Reduced Words in the Affine Symmetric Group and Non-Abelian Fourier Transforms

PhD Dissertation Final Oral Defense

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Overview

- Factorization in Lie Groups and the Symmetric Group.
- A Non-Abelian Fourier Transform.
- The Affine Symmetric Group.
- Reduced Words in the Affine Symmetric Group.

The Group $SU(n)$ and its Lie Algebra $\mathfrak{su}(n)$

- $SU(n)$ is the group of unitary matrices with determinant one,

$$SU(n) = \left\{ g \in GL(n, \mathbb{C}) \,\middle|\, \det(g) = 1,\ gg^* = I \right\}.$$

- Its Lie algebra $\mathfrak{su}(n)$ is the vector space of traceless Hermitian matrices,

$$\mathfrak{su}(n) = \left\{ X \in M_{n\times n} \,\middle|\, \exp(X) \in SU(n) \right\}.$$

The Group $SU(n)$ and its Lie Algebra $\mathfrak{su}(n)$

- $SU(n)$ is a smooth manifold, and group multiplication is smooth.
- For example, $SU(2) \cong S^3 \cong \left(S^1 \to S^3 \to S^2\right)$.
- The Lie algebra $\mathfrak{su}(n)$ is the tangent space to $SU(n)$ at the identity,

$$\mathfrak{su}(n) = \left\{ X \in M_{n \times n} \;\middle|\; \exp(X) \in SU(n) \right\}.$$

- Lie theory is full of beautiful connections between combinatorics, geometry, and topology.

Decomposition in $\mathfrak{su}(n)$: Example: $\mathfrak{su}(3)$.

$$
\begin{aligned}
X &= \begin{pmatrix} i\theta_1 & \zeta_{12} & \zeta_{13} \\ -\bar{\zeta}_{12} & i\left(\theta_2 - \theta_1\right) & \zeta_{23} \\ -\bar{\zeta}_{13} & -\bar{\zeta}_{23} & -i\theta_2 \end{pmatrix} \\
&= \begin{pmatrix} & \zeta_{12} & \\ -\bar{\zeta}_{12} & & \\ & & \end{pmatrix} + \begin{pmatrix} & & \zeta_{13} \\ & & \\ -\bar{\zeta}_{13} & & \end{pmatrix} + \begin{pmatrix} & & \\ & & \zeta_{23} \\ & -\bar{\zeta}_{23} & \end{pmatrix} \\
&\quad + \begin{pmatrix} i\theta_1 & & \\ & -i\theta_1 & \\ & & \end{pmatrix} + \begin{pmatrix} & & \\ & i\theta_2 & \\ & & -i\theta_2 \end{pmatrix}.
\end{aligned}
$$

Decomposition in $\mathfrak{su}(n)$.

Define $f_{ij} : M_{2\times 2} \longrightarrow M_{n\times n}$ by

$$f_{ij} : \begin{pmatrix} x & y \\ z & w \end{pmatrix} \longmapsto \begin{array}{cc} & \begin{array}{ccccc} \phantom{\mathbf{0}} & i & \phantom{\mathbf{0}} & j & \phantom{\mathbf{0}} \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} \mathbf{0} & & & & \\ & x & & y & \\ & & \mathbf{0} & & \\ & z & & w & \\ & & & & \mathbf{0} \end{pmatrix} \end{array}.$$

Decomposition in $\mathfrak{su}(n)$.

- In $\mathfrak{su}(3)$, we have

$$X = \begin{pmatrix} & \zeta_{12} & \\ -\bar{\zeta}_{12} & & \\ & & \end{pmatrix} + \begin{pmatrix} & & \zeta_{13} \\ & & \\ -\bar{\zeta}_{13} & & \end{pmatrix} + \begin{pmatrix} & & \\ & & \zeta_{23} \\ & -\bar{\zeta}_{23} & \end{pmatrix}$$
$$+ \begin{pmatrix} i\theta_1 & & \\ & -i\theta_1 & \\ & & \end{pmatrix} + \begin{pmatrix} & & \\ & i\theta_2 & \\ & & -i\theta_2 \end{pmatrix}.$$

- In $\mathfrak{su}(n)$, we have

$$X = \sum_{1 \leq i < j \leq n} f_{ij}\left(\begin{pmatrix} 0 & \zeta_{ij} \\ -\bar{\zeta}_{ij} & 0 \end{pmatrix}\right) + \sum_{k \in [n]} f_{k,k+1}\left(\begin{pmatrix} i\theta_k & 0 \\ 0 & -i\theta_k \end{pmatrix}\right).$$

Factorization in $SU(n)$.

Define $F_{ij} : GL(2, \mathbb{C}) \longrightarrow GL(n, \mathbb{C})$ by

$$F_{ij} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{array}{cc} & \begin{array}{ccccc} \phantom{I_{i-1}} & i & \phantom{I_{i-j-1}} & j & \phantom{I_{n-i-1}} \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} I_{i-1} & & & & \\ & a & & b & \\ & & I_{i-j-1} & & \\ & c & & d & \\ & & & & I_{n-i-1} \end{pmatrix} \end{array}.$$

Factorization in $SU(n)$.

