so neither H^{-1} nor $\mathcal{U}\left(H^{-1}\right)$ act naturally on $\Omega^{1/2}(S^1)$ a priori. However, the polynomial sections of $\kappa^{-1}|_{S^1}$, denoted by H^{-1}_{poly}, do extend to S^1, and polynomial sections of H^{-s}, H^{-s}_{poly}, are mapped into $\mathcal{U}\left(H^{-1}_{poly}\right)$ by the embedding from §3. Since H^{-s}_{poly} is dense in $H^{-s}_{L^2}$, the action of H^{-s}_{poly} on $\Omega^{1/2}(S^1)$ extends to an action of $H^{-s}_{L^2}$.

3. The Equivariant Cross-Section of $S^s(H^{-1}) \to H^{-s}$

$S^s\left(H^{-1}\right)$ sits inside $T^s\left(H^{-1}\right)$, the space of tensors of order s. We will write a monomial in $S^s\left(H^{-1}\right)$ as

$$\bigodot_{i=1}^{s} f_i(z)\frac{\partial}{\partial z} = \frac{1}{s!}\sum_{\sigma\in S_s}\left(f_{\sigma(1)}(z)\frac{\partial}{\partial z}\otimes\ldots\otimes f_{\sigma(s)}(z)\frac{\partial}{\partial z}\right),$$

where S_s is the symmetric group on s elements. For each s, $S^s\left(H^{-1}(\triangle)\right)$ projects onto $H^{-s}(\triangle)$ via the map

$$\pi_s : \bigodot_{i=1}^{s} f_i(z)\frac{\partial}{\partial z} \longmapsto \prod_{i=1}^{s} f_i(z)\left(\frac{\partial}{\partial z}\right)^s.$$

Let $d_p = z^p\frac{\partial}{\partial z}$. We refer to p as the *power* of d_p. The vectors

$$\left\{\bigodot_{i=1}^{s} d_{p_i} \;\middle|\; p_1 \geq p_2 \geq \ldots \geq p_s, \quad p_i\in\mathbb{N}\right\}$$

are a basis for $S^s\left(H^{-1}\right)$. We refer to $\sum_{i=1}^{s} p_i$ as the *total degree* of such a basis vector.

The infinitesimal action of $\mathfrak{sl}(2,\mathbb{C})$ on H^m, in terms of the coordinate f, is given by the basis

$$A^- = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \longmapsto -\frac{\partial}{\partial z}, \qquad A^+ = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \longmapsto z^2\frac{\partial}{\partial z} + 2mz,$$

$$H = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \longmapsto 2z\frac{\partial}{\partial z} + 2m.$$

The action of $\mathfrak{sl}(2,\mathbb{C})$ on $T^s\left(H^{-1}\right)$ is by a Liebniz rule, and preserves the subspace of symmetric tensors. We denote this action by $\pi_\odot(X)$. The following lemma is key.

Lemma 3.1. $\left(\pi_\odot\left(A^+\right)\right)^{2s}(d_0)^{\odot s} = C_s\,(d_2)^{\odot s}$*, for some constant* $C_s\in\mathbb{R}$.

Proof. The raising operator $\pi_\odot\left(A^+\right)$ maps a vector of total degree p into a linear combination of vectors having total degree $p+1$. Thus,

$$\left(\pi_\odot\left(A^+\right)\right)^{2s}(d_0)^{\odot s} = \sum_{p\in\mathcal{P}^{2s}_s} a_p \bigodot_{i=1}^{s} d_{p(i)},$$

where $\mathcal{P}^{2s}_s$ is the set of partitions of $2s$ into s or fewer parts, arranged so that $p(i) \geq p(i+1)$. At the same time, $A^+ d_2 = 0$, so none of the parts $p(i)$

can be greater than 2. But there is only one partition of $2s$ into s or fewer parts less than or equal to 2, namely $p(i) = 2$ for all i. □

Proposition 3.2. *For each s, there is a $PSU(1,1)$-equivariant cross-section of $S^s\left(H^{-1}\right) \longrightarrow H^{-s}$.*

Proof. We map the basis elements of $H^{-s}(\triangle)$, namely the set $\left\{z^p\left(\frac{\partial}{\partial z}\right)^s\right\}_{p=0}^{\infty}$, into $S^s\left(H^{-1}\right)$, in such a way that the resulting cross-section is equivariant.

