# A DEFINITION OF SCALAR CURVATURE ON SIMPLICIAL APPROXIMATIONS TO RIEMANNIAN MANIFOLDS

BENJAMIN PITTMAN-POLLETTA

Contents



## 1. Introduction

There are many instances in which it is useful to approximate a Riemannian manifold by a simplicial complex, which is a piecewise flat space constructed, roughly, by gluing together Euclidean polygons of various dimensions. In these situations, it is also useful to equip such simplicial complexes with some of the geometric structures of the corresponding smooth manifold. For example, we might like to equip a simplicial complex with a connection, metric, or curvature. Situations in which questions like this arise include the lattice approach to quantum gravity using path integrals ([7], [5]), computational gravity, and computation of geometric flows. In all these cases, we wish to analyze or simulate evolutions of manifolds derived from action functionals involving geometric quantities.

This paper is an exposition of [2], which introduced a measure on an approximating simplicial complex which converges (in measure) to the scalar

*Date*: March, 2007.

curvature of the manifold that the simplicial complex is meant to approximate. This measure is defined on a simplicial complex of any dimension, which does not have to be an approximation of a smooth manifold.

The organization of the paper is as follows. Section 2 is a review of Riemannian geometry. Some basic results are proven, including the Taylor expansions we will use frequently in the rest of the paper. The exposition follows [6]. Euclidean and constant-curvature laws of cosines are presented in section 3, and section 4 treats simplicial complexes. Section 5 gives the motivation behind and definition of our simplicial scalar curvature measure. Section 6 is a full treatment of convergence in dimension 2, and Section 7 gives a very brief, heuristic treatment of convergence in dimension $n$. The Einstein summation convention is used for indices repeated above and below.

## 2. Riemannian Geometry

I will assume familiarity with differentiable manifolds and tensor calculus. Since curvature is a concept associated with Riemannian manifolds, I will begin with a review of these manifolds and their geometry. As mentioned, I will follow the exposition in [6], and the Einstein summation convention will be employed.

A Riemannian manifold is a manifold which is equipped with concepts of length and angle. These come from a set of smoothly varying inner products on tangent spaces, which might be thought of as an inner product field. It is called a metric, since in the end it gives distances on the manifold, and is defined as follows.

**Definition 2.1.** On a differentiable manifold $M$, a **Riemannian metric** is a symmetric, nondegenerate 2-tensor field, $g$. A **Riemannian manifold** is an ordered pair of a smooth manifold and a Riemannian metric, $(M, g)$.

**Theorem 2.2.** *Every differentiable manifold has a Riemannian metric.*

*Proof.* In each coordinate chart $(U_\alpha, x_\alpha)$ we can define $g_\alpha$ as

$$g_\alpha = \sum_i \phi^i \otimes \phi^i,$$

where $\{\phi^i\}$ is a basis for the cotangent space on $U$, so that for vector fields $X = X^i \partial_i$ and $Y = Y^i \partial_i$,

$$g_\alpha(X, Y) = \sum_i X^i Y^i.$$

Now, being locally compact, Hausdorff, and $\sigma$-compact, i.e. a countable union of compact sets, $M$ supports a smooth partition of unity subordinate to its open cover by coordinate neighborhoods $\{U_\alpha\}$, let's call it $\{\rho_\alpha\}$. We can define a metric globally on $M$ by

$$g = \sum_\alpha \rho_\alpha g_\alpha.$$

By construction, $g$ is smooth, symmetric, nondegenerate, and linear over $C^\infty(M)$ - i.e. tensorial - in both its arguments. $\square$

If $\{E_i\}$ is a local frame at a point $p$ on an $n$-manifold $M$, and $\{\phi^i\}$ is the dual coframe, then an arbitrary Riemannian metric on $M$ can be written locally as

$$g = g_{ij}\phi^i \otimes \phi^j.$$

The coefficients $g_{ij}$ are $\frac{n(n+1)}{2}$ smooth functions on $M$, due to the fact that $g$ is symmetric. The components of the inverse of $g$ (as a matrix) are written $g^{ij}$.

Maps which preserve Riemannian metrics are called isometries.

**Definition 2.3.** An **isometry** between Riemannian manifolds $(M, g)$ and $(\tilde{M}, \tilde{g})$ is a diffeomorphism $f : M \to \tilde{M}$ with the property that $f^*\tilde{g} = g$. $M$ and $\tilde{M}$ are then said to be **isometric**.

If $f$ is an isometry sending $M$ into itself, then we say $f$ is an isometry of $M$. The isometries of $M$ form a group, called, naturally, the isometry group of $M$.

If $\tilde{M}$ is an immersed submanifold of a Riemannian manifold $(M, g)$, then the immersion $\iota$ induces a metric on $\tilde{M}$, called the **induced metric** $\tilde{g} = \iota^* g$.