- In $\mathfrak{su}(n)$, we have

$$X = \sum_{1\leq i<j\leq n} f_{ij}\left(\begin{pmatrix} 0 & \zeta_{ij} \\ -\bar{\zeta}_{ij} & 0 \end{pmatrix}\right) + \sum_{k\in[n]} f_{k,k+1}\left(\begin{pmatrix} i\theta_k & 0 \\ 0 & -i\theta_k \end{pmatrix}\right).$$

- In $SU(n)$, generically we should have

$$g =$$

$$\prod_{1\leq i<j\leq n} F_{ij}\left(N(\zeta_{ij})\begin{pmatrix} 1 & \zeta_{ij} \\ -\bar{\zeta}_{ij} & 1 \end{pmatrix}\right) \prod_{k\in[n]} F_{k,k+1}\left(\begin{pmatrix} e^{i\theta_k} & 0 \\ 0 & e^{-i\theta_k} \end{pmatrix}\right)$$

for $N(\zeta_{ij}) = \frac{1}{\sqrt{1+|\zeta_{ij}|^2}}$.

- BUT, to properly define this we must order the first product!

Factorization in $SU(n)$.

- Which orderings on pairs (i,j) yield diffeomorphisms is unknown.
- We will obtain an ordering from a choice of reduced word for the longest permutation in S_n.
- This choice has many applications, to topology, Poisson geometry, and measure theory.
- There are many connections between S_n and matrix groups: representation theory, Schubert and Grassmann varieties, total positivity and quantum groups, Poisson Lie groups, ...

An Ordering on (i,j) From the Symmetric Group

- Let S_n be the permutation group on n letters.
- Let

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix} = (1,n)(2,n-1)\cdots\left(\lfloor \tfrac{n-1}{2} \rfloor, \lceil \tfrac{n+1}{2} \rceil\right).$$

- Let $s_i = (i, i+1)$, the transposition of letters i and $i+1$.
- A **reduced decomposition** of w_0 is an expression

$$\mathrm{w}_0 = s_{i_{\binom{n}{2}}} \cdots s_{i_1} = \left(i_{\binom{n}{2}}, i_{\binom{n}{2}} + 1\right) \cdots (i_1, i_1 + 1).$$

- By a result of Stanley, the number of reduced decompositions of w_0 in S_n is

$$\frac{\binom{n}{2}!}{1^{n-1} 3^{n-2} \dots (2n-3)^1}.$$

An Ordering From the Symmetric Group: Example: S_3.

- For S_3,

$$w_0 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1,3).$$

- We have reduced words $\mathrm{w}_0 = s_1 s_2 s_1$ and $\mathrm{w}_0' = s_2 s_1 s_2$:

$$(1,2,3) \xrightarrow{s_1} (2,1,3) \xrightarrow{s_2} (2,3,1) \xrightarrow{s_1} (3,2,1)$$
$$(1,2,3) \xrightarrow{s_2} (1,3,2) \xrightarrow{s_1} (3,1,2) \xrightarrow{s_2} (3,2,1),$$

An Ordering on (i, j) From the Symmetric Group

- S_n acts linearly on $\mathbb{R}^n$, by $\sigma : e_i \longmapsto e_{\sigma(i)}$.
- It is the group generated by reflections through hyperplanes

$$H_{ij} = \{x_i = x_j\}.$$

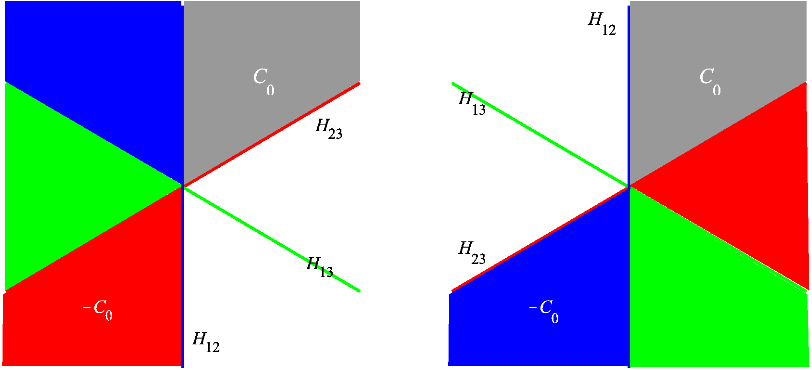
- The fundamental domains are chambers in $\mathbb{R}^n$, such as

$$C_0 = \left\{ \vec{x} \in \mathbb{R}^n \,\middle|\, x_1 > x_2 > \ldots > x_n > 0 \right\}.$$

- A **reduced decomposition** of w_0 is a walk through adjacent chambers from C_0 to $-C_0$, passing over $H_{I(m),J(m)}$ at step m.

An Ordering From the Symmetric Group: Example: S_3

(Projection onto $(1,1,1)^{\perp}$.)



The reduced word $\mathrm{w}_0 = s_1 s_2 s_1$. The reduced word $\mathrm{w}_0' = s_2 s_1 s_2$.