It is clear from the action of $\pi_\odot(H)$ on $S^s\left(H^{-1}\right)$ that $z^p\left(\frac{\partial}{\partial z}\right)^s$ must be mapped to an element having total degree p. Thus, we map $\left(\frac{\partial}{\partial z}\right)^s$ to the only monomial of total degree zero, the vector $(d_0)^{\odot s}$. We map $z^{2s+1}\left(\frac{\partial}{\partial z}\right)^s$ into a vector v_{2s+1} of total degree $2s+1$. We obtain the images of all other basis vectors by applying the raising operator. The resulting cross-section will be equivariant provided that

$$\pi_\odot\left(A^-\right) v_{2s+1} = \left(\pi_\odot\left(A^+\right)\right)^{2s}(d_0)^{\odot s} = C_s\,(d_2)^{\odot s}. \tag{3}$$

We take

$$v_{2s+1} = -\frac{C_s}{3} d_3 \odot (d_2)^{\odot(s-1)} - \sum_{4\le p\le 2s+1} d_p \odot \sum_{\substack{m+n\le s-1\\ 2m+n=2s+1-p}} A_{pn}\,(d_2)^{\odot m} \odot (d_1)^{\odot n}.$$

For example, for $s = 2$, we have

$$-\frac{3}{C_2} v_5 = d_3 \odot d_2 - \frac{1}{2} d_4 \odot d_1 + \frac{1}{10} d_5.$$

The monomials in v_{2s+1} all have total degree $2s+1$. The largest power p ranges from 3 to $2s+1$. Among basis elements with highest power p, we take all those where the remaining powers are no greater than two. Thus the inner sum is really over partitions of $2s+1-p$ into $s-1$ or fewer parts, each part being less than or equal to 2. The coefficients A_{pn} are indexed by the highest power and n, the (symmetric tensor) exponent of d_1. If p is even, n is odd, and vice-versa. Modulo parity, n ranges from 0 to the minimum of $p-3$ and $2s+1-p$. Thus the range of n increases until $p = s+2$, and then decreases to $n = 0$ when $p = 2s+1$.

Now,

$$\begin{aligned}\pi_\odot\left(A^-\right) v_{2s+1} =& C_s\,(d_2)^{\odot(s)} + \left(\frac{2C_s(s-1)}{3} + 4A_{41}\right) d_3 \odot (d_2)^{\odot(s-2)} \odot d_1\\ &+ \sum_{4\le p\le 2s} d_p \odot \sum_{\substack{m+n\le s-1\\ 2m+n=2s-p}} B_{pn}\,(d_2)^{\odot m} \odot (d_1)^{\odot n},\end{aligned}$$

where

$$B_{pn} = (p+1)A_{(p+1)n} + 2(m+1)A_{p(n-1)} + nA_{p(n+1)},$$

and we take A_{pn} to be zero if p and n are not a pair of indices appearing in v_{2s+1}. Thus, if we choose $A_{41} = -\frac{C_s(s-1)}{6}$, and define

$$A_{(p+1)n} = -\frac{1}{p+1}\left((2s-p-n+2)A_{p(n-1)} + nA_{p(n+1)}\right)$$

for all other p and n in the given range, we obtain (3). This recursion relation is linear, and terminates at $p = 2s+1$. Thus it can be solved, and v_{2s+1} exists and satisfies (3). This proves the Proposition. □

4. Lowest Weight Vectors in $\mathcal{L}_2\left(H^{1/2}_{L^2}(\triangle^*), H^{1/2}_{L^2}\right)$

The action of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathcal{L}_2\left(H^{1/2}_{L^2}(\triangle^*), H^{1/2}_{L^2}\right) = H^{1/2}_{L^2} \otimes H^{1/2}_{L^2}$ is again by a Liebniz rule, and will be denoted $\pi_\otimes(X)$. The irreducible subspaces of $H^{1/2}_{L^2} \otimes H^{1/2}_{L^2}$ are lowest-weight representations of $SU(1,1)$. We now identify the vectors in $H^{1/2}_{L^2} \otimes H^{1/2}_{L^2}$ annihilated by $\pi_\otimes(A^-)$.