2.1. **Connections.** To see intuitively how a Riemannian metric can be used to define distances on M, note that we can compute the length of any curve $\gamma : I \to M$ in $M$ as

$$l(\gamma) = \int_I \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}dt.$$

Given two points in $M$, we can define the distance between them as the infimum of the lengths of all paths connecting the two points. If length-minimizing paths exist, they are called **geodesics**. These are the analogs of straight lines on Riemannian manifolds.

Straight lines in Euclidean space are certainly length minimizing, but they are also characterized by having zero acceleration. Indeed, since the curvature of a plane curve $\gamma$ is just the magnitude of its acceleration, $\kappa(t) = \|\ddot{\gamma}(t)\|$, its straightness is a consequence of the fact that it has no acceleration.

It turns out that this is the property of straight lines in $\mathbb{R}^n$ which is most convenient to abstract to geodesics on Riemannian manifolds. That is, we'd like to define geodesics as curves with no acceleration. First, however, we need to define acceleration on an arbitraty Riemannian manifold. In Euclidean space, acceleration is the directional derivative of the tangent vector field to a curve, along that curve. So, we have to find a way to take directional derivatives of vector fields on abstract manifolds. The tool we use to do this is called a **connection**. It will allow us to take the derivative of one vector field with respect to another, and the resulting object will be a third vector field.

What problems do we encounter when attempting to take the directional derivative of a vector field intrinsically? Say we have a two-dimensional surface $S$ embedded in $\mathbb{R}^3$, a curve $\gamma$ in $S$, and a vector field $X$ along $\gamma$, which is tangent to $S$ at every point. We are perfectly capable of computing the derivative of $X$ along $\gamma$, $\dot{X}$, but this new vector field may not be tangent to $S$, that is, it may not define a vector field on $S$.

One way to rectify the problem is to project $\dot{X}$ back onto $TS$. Then, we can define the directional derivative as the new (tangent) vector field resulting from this projection. This is an example of a connection, called the **tangential connection** for the embedding. In fact, each embedding of $S$ in $\mathbb{R}^n$ provides us with a different connection on $S$ by repeating this procedure - taking the directional derivative in the ambient space, and then projecting onto $TS$. So we have a great deal of choice in defining intrinsic directional derivatives. In moving to abstract manifolds, the most useful way to define a connection is by its algebraic properties.

**Definition 2.4.** A **linear connection** on a manifold $M$ is a map

$$\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M) : (X, Y) \mapsto \nabla_X Y,$$

where $\mathcal{T}(M)$ is the space of smooth sections of $TM$, which has the following properties:

(1) $\nabla_X Y$ is linear over $C^\infty(M)$ in $X$;
(2) $\nabla_X Y$ is linear over $\mathbb{R}$ in $Y$;
(3) $\nabla$ satisfies the product rule

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$$

for $f \in C^\infty(M)$.

These properties should be intuitive. The directional derivative of a vector field depends on values of the vector field in a small neighborhood around a point, thus (2), but on only a single vector giving the direction of derivation, thus (1). (3) is simply the fact that we want a derivative, or in algebraic terms a derivation, so we need some kind of product rule. The "linear" in the definition refers to the second factor in the domain being $\mathcal{T}(M)$; in general this factor can be the set of smooth sections of an arbitrary vector bundle $E$ over $M$, and we will see how a linear connection gives rise to a connection on the tensor bundles of arbitrary order over $M$.

Why is this object called a connection? Say we want to compute the acceleration of a curve $\gamma$. We must compute the difference quotient

$$\ddot{\gamma}(t) = \lim_{h\to 0} \frac{\dot{\gamma}(t+h) - \dot{\gamma}(t)}{h}.$$

However, $\dot{\gamma}(t+h) \in T_{\gamma(t+h)}M$, while $\dot{\gamma}(t) \in T_{\gamma(t)}M$. So we need a way to *connect* the two tangent spaces - thus the name.

It is revealing (and useful) to see how a connection appears in coordinates. If $M$ is equipped with a connection $\nabla$, and $\{E_i\}$ is a local frame on $U \subset M$, then we can define $n^3$ smooth functions on $U$, $\Gamma_{ij}^k$, by

$$\nabla_{E_i} E_j =: \Gamma_{ij}^k E_k.$$

These are called the **Christoffel symbols** of $\nabla$ with respect to the local frame.

**Lemma 2.5.** *Let $\nabla$ be a connection on $M$, $\{E_i\}$ a local frame on $U \subset M$, and $X, Y \in \mathcal{T}(U)$ with $X = X^i E_i$, $Y = Y^i E_i$. Then*

$$\nabla_X Y = \left( XY^k + X^i Y^j \Gamma_{ij}^k \right) E_k.$$

The proof is a simple application of the connection properties.