The Resulting Factorization in $SU(n)$

Given $\mathrm{w}_0 = s_{i_{\binom{n}{2}}} \dots s_{i_1}$, $I(m)$, $J(m)$, generically we have (Soibelman, Bott & Samelson)

$$g = \overleftarrow{\prod_{m=1}^{\binom{n}{2}}} F_{I(m),J(m)} \left(N\left(\zeta_m\right) \begin{pmatrix} 1 & \zeta_m \\ -\bar{\zeta}_m & 1 \end{pmatrix} \right)$$
$$\times \prod_{k=1}^{n-1} F_{k,k+1} \left(\begin{pmatrix} e^{i\theta_k} & 0 \\ 0 & e^{-i\theta_k} \end{pmatrix} \right).$$

The Resulting Factorization in $SU(3)$

Given $\mathrm{w}_0 = s_1 s_2 s_1$, generically we have

$$g = \begin{pmatrix} 1 & & \\ & N(\zeta_3) & -N(\zeta_3)\bar{\zeta}_3 \\ & N(\zeta_3)\zeta_3 & N(\zeta_3) \end{pmatrix} \begin{pmatrix} N(\zeta_2) & & -N(\zeta_2)\bar{\zeta}_2 \\ & 1 & \\ N(\zeta_2)\zeta_2 & & N(\zeta_2) \end{pmatrix}$$

$$\times \begin{pmatrix} N(\zeta_1) & -N(\zeta_1)\bar{\zeta}_1 & \\ N(\zeta_1)\zeta_1 & N(\zeta_1) & \\ & & 1 \end{pmatrix} \begin{pmatrix} e^{it_1} & & \\ & e^{i(t_2-t_1)} & \\ & & e^{-it_2} \end{pmatrix}.$$

Factoring $SU(n)$ as a Product of Spheres and Circles

- The factor
$$N(\zeta)\begin{pmatrix} 1 & \zeta \\ -\bar{\zeta} & 1 \end{pmatrix}$$
is an element of $SU(2)/S^1 \cong S^2$.
- The factor
$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$
is an element of S^1.
- Thus, a generic subset of $SU(n)$ can be factored into a product of two-spheres and circles.

- The combinatorics of the map between $\{\zeta_m, \theta_k\}$ and the entries of g is complicated.
- There is a canonical reduced word for w_0 making this combinatorics simple, the lexicographically minimal reduced word:

$$\mathrm{w}_0 = (s_1 s_2 \ldots s_{n-1})\,(s_1 s_2 \ldots s_{n-2}) \ldots (s_1 s_2)\, s_1 .$$

The Lexicographically Minimal Reduced Word

- $\mathrm{w}_0 = (s_1 s_2 \dots s_{n-1})\,(s_1 s_2 \dots s_{n-2}) \dots (s_1 s_2)\, s_1.$
- It can be read off of the "upper triangle":

$$\mathrm{w}_0 \leftrightarrow \begin{pmatrix} & s_1 & s_2 & s_3 & \cdots & & s_{n-1} \\ & & s_1 & s_2 & \cdots & & s_{n-2} \\ & & & s_1 & \cdots & & s_{n-3} \\ & & & & \ddots & & \vdots \\ & & & & & s_1 & s_2 \\ & & & & & & s_1 \end{pmatrix}.$$

The Lexicographically Minimal Reduced Word

- $\mathrm{w}_0 = (s_1 s_2 \dots s_{n-1})(s_1 s_2 \dots s_{n-2}) \dots (s_1 s_2) s_1$.
- $I(m)$ and $J(m)$ are the corresponding row, column indices:

$$(I(m), J(m)) \leftrightarrow \begin{pmatrix} & (1,2) & (1,3) & (1,4) & \dots & (1,n) \\ & & (2,3) & (2,4) & \dots & (2,n) \\ & & & (3,4) & \dots & (3,n) \\ & & & & \ddots & \vdots \\ & & & & & (n-1,n) \end{pmatrix}.$$

The Loop Group $LSU(n)$

- $LSU(n) = C^{\infty}\left(S^1, SU(n)\right)$ is a group, with product
$$(gh)(z) = g(z)h(z)$$
for $z \in S^1$.
- $LSU(n)$ is a Fréchet manifold (locally Fréchet), modeled on the Fréchet space $L\mathfrak{su}(n) = C^{\infty}\left(S^1, \mathfrak{su}(n)\right)$.
- Multiplication is smooth, so $LSU(n)$ is an infinite-dimensional Lie group.
- Elements are “loops” in $SU(n)$.

The Loop Group $LSU(2)$

We can even visualize a (finite) cyclic subgroup of loops in $SU(2) \cong S^3$ (Caine).

Decomposition in $L\mathfrak{su}(n)$: Example: $L\mathfrak{su}(3)$.