Proposition 4.1. *The set of vectors*

$$\left\{ l_s := \sum_{i=0}^{s} (-1)^i \begin{pmatrix} s \\ i \end{pmatrix} z^{s-i}(dz)^{1/2} \otimes z^i (dz)^{1/2} \right\}_{s=0}^{\infty}$$

are annihilated by the operator $\pi_\otimes(A^-)$. *The vector* l_s *has weight* $2(s+1)$.

Proof. Let $b_p = z^p(dz)^{1/2}$. Applying $\pi_\otimes(A^-)$ to $b_{s-i}\otimes b_i$ results in two terms unless $i=0$ or $i=s$, so we pull these cases out of the sum l_s. Thus

$$-\pi_\otimes\left(A^-\right)[l_s] = s b_{s-1}\otimes b_0 + \sum_{i=1}^{s-1}(-1)^i \begin{pmatrix} s \\ i \end{pmatrix} \pi_\otimes\left(A^-\right)[b_{s-i}\otimes b_i] + (-1)^s s b_0 \otimes b_{s-1}.$$

The middle term is

$$\begin{aligned}
&\sum_{i=1}^{s-1}(-1)^i \begin{pmatrix} s \\ i \end{pmatrix} \left[(s-i) b_{s-i-1}\otimes b_i + i b_{s-i}\otimes b_{i-1}\right] \\
&= \sum_{i=1}^{s-1}(-1)^i \frac{s\dots(s-i)}{i!} b_{s-i-1}\otimes b_i + \sum_{i=1}^{s-1}(-1)^i \frac{s\dots(s-i+1)}{(i-1)!} b_{s-i}\otimes b_{i-1} \\
&= \sum_{i=1}^{s-1}(-1)^i \frac{s\dots(s-i)}{i!} b_{s-1-i}\otimes b_i + \sum_{j=0}^{s-2}(-1)^{j+1} \frac{s\dots(s-j)}{j!} b_{s-j-1}\otimes b_j \\
&= -s b_{s-1}\otimes b_0 + \sum_{j=1}^{s-2}\left[(-1)^j + (-1)^{j+1}\right] \frac{s\dots(s-j)}{j!} b_{s-1-j}\otimes b_j \\
&\quad + (-1)^{s-1} s b_0\otimes b_{s-1} \\
&= -s b_{s-1}\otimes b_0 + (-1)^{s-1} s b_0 \otimes b_{s-1}.
\end{aligned}$$

Thus,

$$A^-[l_s] = s b_{s-1}\otimes b_0 - s b_{s-1}\otimes b_0 + (-1)^{s-1} s b_0\otimes b_{s-1} + (-1)^s s b_0 \otimes b_{s-1} = 0.$$

To prove the second claim, simply notice that

$$H[b_{s-i}\otimes b_i] = 2(s-i+i+1) b_{s-i}\otimes b_i = 2(s+1) b_{s-i}\otimes b_i.$$

□

5. An Explicit Formula for B_s

We use the following lemma to prove Theorem 1.1. For $x(z)\left(\frac{\partial}{\partial z}\right)^s \in H^{-s}_{L^2}$, let

$$\mathcal{O}_j\left(x\right) = \mathcal{P}_+ x^{(s-j)}\left(\frac{\partial}{\partial z}\right)^j \in \mathcal{L}_2\left(H^{1/2}_{L^2}(\triangle^*), H^{1/2}_{L^2}\right).$$