**Proposition 2.6.** *Every smooth manifold has a connection.*

*Proof.* Cover a $M$ by coordinate charts $\{U_\alpha\}$, and let $\{\partial_i^\alpha\}$ be a basis for $TM$ on $U_\alpha$, i.e. a coordinate frame. Then, we can define a linear connection $\nabla^\alpha$ on $U_\alpha$ using Lemma 2.5. First, we choose $n^3$ smooth functions on $U_\alpha$, and label them $\Gamma_{ij}^{\alpha,k}$. Then, for $X = X^i\partial_i^\alpha, Y = Y^i\partial_i^\alpha \in \mathcal{T}(M)$, we define

$$\nabla_X^\alpha Y = \left(X^i\partial_i^\alpha Y^k + X^iY^j\Gamma_{ij}^{\alpha,k}\right)\partial_k^\alpha.$$

Now, our manifold supports a partition of unity subordinate to $\{U_\alpha\}$, call it $\{\rho_\alpha\}$; we use it to patch together our coordinate connections. Define

$$\nabla = \sum_\alpha \rho_\alpha \nabla^\alpha.$$

Now, this is clearly smooth, tensorial in $X$, and $\mathbb{R}$-linear in $Y$. We must show that the product rule holds. But,

$$\begin{aligned}
\nabla_X(fY) &= \sum_\alpha \rho_\alpha \nabla_X^\alpha(fY) \\
&= \sum_\alpha \rho_\alpha \left(X^i\partial_i^\alpha(fY)^k + X^i(fY)^j\Gamma_{ij}^{\alpha,k}\right)\partial_k^\alpha \\
&= \sum_\alpha \rho_\alpha \left(Y^kX^i\partial_i^\alpha f + fX^i\partial_i^\alpha Y^k + fX^iY^j\Gamma_{ij}^{\alpha,k}\right)\partial_k^\alpha \\
&= \sum_\alpha \rho_\alpha \left[X^i\partial_i^\alpha fY^k\partial_k^\alpha + f\left(X^i\partial_i^\alpha Y^k + X^iY^j\Gamma_{ij}^{\alpha,k}\right)\partial_k^\alpha\right] \\
&= \sum_\alpha \rho_\alpha \left[(Xf)Y + f\nabla_X^\alpha Y\right] \\
&= (Xf)Y + f\sum_\alpha \rho_\alpha \nabla_X^\alpha Y \\
&= (Xf)Y + f\nabla_X Y.
\end{aligned}$$

□

Given a linear connection $\nabla$, we can define a unique connection on tensor bundles of arbitrary degree, which restricts to $\nabla$ on $TM$.

**Proposition 2.7.** *Given a linear connection $\nabla$ on a manifold $M$, there is a unique connection $\tilde{\nabla}$ on $T_l^kM$, so that if $X \in TM$, $\tilde{\nabla}$ has the following properties:*

(1) *If $Y \in TM$, $\tilde{\nabla}_X Y = \nabla_X Y$;*
(2) *If $f \in T^0M$, then $\nabla_X f = Xf$;*
(3) *If $F \in T_l^kM$, $G \in T_j^iM$, then*

$$\tilde{\nabla}_X(F \otimes G) = \left(\tilde{\nabla}_X F\right) \otimes G + F \otimes \left(\tilde{\nabla}_X G\right);$$

(4) *If $F \in T_l^kM$, then $\tilde{\nabla}_X(trF) = tr\left(\tilde{\nabla}_X F\right)$;*
(5) *If $\omega \in T^*M$, $Y \in TM$, then*

$$\tilde{\nabla}_X(\omega(Y)) = \left(\tilde{\nabla}_X\omega\right)(Y) + \omega(\tilde{\nabla}_X Y);$$

(6) *If* $F \in T^k_l M$, $Y_i \in TM$ *for* $i = 1, \dots, k$, *and* $\omega^j \in T^*M$ *for* $j = 1, \dots, l$, *then*

$$\left(\tilde{\nabla}_X F\right)(\omega^1, \dots, \omega^l, Y_1, \dots, Y_k) = X(F(\omega^1, \dots, \omega^l, Y_1, \dots, Y_k)) - \sum_j F(\omega^1, \dots, \tilde{\nabla} j_X \omega^j, \dots, \omega^l, Y_1, \dots, Y_k) - \sum_i F(\omega^1, \dots, \omega^l, Y_1, \dots, \tilde{\nabla}_X Y_i, \dots, Y_k).$$

Henceforth, we will refer to this connection as $\nabla$ also. If $X \in TM$, $F \in T^k_l M$, then $\nabla_X F$ is linear over $C^\infty(M)$ in $X$, so by letting $X$ vary we can think of $\nabla_{(\cdot)} F$ as a tensor of degree $\binom{k+1}{l}$, called the **total covariant derivative** of $F$, and written $\nabla F$.