Now, our entries are smooth functions:

$$X(z) = \begin{pmatrix} i\theta_1(z) & \zeta_{12}(z) & \zeta_{13}(z) \\ -\zeta_{12}^*(z) & i\left(\theta_2(z) - \theta_1(z)\right) & \zeta_{23}(z) \\ -\zeta_{13}^*(z) & -\zeta_{23}^*(z) & -i\theta_2(z) \end{pmatrix}.$$

Taking the Fourier transform of each entry, for example $\zeta_{12}(z) = \sum_{p\in\mathbb{Z}} \zeta_{p;12} z^p$, we have the decomposition

$$X(z) = \sum_{p\in\mathbb{Z}} z^p \left(\begin{pmatrix} & \zeta_{p;12} & \\ -\zeta_{p;12}^* & & \\ & & \end{pmatrix} + \begin{pmatrix} & & \zeta_{p;13} \\ & & \\ -\zeta_{p;13}^* & & \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} & & \\ & & -\zeta_{p;23}^* \\ & -\zeta_{p;23}^* & \end{pmatrix} + \begin{pmatrix} i\theta_{p;1} & & \\ & -i\theta_{p;1} & \\ & & \end{pmatrix} + \begin{pmatrix} & & \\ & i\theta_{p;2} & \\ & & -i\theta_{p;2} \end{pmatrix} \right).$$

Decomposition in $L\mathfrak{su}(n)$.

Define $f_{p;ij} : M_{2\times 2} \longrightarrow LM_{n\times n}$ by

$$f_{p;ij} : \begin{pmatrix} x & y \\ u & w \end{pmatrix} \longmapsto \begin{cases} f_{ij}\left(\begin{pmatrix} x & yz^{p} \\ uz^{-p} & w \end{pmatrix}\right) & p \leq 0, \\ f_{ij}\left(\begin{pmatrix} x & uz^{-p} \\ yz^{p} & w \end{pmatrix}\right) & p \geq 1. \end{cases}$$

Decomposition in $\mathfrak{su}(n)$.

In $L\mathfrak{su}(n)$, we have

$$X(z) = \sum_{p\in\mathbb{Z}} \sum_{1\le i<j\le n} f_{p;ij}\left(\begin{pmatrix} 0 & \zeta_{p;ij} \\ -\bar{\zeta}_{p;ij} & 0 \end{pmatrix}\right) + \sum_{p\in\mathbb{Z}} \sum_{k=1}^{n-1} f_{k,k+1}\left(\begin{pmatrix} i\theta_{p;k} z^p & 0 \\ 0 & -i\theta_{p;k} z^p \end{pmatrix}\right).$$

Factorization in $LSU(n)$: a Non-Abelian Fourier Transform.

Define $F_{p;ij} : GL(2,\mathbb{C}) \longrightarrow LGL(n,\mathbb{C})$ by

$$F_{p;ij} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{cases} F_{ij}\left(\begin{pmatrix} a & bz^{p} \\ cz^{-p} & d \end{pmatrix}\right) & p \leq 0, \\ F_{ij}\left(\begin{pmatrix} a & cz^{-p} \\ bz^{p} & d \end{pmatrix}\right) & p \geq 1. \end{cases}$$

Factorization in $LSU(n)$: a Non-Abelian Fourier Transform.

In $LSU(n)$, we seek

$$g(z) = \prod_{p\in\mathbb{Z}} \prod_{1\le i<j\le n} F_{p;ij}\left(N\left(\zeta_{p;ij}\right) \begin{pmatrix} 1 & \zeta_{p;ij} \\ -\bar{\zeta}_{p;ij} & 1 \end{pmatrix} \right)$$
$$\times \prod_{p\in\mathbb{Z}} \prod_{k=1}^{n-1} F_{k,k+1}\left(\begin{pmatrix} \exp\left(i\theta_{p;k} z^p\right) & 0 \\ 0 & \exp\left(-i\theta_{p;k} z^p\right) \end{pmatrix} \right).$$

To define the first product, it also must be ordered; now we need an order on triples (p, i, j), and we get it from the affine symmetric group $\tilde{S}_n$.

The Affine Symmetric Group

- Let S_∞ be the group of permutations of $\mathbb{Z}$.
- The affine symmetric group $\tilde{S}_n$ is the subgroup of S_∞ whose elements satisfy
$$p(i+n) = p(i) + n.$$
- $\tilde{S}_n$ is generated by the transpositions s_i for $i = 0, 1, \ldots, n$, where s_i is the transposition
$$(i + kn, i + 1 + kn) \text{ for all } k \in \mathbb{Z}.$$

The Affine Symmetric Group: Example: $\tilde{S}_3$.

- $\tilde{S}_3$ is the group of permutations of $\mathbb{Z}$ satisfying

$$p(i+3) = p(i) + 3.$$

- It is generated by transpositions s_0, s_1, and s_2, where

$$\begin{aligned}
s_0(\ldots \mid -2,\ -1,\ 0,\ \mid 1,\ 2,\ 3 \mid 4,\ 5,\ 6 \mid \ldots) \\
= (\ldots \mid -\mathbf{3},\ -1,\ \mathbf{1} \mid \mathbf{0},\ 2,\ \mathbf{4} \mid \mathbf{3},\ 5,\ \mathbf{7} \mid \ldots), \\
s_1(\ldots \mid -2,\ -1,\ 0 \mid 1,\ 2,\ 3 \mid 4,\ 5,\ 6 \mid \ldots) \\
= (\ldots \mid -\mathbf{1},\ -\mathbf{2},\ 0 \mid \mathbf{2},\ \mathbf{1},\ 3 \mid \mathbf{5},\ \mathbf{4},\ 6 \mid \ldots), \\
s_2(\ldots \mid -2,\ -1,\ 0 \mid 1,\ 2,\ 3 \mid 4,\ 5,\ 6 \mid \ldots) \\
= (\ldots \mid -2,\ \mathbf{0},\ -\mathbf{1} \mid 1,\ \mathbf{3},\ \mathbf{2} \mid 4,\ \mathbf{6},\ \mathbf{5} \mid \ldots).
\end{aligned}$$