Lemma 5.1. *For $k > s-j$,*

$$\mathcal{O}_j\left(z^k\right) = \sum_{i=0}^{k-s-1}(-1)^j\frac{(i+j)!}{i!}\frac{k!}{(k-s+j)!}z^{(k-s-1)-i}(dz)^{1/2}\otimes z^i(dz)^{1/2}.$$

Proof. Let $f(z)(dz)^{1/2} \in H^{1/2}_{poly}(\triangle^*)$, with $f(z) = \sum_{n=1}^N f_{-n}z^{-n}$ for $N > k-s$. Then,

$$\left(\frac{\partial}{\partial z}\right)^j f(z) = \sum_{n=1}^N (-1)^j\frac{(n+j-1)!}{(n-1)!}f_{-n}z^{-(n+j)}.$$

Also,

$$\left(\frac{\partial}{\partial z}\right)^{s-j} z^k = \frac{k!}{(k-s+j)!}z^{k-s+j}.$$

Thus,

$$\left[\left(\frac{\partial}{\partial z}\right)^{s-j} z^k\right]\left[\left(\frac{\partial}{\partial z}\right)^j f(z)\right] = \sum_{n=1}^N (-1)^j\frac{(n+j-1)!}{(n-1)!}\frac{k!}{(k-s+j)!}f_{-n}z^{k-s-n},$$

and so

$$\mathcal{O}_j\left(z^k\right)f(z)(dz)^{1/2} = \sum_{n=1}^{k-s}(-1)^j\frac{(n+j-1)!}{(n-1)!}\frac{k!}{(k-s+j)!}f_{-n}z^{k-s-n}(dz)^{1/2}.$$

Since

$$f_{-n}z^{k-s-n}(dz)^{1/2} = \left(z^{k-s-n}(dz)^{1/2}\otimes z^{n-1}(dz)^{1/2}\right)f(z)(dz)^{1/2},$$

we obtain the required formula after reindexing. But this formula depends on only the first $k-n$ coefficients of f, so it applies to all of $H^{1/2}_{L^2}(\triangle^*)$. □

Proof of Theorem 1.1. By Proposition 3.2, we know $B_s(v) = \mathcal{P}_+L_s(v)$, where

$$L_s(v) = \sum_{j=0}^s c_j(v)\left(\frac{\partial}{\partial z}\right)^j.$$

The cases $s=0$ and $s=1$ suggest that $c_j(v) = a_jv^{(s-j)}$. Since B_s is unique, we need only find coefficients a_j which satisfy

$$\mathcal{P}_+L_s\left(z^{2s+1}\right) = l_s. \tag{4}$$

By Proposition 4.1 and Lemma 5.1, (4) is equivalent to

$$\sum_{j=0}^s a_j\sum_{i=0}^s(-1)^j\frac{(i+j)!}{i!}\frac{(2s+1)!}{(s+j+1)!}z^{s-i}(dz)^{1/2}\otimes z^i(dz)^{1/2}$$
$$= \sum_{i=0}^s(-1)^i\begin{pmatrix} s \\ i\end{pmatrix}z^{s-i}(dz)^{1/2}\otimes z^i(dz)^{1/2}.$$

In other words, the a_j's must solve the linear system

$$M_s\vec{a}_s := \left(\left((-1)^j \tfrac{(i+j)!}{i!} \tfrac{(2s+1)!}{(s+j+1)!}\right)_{i,j=0}^s \right) \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_s \end{pmatrix} = \begin{pmatrix} 1 \\ -s \\ \vdots \\ (-1)^s \end{pmatrix} =: \vec{l}_s.$$

Pleasantly, the matrix M_s is easily factored. First, we rewrite M_s as

$$N_s D_s := \left(\frac{(i+j)!}{i!j!}\right)_{i,j=0}^s diag\left((-1)^j j! \frac{(2s+1)!}{(s+j+1)!}\right)_{j=0}^s.$$