2.1.1. *Geodesics.* Now that we have a concept of directional derivative, we can do what we set out to do - write down the acceleration of a curve:

$$\ddot{\gamma}(t) = \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t).$$

From here, given only a connection, it is an easy step to define geodesics on $M$.

**Definition 2.8.** Let $M$ be a manifold equipped with a connection $\nabla$. A geodesic on $M$ is a curve $\gamma$ satisfying the equation

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0.$$

If $\{x^i\}$ are local coordinates on $U \subset M$, then the coordinate functions $\{\gamma^i(t)\}$, where $\gamma(t) = \left(\gamma^1(t), \dots, \gamma^n(t)\right)$, satisfy the equation

$$\ddot{\gamma}^k(t) + \dot{\gamma}^i(t)\dot{\gamma}^j(t)\Gamma^k_{ij}(\gamma(t)) = 0.$$

**Theorem 2.9.** *Given a point* $p \in M$ *and a vector* $X \in T_p M$, *there is a unique geodesic* $\gamma_X$ *passing through* $p$ *with* $\dot{\gamma}_X(p) = X$.

The preceeding theorem is a consequence of the existence and uniqueness theorem for ordinary differential equations.

2.1.2. *Parallel Translation.* We also hoped that a connection would allow us to "connect" tangent spaces at different points. **Parallel translation** will allow us to do this.

**Definition 2.10.** A vector field $X$ defined along a curve $\gamma$ is said to be **parallel** along $\gamma$ if

$$\nabla_{\dot{\gamma}(t)} X = 0.$$

In fact, we can extend any vector on a curve to a vector field parallel on that curve.

**Theorem 2.11.** *Given a curve* $\gamma : I \to M$, $t_0 \in I$, *and a vector* $X_0 \in T_{\gamma_0} M$, *there is a unique vector field* $X$ *that is parallel along* $\gamma$ *such that* $X_{\gamma(t_0)} = X_0$.

This theorem is a consequence of the existence and uniqueness theorem for *linear* ordinary differential equations. For $t \in I$, the vectors $X_t$ from the theorem are called the **parallel translates** of $X_0$ along $\gamma$. Parallel translation defines an operator between the tangent spaces at different points.

**Definition 2.12.** Let $p, q \in M$ and $\gamma : I \to M$ be such that there are $s, t \in I$ with $\gamma(s) = p$, $\gamma(t) = q$. Then we can define an operator

$$P^{\gamma}_{pq} : T_pM \to T_qM$$

which takes $X \in T_pM$ to its parallel translate along $\gamma$ at $q$.

2.2. **The Riemannian Connection & Riemannian Geodesics.** Using a linear connection, we can define geodesics on a manifold even if it is not equipped with a Riemannian metric. Unfortunately, each connection on the manifold defines a different set of geodesics. The one property of straight lines in Euclidean space we haven't talked about is their uniqueness; given two points in $E^n$, there is only one straight line connecting them. How do we pick out a particular connection to use in our definition of geodesics?

It turns out that on a Riemannian manifold $(M, g)$, we can single out a connection which has two properties: **compatibility with the metric** and **symmetry**.

**Theorem 2.13.** *Let $(M, g)$ be a Riemannian manifold. There is a unique linear connection $\nabla$ on $M$ such that for $X, Y, Z \in TM$,*

1. $\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ *(compatibility),*
2. $\nabla_X Y - \nabla_Y X = [X, Y]$ *(symmetry).*

*If $\{x^i\}$ is a local coordinate system for $U \subset M$, then on $U$ the Christoffel symbols of this connection are*

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$$

It turns out that the tangential connection for an embedding, defined above, is always compatible with the metric induced by the embedding. Compatibility confers several nice properties on the Riemannian connection.

**Proposition 2.14.** *Let $(M, g)$ be a Riemannian manifold equipped with a linear connection $\nabla$. Then the following statements are equivalent:*

1. $\nabla$ *is compatible with $g$;*
2. $\nabla g$*=0;*
3. *If $X, Y$ are parallel vector fields along a curve $\gamma$, then $g(X, Y)$ is constant.*

The Riemannian connection is natural in the sense that it is preserved by isometries.

**Proposition 2.15.** *Let $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometry between Riemannian manifolds, and let $\nabla$, $\tilde{\nabla}$ be the Riemannian connections on $M$, $\tilde{M}$ respectively. Then, for $X, Y \in TM$,*

$$\phi_* \left(\nabla_X Y\right) = \tilde{\nabla}_{\phi_* X} \left(\phi_* Y\right).$$

Geodesics defined by the Riemannian connection are called **Riemannian geodesics**. If we have a Riemannian manifold in mind, we will often drop the adjective Riemannian and refer to these special geodesics by their surname. We can define the speed of a path $\gamma$ on a Riemannian manifold $(M, g)$ to be $\sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}$. Then, it is a corollary of Proposition 2.14.3 that Riemannian geodesics have constant speed.