The Action of $\tilde{S}_n$ on $\mathbb{R}^n$

- $\tilde{S}_n$ acts by affine reflections on $\mathbb{R}^n$ (Lusztig, Björner & Brenti).
- It is the group generated by reflections through hyperplanes

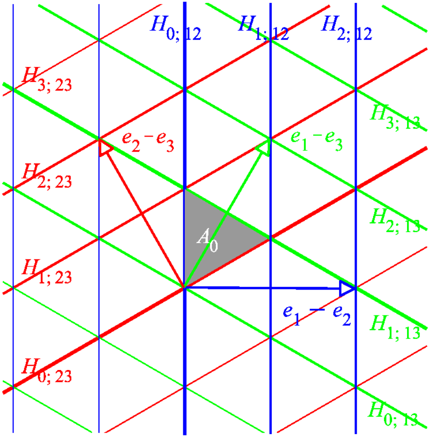
$$H_{p;ij} = \{x_i - x_j = p\}.$$

- The fundamental domains are **alcoves** in $\mathbb{R}^n$, such as

$$A_0 = \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; x_1 > x_2 > \ldots > x_n > 0,\ x_1 - x_n < 1 \right\}.$$

The Action of $\tilde{S}_3$ on $\mathbb{R}^n$

(Projection onto $(1,1,1)^{\perp}$.)



Reduced Words in the Affine Symmetric Group

- A **reduced word** for $w \in \tilde{S}_n$ is an expression

$$w = s_{i_{l(w)}} \dots s_{i_1}$$

  which is as short as possible.
- For a given w, these words are **not unique**.
- An **infinite reduced word** is a sequence $\left\{s_{i_j}\right\}_{j \in \mathbb{N}}$ such that $s_{i_p} \dots s_{i_1}$ is reduced for all $p \in \mathbb{N}$ (Cellini & Papi, Lam & Pylyavskyy, Pickrell & P.-P.).

Reduced Words in the Affine Symmetric Group: Example: $\tilde{S}_3$

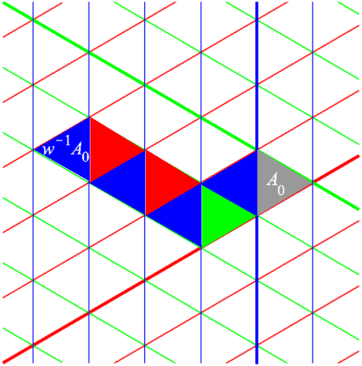
In $\tilde{S}_3$, we have the reduced word $\mathrm{w} = s_2 s_0 s_1 s_2 s_0 s_2 s_1$, given by

$$
\begin{aligned}
&(\ldots,\ 0 \mid 1,\ 2,\ 3 \mid 4, \ldots) \xrightarrow{s_1} (\ldots, 0 \mid \mathbf{2}, \mathbf{1}, 3 \mid \mathbf{5}, \ldots) \\
&\xrightarrow{s_2} (\ldots, -\mathbf{2} \mid 2, \mathbf{3}, \mathbf{1} \mid 5, \ldots) \xrightarrow{s_0} (\ldots, \mathbf{2} \mid -\mathbf{2}, 3, \mathbf{5} \mid \mathbf{1}, \ldots) \\
&\xrightarrow{s_2} (\ldots, \mathbf{0} \mid -2, \mathbf{5}, \mathbf{3} \mid 1, \ldots) \xrightarrow{s_1} (\ldots, 0 \mid \mathbf{5}, -\mathbf{2}, 3 \mid \mathbf{8}, \ldots) \\
&\xrightarrow{s_0} (\ldots, \mathbf{5} \mid \mathbf{0}, -2, \mathbf{8} \mid \mathbf{3}, \ldots) \xrightarrow{s_2} (\ldots, -\mathbf{5} \mid 0, \mathbf{8}, -\mathbf{2} \mid 3, \ldots).
\end{aligned}
$$

Reduced Words and Orderings of (p, i, j)

- In the action on $\mathbb{R}^n$, a reduced word for w corresponds to a walk through adjacent alcoves from A_0 to $w^{-1}A_0$, containing the minimum possible number of alcoves.
- Such a walk is not unique.
- The alcove walk corresponding to a reduced word for w passes over $H_{P(m);I(m),J(m)}$ at step m; thus, it orders **some** triples (p, i, j).