Now, a beautiful combinatorial identity comes into play, and we have

$$N_s = \left(\begin{pmatrix} i+j \\ i \end{pmatrix}\right)_{i,j=0}^s = \left(\begin{pmatrix} i \\ j \end{pmatrix}\right)_{i,j=0}^s \left(\begin{pmatrix} j \\ i \end{pmatrix}\right)_{i,j=0}^s =: L_s U_s,$$

using (5.23) from [1]. Let

$$\tilde{L}_s = \left((-1)^{i+j} \begin{pmatrix} i \\ j \end{pmatrix}\right)_{i,j=0}^s$$

and let $P = L_s \tilde{L}_s$. Then the (i,j)th entry of P is

$$\begin{aligned} p_{ij} &= \sum_{k=j}^{i} (-1)^{k+j} \begin{pmatrix} i \\ k \end{pmatrix} \begin{pmatrix} k \\ j \end{pmatrix} = \begin{pmatrix} i \\ j \end{pmatrix} \sum_{k=j}^{i} (-1)^{k+j} \begin{pmatrix} i-j \\ i-k \end{pmatrix} \\ &= \begin{pmatrix} i \\ j \end{pmatrix} \sum_{k=0}^{i-j} (-1)^k \begin{pmatrix} i-j \\ k \end{pmatrix} = \delta_{ij}, \end{aligned}$$

since $\sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} = (1-1)^n = \delta_{n0}$. Thus $\tilde{L}_s = L_s^{-1}$. Analogously,

$$U_s^{-1} = \left((-1)^{i+j} \begin{pmatrix} j \\ i \end{pmatrix}\right)_{i,j=0}^s.$$

Now,

$$L_s^{-1}\vec{l}_s = \left(\sum_{k=0}^{s} (-1)^{j+k} (-1)^k \begin{pmatrix} j \\ k \end{pmatrix} \begin{pmatrix} s \\ k \end{pmatrix}\right)_{j=0}^s = \left((-1)^j \begin{pmatrix} s+j \\ j \end{pmatrix}\right)_{j=0}^s,$$

again using (5.23) from [1]. Next,

$$\begin{aligned} U_s^{-1} L_s^{-1} \vec{l}_s &= \left((-1)^j \sum_{k=0}^{s} \begin{pmatrix} k \\ j \end{pmatrix} \begin{pmatrix} s+k \\ k \end{pmatrix}\right)_{j=0}^s \\ &= \left((-1)^j \begin{pmatrix} s+j \\ j \end{pmatrix} \sum_{k=j}^{s} \begin{pmatrix} s+k \\ s+j \end{pmatrix}\right)_{j=0}^s \\ &= \left((-1)^j \begin{pmatrix} s+j \\ j \end{pmatrix} \begin{pmatrix} 2s+1 \\ s+j+1 \end{pmatrix}\right)_{j=0}^s, \end{aligned}$$

using upper summation ((5.10) in [1]), and finally,

$$\vec{a}_s = M_s^{-1} l_s = \left(\frac{(s+j+1)!}{j!(2s+1)!} \binom{s+j}{j} \binom{2s+1}{s+j+1} \right)_{j=0}^{s}$$
$$= \left(\frac{1}{s!} \binom{s}{j} \binom{s+j}{j} \right)_{j=0}^{s}.$$

□

6. Binomial Coefficient Identities

The equivariance of B_s implies a pair of identities relating sums of products of binomial coefficients.