2.2.1. *The Exponential Map.* Just as it is nice to have a map between tangent spaces, we would sometimes like to have a map between the tangent space at a point $p$ and the manifold near $p$. Such a map allows us to quantify exactly *how* the tangent space locally linearizes the manifold. On Riemannian manifolds, geodesics give us an intuitive way of constructing such a map.

**Definition 2.16.** Let $\mathcal{E}$ be the set of all $X \in TM$ with the property that $\gamma_X$ is defined on $[0,1]$. Then we define

$$\begin{aligned} \exp : \mathcal{E} &\longrightarrow M \\ V &\longmapsto \gamma_V(1). \end{aligned}$$

For $p \in M$, we denote the restriction of exp to $\mathcal{E}_p := \mathcal{E} \cap T_pM$ by $\exp_p$.

The following proposition summarizes some of the properties of the exponential map.

**Proposition 2.17.**
(1) *$\mathcal{E}_p$ is star-shaped with respect to the origin.*
(2) exp *is smooth.*
(3) *For $X \in TM$, $\gamma_X(t) = exp(tX)$ whenever both sides of the equation are defined.*
(4) *Let $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometry. Then, for $p \in M$ and $X \in \mathcal{E}_p$ such that $\phi_* X \in \tilde{\mathcal{E}}_{\phi(p)}$,*

$$\phi\left(\exp_p(X)\right) = \tilde{\exp}_{\phi(p)}\left(\phi_* X\right).$$

In fact, we can say more. The exponential map is a local diffeomorphism from $T_pM$ to $M$.

**Lemma 2.18.** *Let $p \in M$. Then there are open sets $U, V$ with $0 \in U \subset T_pM$ and $p \in V \subset M$ so that $\exp_p : U \to V$ is a diffeomorphism. $V$ is called a* ***normal neighborhood*** *at $p$.*

*Proof.* This follows from the inverse function theorem, and the fact that we can show $\exp_{p*} : T_0T_pM = T_pM \to T_pM$ is the identity. The calculation is as follows.

$$\exp_{p*} V = \frac{\partial}{\partial t}\Big|_{t=0} \exp(tV) = \frac{\partial}{\partial t}\Big|_{t=0} \gamma_V(t) = V.$$

The first equality is due to the fact that $tV$ is a path in $T_pM$ passing through the origin at $t = 0$ and with tangent vector $V$ there. □

Using this fact about the exponential map, we can use it to define a coordinate chart at any point.

**Definition 2.19.** Let $p \in M$, and let $U$ be a normal neighborhood at $p$. Let $\{E_i\}$ be an orthonormal basis for $T_pM$, and define

$$E : T_pM \longrightarrow \mathbb{R}^n : X^i E_i \longmapsto X^i e_i$$

where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$. Then, we can define a coordinate chart $(U, \phi)$ with the map

$$\phi = E \circ \exp_p^{-1} : U \to \mathbb{R}^n.$$

Then $(U, \phi)$ is called a **normal coordinate chart** centered at $p$. The coordinate functions $\{x^i\}$ are called simply **normal coordinates**, the **radial distance function** is

$$r = \left( \sum_i (x^i)^2 \right)^{\frac{1}{2}}$$

and the **radial vector field** is

$$\frac{\partial}{\partial r} = x^i \frac{\partial}{\partial x^i}.$$

Normal coordinates are endlessly useful in performing Taylor expansions, partially because of the following properties.

**Proposition 2.20.** *Let $(U, \{x^i\})$ be a normal coordinate chart centered at $p$. Then,*

1. *The coordinates of $p$ are $p = (0, \dots, 0)$.*
2. *Let $X = X^i \partial_i \in T_pM$. Then, the geodesic starting at $p$ in the direction of $X$ is given in coordinates as $\gamma_X(t) = (tX^1, \dots, tX^n)$ so long as $\gamma_X(t) \in U$.*
3. *At $p$, $g = \sum_i dx^i \otimes dx^i$. In other words, at $p$, $g_{ij} = \delta_{ij}$.*
4. *The first partial derivates of the $g_{ij}$ and the Christoffel symbols $\Gamma^k_{ij}$ vanish at $p$.*

In light of Proposition 2.20.2, we can think of normal coordinates as radial geodesic coordinates.

2.3. **Curvature.** We define curvature as the degree of obstruction to parallel translation of an orthonormal frame.