A Reduced Word in $\tilde{S}_3$



Minimal alcove walk for

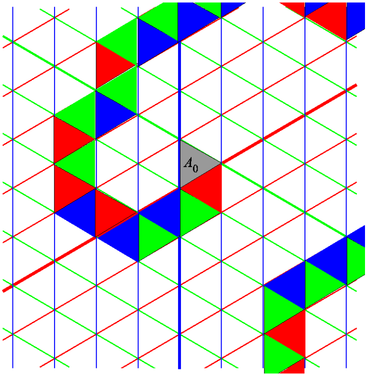
$$w = s_2 s_0 s_1 s_2 s_0 s_2 s_1,$$

crossing

$$H_{0;12},\ H_{0;13},\ H_{-1;12},\ H_{1;23},\ H_{-2;12},\ H_{2;23},\ H_{-3;12}.$$

A Longest Element in $\tilde{S}_n$ to Order Our Product?

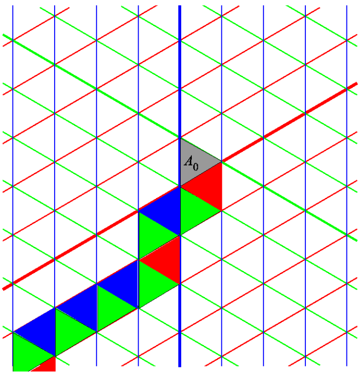
- To order the product in our factorization, we'd like an order on **all** (p, i, j).
- This means we want a minimal alcove walk passing over **all** the hyperplanes $H_{p;ij}$.
- Thus, we need at least an infinite, minimal alcove walk.



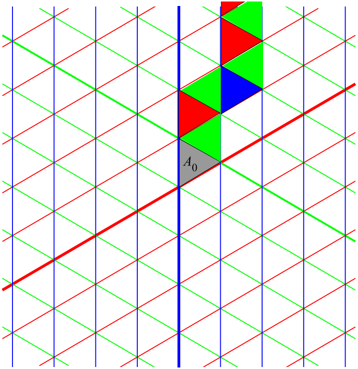
Infinite, but **not** minimal.

Two "Longest Elements"

At best, we can cross $H_{p;ij}$ for all $p \leq 0$, or for all $p \geq 1$.



Crossing $p \leq 0$.



Crossing $p > 0$.

A Change to Our Factorization for $LSU(n)$.

Now, we seek

$$g(z) = g_1^*(z)\lambda(z)g_2(z),$$

where

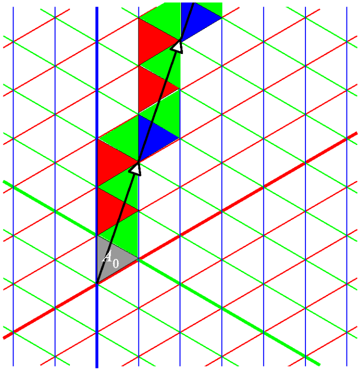
$$g_1(z) = \prod_{p\leq 0}\prod_{1\leq i<j\leq n} F_{p;ij}\left(N\left(\eta_{p;ij}\right)\begin{pmatrix} 1 & \eta_{p;ij} \\ -\bar{\eta}_{p;ij} & 1 \end{pmatrix}\right),$$

$$\lambda(z) = \prod_{p\in\mathbb{Z}}\prod_{k=1}^{n-1} F_{k,k+1}\left(\begin{pmatrix} \exp\left(i\theta_{p;k}z^p\right) & 0 \\ 0 & \exp\left(-i\theta_{p;k}z^p\right) \end{pmatrix}\right),$$

$$g_2(z) = \prod_{p\geq 1}\prod_{1\leq i<j\leq n} F_{p;ij}\left(N\left(\zeta_{p;ij}\right)\begin{pmatrix} 1 & \zeta_{p;ij} \\ -\bar{\zeta}_{p;ij} & 1 \end{pmatrix}\right).$$

Periodic Infinite Reduced Sequences

- To do analysis, our infinite reduced walk must be periodic.
- A periodic walk can be constructed by repeated translation.
- There are infinitely many possibilities...



Non-Abelian Fourier Transforms.

Fix periodic orderings $\{P'(m), I'(m), J'(m)\}$, $\{P(m), I(m), J(m)\}$, with $P'(m) \leq 0 < P(m)$.
Generically, we have (Pickrell & P.-P.),

$$g(z) = g_1^*(z)\lambda(z)g_2(z),$$

where for rapidly decreasing sequences $\{\eta_m\}_{m\in\mathbb{N}}$, $\{\zeta_m\}_{m\in\mathbb{N}}$,

$$g_1(z) = \lim_{M\longrightarrow\infty} \prod_{m=1}^{M} F_{P'(m);I'(m),J'(m)}\left(N\left(\eta_m\right)\begin{pmatrix} 1 & \eta_m \\ -\bar{\eta}_m & 1 \end{pmatrix}\right),$$

$$g_2(z) = \lim_{M\longrightarrow\infty} \prod_{m=1}^{M} F_{P(m);I(m),J(m)}\left(N\left(\zeta_m\right)\begin{pmatrix} 1 & \zeta_m \\ -\bar{z}\eta_m & 1 \end{pmatrix}\right),$$

and λ is smooth, diagonal, and unitary.