Proposition 6.1. *For $s \in \mathbb{N}$, $k \geq 2s+1$, and $l = 0, \dots, k-s$,*

$$\sum_{j=0}^{s} (-1)^j \binom{s+j}{j} \binom{k}{s-j} \left[\binom{l+j}{j} (k-s) - \binom{l+j-1}{j-1} l \right]$$
$$= \sum_{j=0}^{s} (-1)^j \binom{s+j}{j} \binom{k+1}{s-j} \binom{l+j}{j} (k-2s),$$

and for $s \in \mathbb{N}$, $i+j \geq s$,

$$\sum_{l=0}^{s} (-1)^l \binom{s+l}{l} \binom{j+l}{l} \binom{i+j+s+1}{s-l}$$
$$= \sum_{l=0}^{s} (-1)^l \binom{s}{j} \binom{j}{l} \binom{i}{s-l}.$$

Proof. To prove the first identity, expand the equation

$$B_s \left(A^+ z^k \right) = \pi_\otimes \left(A^+ \right) B_s \left(z^k \right)$$

in the basis $\{z^i (dz)^{1/2} \otimes z^j (dz)^{1/2}\}$; to prove the second, expand the equation

$$B_s \left(\left(A^+ \right)^{(i+j-s)} z^{2s+1} \right) = \left(\pi_\otimes \left(A^+ \right) \right)^{(i+j-s)} B_s \left(z^{2s+1} \right).$$

□

7. Connections With Prior Work

The *transvectant* of order s is the essentially unique equivariant map $\tau_s : H^{1/2}_{L^2} \otimes H^{1/2}_{L^2} \longrightarrow H^{s+1}_{L^2}$. Let $\theta = f(z)(dz)^{1/2}, \eta = g(z)(dz)^{1/2} \in H^{1/2}_{L^2}$. Then

$$\tau_s(\theta \otimes \eta) = \tau_s(f,g)(dz)^{s+1} = \left(\sum_{j=1}^{s} (-1)^j \binom{s}{j}^2 f^{(s-j)} g^{(j)} \right) (dz)^{s+1}.$$

This map is known in classical invariant theory. In [2] it is used to construct higher-order Hankel bilinear forms on the space $H^{1/2}_{L^2} \otimes H^{1/2}_{L^2}$. If $\nu = x(z) \left(\frac{\partial}{\partial z} \right)^s \in H^{-s}_{L^2}$ and $\mu = y(z)(dz)^{s+1} \in H^{s+1}_{L^2}$, then $\bar{\nu}\mu = \bar{x} y dz$ is a one-density on S^1, and so $\left(H^{s+1}_{L^2}\right)^* = H^{-s}_{L^2}$. As in [2], define the Hankel form of order $s+1$ with symbol x to be

$$K_s(x)[f,g] = \int_{S^1} \bar{\nu}\tau_s(\theta \otimes \eta) = \int_{S^1} \bar{x}\tau_s(f,g)dz.$$

Since $B_s(x)\bar{\theta} \in H^{1/2}_{L^2}$, we can define another bilinear form by

$$\tilde{K}_s(x)[f,g] = \langle B_s(x)\bar{\theta}, \eta\rangle_{1/2} = \int_{S^1} \overline{B_s(x)\bar{f}} g d\theta,$$

where $\langle , \rangle_{1/2}$ is the inner product on $H^{1/2}_{L^2}$. $K_s(x)$ and $\tilde{K}_s(x)$ are easily seen to be equivalent. Another expression for $\tilde{K}_s(x)$ is

$$\tilde{K}_s(x)[f,g] = \langle B_s(x), \theta \otimes \eta\rangle_{\otimes} = tr\left(B_s(x)(\theta \otimes \eta)^*\right),$$

where $\langle , \rangle_{\otimes}$ is the inner product on $H^{1/2}_{L^2} \otimes H^{1/2}_{L^2}$. Thus, B_s is the adjoint of the map τ_s, and we have the diagram

$$\begin{array}{ccc} H^{-s}_{L^2} & & \\ I_s \downarrow & \searrow^{B_s} & \\ H^{s+1}_{L^2} & \xleftarrow{\tau_s} & \mathcal{L}_2(H^-, H^+). \end{array}$$

References

[1] Graham R.L., Knuth D.E., Patashnik O. Concrete Mathematics. 2nd ed. Addison-Wesley, 1994.

[2] Janson S. & Peetre J. "A new generalization of hankel operators (the case of higher weights)." *Math. Nach.* Volume 132, No. 1, pp. 313-328. 1987.