**Definition 2.21.** The **Riemannian curvature endomorphism** is the map

$$R : \mathcal{T}(M)^3 \longrightarrow \mathcal{T}(M) : (X, Y, Z) \longmapsto R(X, Y)Z$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The curvature endomorphism is tensorial in all its inputs. By pairing it with another vector field via the metric, we can make it a covariant 4-tensor; this is what is commonly known as the curvature tensor.

**Definition 2.22.** The **Riemannian curvature tensor** is the covariant 4-tensor field

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

Given bases $\{\partial_i\}$ and $\{\phi^j\}$ for the tangent and cotangent spaces, respectively, we can write both the curvature endomorphism and the curvature tensor in coordinates, defining

$$R^l_{ijk} = \phi^l \left( R \left( \partial_i, \partial_j \right) \partial_k \right)$$

and

$$R_{ijkl} = R \left( \partial_i, \partial_j, \partial_k, \partial_l \right) = g_{lm} R^m_{ijk}.$$

It is a fact that the curvature of Euclidean space is zero.

**Theorem 2.23.** *A Riemannian manifold is locally isometric to Euclidean space if and only if its curvature tensor vanishes identically.*

We will most commonly encounter the curvature tensor in the form of a number of contractions and derived quantities, defined below.

**Definition 2.24.** Let $(M, g)$ be a Riemannian manifold and $R$ its associated curvature tensor. Also let $p \in M$, $X, Y \in T_pM$, and let $\{x^i\}$ be coordinates around $p$ and $\{\partial_i\}$ and $\{\phi^j\}$ be *orthonormal* bases for the tangent and cotangent spaces.

(1) The **sectional curvature** is the map $K : T_pM \times T_pM \to \mathbb{R}$ defined as
$$K(X,Y) = \frac{R(X,Y,Y,X)}{|X|^2|Y|^2 - g(X,Y)^2}.$$
(2) The **Ricci curvature** is the symmetric 2-tensor field defined as
$$\begin{aligned} Rc(X,Y) =& \phi^i \left(R\left(\partial_i, X\right) Y\right) \\ =& \sum_i R(\partial_i, X, Y, \partial_i). \end{aligned}$$
Thus its coefficients are $R_{ij} = g^{kl} R_{kijl}$.
(3) The **scalar curvature** is the $C^\infty$ function
$$\begin{aligned} R^2 =& \sum_i Rc(\partial_i, \partial_i) \\ =& \sum_{i,j} R\left(\partial_i, \partial_j, \partial_j, \partial_i\right). \end{aligned}$$
Its value is $R^2 = g^{ij} g^{kl} R_{kijl}$.

2.4. **Jacobi Fields.** Jacobi fields are vector fields which give us an idea of how curvature affects nearby geodesics. More precisely, they are vector fields associated to variations of geodesics.

**Definition 2.25.** A **variation through geodesics** of a geodesic $\gamma : [a,b] \to M$ on a Riemannian manifold $(M, g)$ is a map $\Gamma : (-\epsilon, \epsilon) \times [a,b] \to M$ such that:

(1) $\Gamma(0,t) = \gamma(t)$;
(2) $\Gamma$ is smooth as a map from $\mathbb{R}^2$ to $M$;
(3) For each $s \in (-\epsilon, \epsilon)$, the **main curve** $\Gamma_s : [a,b] \to M$ is a geodesic.

The tangent vectors to the **transverse curves** $\Gamma_t : (-\epsilon, \epsilon) \to M$ which lie along $\gamma$ form the **variation field** associated to $\Gamma$.

**Definition 2.26.** If $\Gamma(s,t)$ is a variation through geodesics of a geodesic $\gamma(t)$, then the variation field of $\Gamma$ along $\gamma$ is
$$V(t) = \frac{\partial}{\partial s}\Gamma(s,t)\big|_{(0,t)}.$$

The variation field of a variation through geodesics is called a **Jacobi field**. Jacobi fields satisfy the following differential equation.

**Theorem 2.27** (The Jacobi Equation)**.** *Let $V(t)$ be a vector field along a geodesic $\gamma(t)$. Then $V$ is a Jacobi field if and only if*

$$D_t^2 V = -R\left(V, \dot{\gamma}\right)\dot{\gamma}.$$

Like geodesics, Jacobi fields are unique given initial conditions. The space of Jacobi fields spans (with coefficients in $C^\infty(\gamma)$) the space of vector fields along $\gamma$.

**Proposition 2.28.** *Let $\gamma : I \to M$ be a geodesic, let $a \in I$, and let $p = \gamma(a)$. Then for all $X, Y \in T_pM$, there exists a unique Jacobi field $J(t)$ along $\gamma$ satisfying*

$$J(a) = X$$
$$D_t J(a) = Y.$$

*Thus, the space of Jacobi fields has dimension $2n$.*

Jacobi fields look especially nice in normal coordinates.