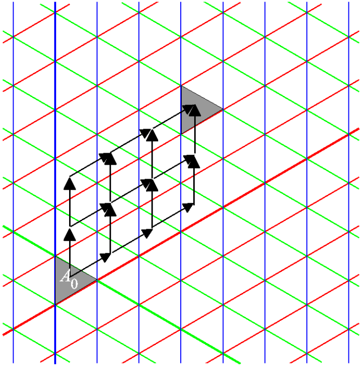
The Combinatorics of Minimal Reduced Words in $\tilde{S}_n$

The dependence on a choice of infinite reduced word prompts several questions.

- How many reduced words are there for a given finite element? (Big business: Stanley, Lam, Billey, ...)
- Can we produce examples of periodic infinite reduced words? (Lam & Pylyavskyy)
- Can we pick a canonical infinite reduced word? (Lam & Pylyavskyy)
- Can we understand the combinatorics of the map from $\{\eta_m, \theta_{m;k}, \zeta_m\}_{m\in\mathbb{Z},\ k=1,\ldots,n}$ to the entries of $g(z)$?

The Case of $\tilde{S}_3$

For certain elements of $\tilde{S}_3$, all minimal alcove walks are determined by successive choices to go right or left.



The Case of $\tilde{S}_3$

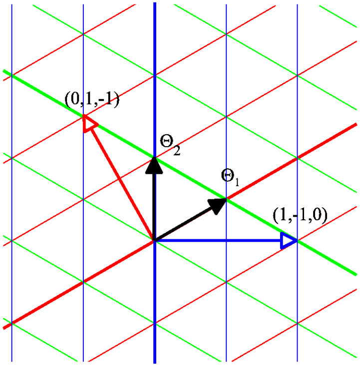
These right and left moves are moves by vectors

$$\Theta_1 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$$

$$\Theta_2 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$$

dual to the basis for $(1,1,1)^{\perp}$

$$\{(1,-1,0),\ (0,1,-1)\}\ .$$



The Case of $\tilde{S}_3$

Let A be an alcove with

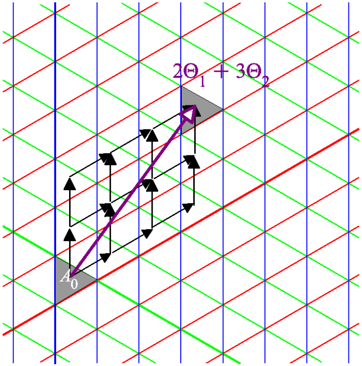
$$\mathrm{center}(A) \\ = \mathrm{center}\left(A_0\right) + a\Theta_1 + b\Theta_2$$

for integers $a > 0$, $b > 0$.

Then the number of minimal alcove walks to A is

$$\binom{a+b}{a}.$$

(P.-P., not generalizable?)



The Case of $\tilde{S}_3$

Moves by Θ_1 and Θ_2 correspond to reduced words

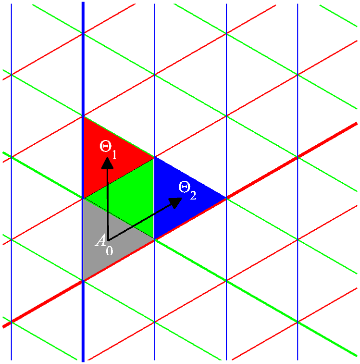
$$\mathrm{W}_1 = s_2 s_0, \qquad \mathrm{W}_2 = s_1 s_0.$$

These can be read off:

$$\begin{pmatrix} s_2 & s_0 \\ & \end{pmatrix}, \qquad \begin{pmatrix} & s_0 \\ & s_1 \end{pmatrix}.$$

So can the corresponding indices:

$$\begin{pmatrix} \color{blue}{(1,2)} & \color{green}{(1,3)} \\ & \end{pmatrix}, \qquad \begin{pmatrix} & \color{green}{(1,3)} \\ & \color{red}{(1,2)} \end{pmatrix}.$$



The Case of $\tilde{S}_3$

If A is an alcove with

$$\mathrm{center}(A) - \mathrm{center}\,(A_0) = a\Theta_1 + b\Theta_2,$$

the minimal walk from A_0 to A corresponding to

$$\vec{c} = (c_1, \ldots, c_{a+b}) \text{ with } c_i \in \{1, 2\}$$

has reduced word

$$\mathrm{W}(\vec{c}) = \mathrm{W}_{\mathrm{c_{a+b}}}^{\left(\sum_{i=1}^{a+b-1} \sigma_i\right)} \ldots \mathrm{W}_{\mathrm{c_3}}^{(\sigma_1+\sigma_2)} \mathrm{W}_{\mathrm{c_2}}^{\sigma_1} \mathrm{W}_{\sigma_1},$$

where $\mathrm{w}^{(k)}$ has "rotated indices" (mod n),

$$\mathrm{w}^{(k)} = \left(s_{i_{l(w)}} \ldots s_{i_1}\right)^{(k)} = s_{i_{l(w)}+k} \ldots s_{i_1+k}.$$

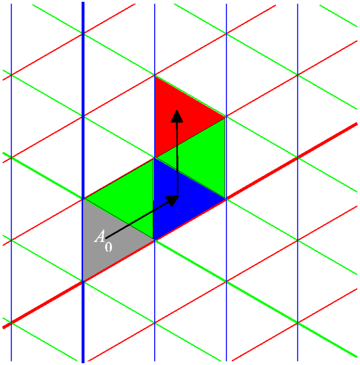
An Analogue of the Lexicographically Minimal Reduced Word for $\tilde{S}_3$?

Following Θ_1 by Θ_2 gives us a walk of shortest possible length into C_0, with reduced word

$$\mathrm{W}_2^{(1)}\mathrm{W}_1 = s_{1+1}s_{0+1}s_2s_0 = s_2s_1s_2s_0.$$

This word and the corresponding indices can be read off:

s_2	s_0	$(1,2)$	$(1,3)$
	s_1		$(1,3)$
	s_2		$(1,2)$



An Analogue of the Lexicographically Minimal Reduced Word for $\tilde{S}_n$?

This can be generalized to $\tilde{S}_n$, where there are $n-1$ vectors Θ_l (P.-P., others?).

Reduced words W_l for these moves, and indices $(I(m), J(m))$ can be read off:

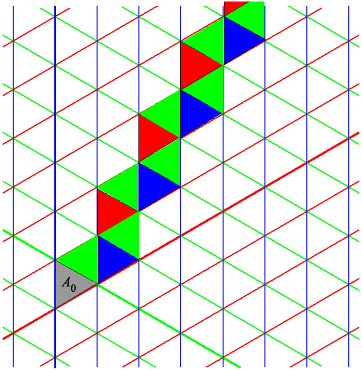
$$\begin{array}{ccccc} s_{l+1} & s_{l+2} & \dots & s_{n-1} & s_0 \\ s_{l+2} & s_{l+3} & \dots & s_0 & s_1 \\ \vdots & & & & \vdots \\ s_{2l+1} & s_{2l+2} & \dots & s_{l-2} & s_{l-1} \end{array} \qquad \begin{array}{cccc} (1, l+1) & (1, l+2) & \dots & (1, n) \\ (2, l+1) & (2, l+2) & \dots & (2, n) \\ \vdots & & & \vdots \\ (l, l+1) & (l, l+2) & \dots & (l, n) \end{array}$$

(Indices are read mod n.)

The corresponding "lexicographically minimal word" is

$$\mathrm{W}_0 = \mathrm{W}_{\mathrm{n}-1}{}^{\left(\binom{n-1}{2}\right)} \dots \mathrm{W}_3{}^{(3)} \mathrm{W}_2{}^{(1)} \mathrm{W}_1.$$

A Canonical Minimal Infinite Reduced Word in $\tilde{S}_3$



For $\tilde{S}_3$, repeating our "lexicographically minimal reduced word" gives a sequence crossing all $p \geq 1$:

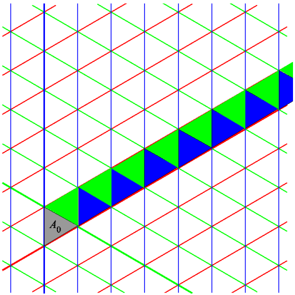
$$s_0, s_2, s_1, s_2, s_0, s_2, s_1, s_2, \ldots$$

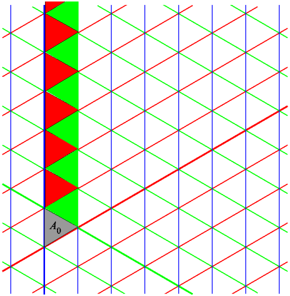
It crosses

$$H_{1;13},\ H_{1;12},\ H_{2;13},\ H_{1;23},$$
$$H_{3;13},\ H_{2;12},\ H_{4;13},\ H_{2;23},$$
$$H_{5;13},\ H_{3;12},\ H_{6;13}, \ldots$$

A Collection of Infinite Reduced Words and an Alternate Factorization

Why stop at two separate infinite reduced words?





Four infinite reduced words gives a factorization (P.-P.)

$$g(z) = g_1(z)g_2(z)\lambda(z)g_2'(z)g_1'(z).$$

Some Open Questions

- Can we understand the combinatorics of the map from $\{\eta_m, \theta_{p;k}, \zeta_m\}$ to the entries of g?
- What is the largest domain on which the correspondence between the sequence space of coordinates $\{\eta_m, \theta_{p;k}, \zeta_m\}$ and $LSU(n)$ holds?
- The unique translation invariant probability measure on $SU(n)$ is a product in the coordinates $\{\zeta_m, \theta_k\}$. Can the same be said for the translation-invariant measure on $LSU(n)$? (This measure is involved in the study of conformal field theory.)
- Can we classify all the possible factorization results for $LSU(n)$?
- Can we study the random geometry of $\tilde{S}_n$? (Grabiner, Biane,...)

Thanks!

- Thanks to all of you for coming!
- Thanks very much to my committee.
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