**Lemma 2.29.** *Let $(U, \{x^i\})$ be normal coordinates around $p \in M$, and let $\gamma$ be a radial geodesic starting at $p$. For $W = W^i\partial_i \in T_pM$, the Jacobi field $J(t)$ with $J(0) = 0$ and $D_tJ(0) = W$ is given in normal coordinates as*

$$J(t) = tW^i\partial_i.$$

2.5. **Taylor Expansions.** Using Jacobi fields and the various curvatures we can obtain a number of useful Taylor expansions. The starting point is the following Taylor expansion of the metric in terms of the curvature tensor.

**Proposition 2.30.** *Let $(U, \{x^i\})$ be a normal coordinate chart centered at $p \in U \subset M$. Then, for $x \in U$,*

$$g(x) = \delta_{ij} - \frac{1}{3}R_{iklj}(p)x^kx^l + \frac{1}{6}\nabla_m R_{iklj}(p)x^kx^lx^m + O(|x|^4).$$

*Proof.* Let $\{x^i\}$ be Riemannian normal coordinates around a point $p$. Then, we can use Taylor's theorem to expand the metric around $p$ as

$$g_{ij}(x) = g_{ij}(p) + \partial_k g_{ij}(p)x^k + \partial_k\partial_l g_{ij}(p)x^kx^l + \partial_k\partial_l\partial_m g_{ij}(p)x^kx^lx^m + O(|x|^4).$$

To find the partial derivatives of $g_{ij}$, we will take the directional derivative along a radial geodesic of the norm of a Jacobi field. Because of the Jacobi equation, terms involving the curvature endomorphism will appear. Because we choose our Jacobi field to vanish at $p$, most of the terms will vanish when we evaluate the partial derivatives there.

We can concern ourselves only with Jacobi fields because because the Jacobi fields vanishing at $p$ provide a basis for $T_qM \subset TU$ so long as $q \neq p$. Thus, if $\{X_i\}$ is a basis for $T_qM$, we can expand any vector $Y$ at $q \in U$ as a linear combination of the vectors $J_i(q)$, where $J_i$ are Jacobi fields along the radial geodesic $\gamma_V(t) = \left(tV^1, \ldots, tV^n\right)$ connecting $p$ to $q$, and for each $i$ $J_i(p) = 0$ and $D_tJ_i(p) = X_i$.

Take $\gamma_V(t) = t\sum_i V^ix^i$ to be the radial geodesic with $\gamma_V(0) = p$ and $\dot{\gamma}_V(0) = V^i\partial_i$. Also, let $V = V^i\partial_i$ be the vector field corresponding to $\dot{\gamma}$. Let $J(t) = tW^i\partial_i$ be a Jacobi field along $\gamma_V$. In normal coordinates the derivative along $\gamma$ is just the derivative with respect to $t$. By compatibility with the metric, we have

$$\frac{\partial}{\partial t}|J(t)|^2 = \frac{\partial}{\partial t}g(J(t), J(t)) = 2g\left(D_t J(t), J(t)\right) = 2g(W, J(t)).$$

Also, by the Jacobi equation,

$$\begin{aligned}\frac{\partial^2}{\partial t^2}|J(t)|^2 &= 2g\left(D_t^2 J(t), J(t)\right) + 2\left|D_t J(t)\right|^2\\ &= -2g\left(R\left(J(t), \dot{\gamma}(t)\right)\dot{\gamma}(t), J(t)\right) + 2|W|^2\\ &= -2R(J(t), V, V, J(t)) + 2|W|^2.\end{aligned}$$

Then,

$$\begin{aligned}\frac{\partial^3}{\partial t^3}|J(t)|^2 &= 2g\left(D_t^3 J(t), J(t)\right) + 6g\left(D_t^2 J(t), D_t J(t)\right)\\ &= 2g\left(D_t R(J(t), V)V, J(t)\right) + 6R(J(t), V, V, W).\end{aligned}$$

Next,

$$\begin{aligned}\frac{\partial^4}{\partial t^4}|J(t)|^2 &= 2g\left(D_t^4 J(t), J(t)\right) + 8g\left(D_t^3 J(t), D_t J(t)\right) + 6\left|D_t^2 J(t)\right|^2\\ &= -2g\left(D_t^2 R(W, V)V, J(t)\right) + 8g\left(D_t R(J(t), V)V, W\right) + 12\left|R(J(t), V)V\right|^2.\end{aligned}$$

Now,

$$\begin{aligned}D_t R\left(J(t), V\right)V &= \frac{\partial}{\partial t}R\left(J(t), V\right)V - R\left(D_t J(t), V\right)V\\ &\quad - R\left(J(t), D_t V\right)V - R\left(J(t), V\right)D_t V\\ &= \frac{\partial}{\partial t}R\left(J(t), V\right)V - R\left(W, V\right)V,\end{aligned}$$

so we have

$$\begin{aligned}\frac{\partial^4}{\partial t^4}|J(t)|^2 = &- 2g\left(D_t^2 R(W, V)V, J(t)\right) + 8g\left(\frac{\partial}{\partial t}R(J(t), V)V, W\right)\\ &- 8R(W, V, V, W) + 12\left|R(J(t), V)V\right|^2.\end{aligned}$$

Finally,

$$\begin{aligned}\frac{\partial^5}{\partial t^5}|J(t)|^2 &= 2g\left(D_t^5 J(t), J(t)\right) + 10g\left(D_t^4 J(t), W\right) + 20g\left(D_t^3 J(t), D_t^2 J(t)\right)\\ &= -2g\left(D_t^3 R(J(t), V)V, J(t)\right) - 10g\left(D_t^2 R(J(t), V)V, W\right)\\ &\quad + 20g\left(D_t R(J(t), V)V, R(J(t), V)V\right)\end{aligned}$$

Now,

$$D_t^2 R(J(t), V)V = \frac{\partial^2}{\partial t^2}R(J(t), V)V - 2\frac{\partial}{\partial t}R(D_t J(t), V)V + R(D_t^2 J(t), V)V,$$

so

$$
\begin{aligned}
\frac{\partial^5}{\partial t^5}|J(t)|^2 = 2g\left(D_t^3 R(J(t),V)V, J(t)\right) - 10g\left(\frac{\partial^2}{\partial t^2}R(W,V)V,W\right) \\
+ 20g\left(\frac{\partial}{\partial t}R(W,V)V,W\right) - 10g\left(R(D_t^2 J(t),V)V,W\right) \\
+ 20g\left(D_t R(J(t),V)V, R(J(t),V)V\right).
\end{aligned}
$$

Also,

$$
\begin{aligned}
D_t g\left(R(W,V)V,W\right) =& g\left(D_t R(W,V)V,W\right) + g\left(R(W,V)V, D_t W\right) \\
=& g\left(\frac{\partial}{\partial t}R(W,V)V,W\right) + g\left(R(D_t W,V)V,W\right) \\
&+ g\left(R(W,V)V, R(J(t),V)V\right) \\
=& g\left(\frac{\partial}{\partial t}R(W,V)V,W\right) + g\left(R(R(J(t),V)V,V)V,W\right) \\
&+ g\left(R(W,V)V, R(J(t),V)V\right)
\end{aligned}
$$

meaning that

$$
\begin{aligned}
g\left(\frac{\partial}{\partial t}R(W,V)V,W\right) =& D_t R(W,V,V,W) - g\left(R(R(J(t),V)V,V)V,W\right) \\
&- g\left(R(W,V)V, R(J(t),V)V\right).
\end{aligned}
$$

Substituting this into the expression for the fifth derivative, we have

$$
\begin{aligned}
\frac{\partial^5}{\partial t^5}|J(t)|^2 = 2g\left(D_t^3 R(J(t),V)V, J(t)\right) - 10g(\frac{\partial^2}{\partial t^2}R(W,V)V,W) \\
+ 20 D_t R(W,V,V,W) - 20g\left(R(R(J(t),V)V,V)V,W\right) \\
- 20g\left(R(W,V)V, R(J(t),V)V\right) - 10g\left(R(D_t^2 J(t),V)V,W\right) \\
+ 20g\left(D_t R(J(t),V)V, R(J(t),V)V\right).
\end{aligned}
$$

Evaluating at $t = 0$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}|J(t)|^2\Big|_{t=0} &= \frac{\partial^3}{\partial t^3}|J(t)|^2\Big|_{t=0} = 0 \\
\frac{\partial^2}{\partial t^2}|J(t)|^2\Big|_{t=0} &= 2g\big|_p(W,W) \\
\frac{\partial^4}{\partial t^4}|J(t)|^2\Big|_{t=0} &= -8R\big|_p(W,V,V,W) \\
\frac{\partial^5}{\partial t^5}|J(t)|^2\Big|_{t=0} &= 20D_t\big|_{t=0}R(W,V,V,W)
\end{aligned}
$$

Now, $g(J(t), J(t)) = g_{ij} tW^i tW^j$. So, plugging the above into the Taylor expansion of $g(J(t), J(t))$ with respect to $t$, we have

$$\begin{aligned} t^2 g_{ij}(tV) W^i W^j =& t^2 \frac{2}{2!} g_{ij}(p) W^i W^j - t^4 \frac{8}{4!} R_{iklj}(p) W^i V^k V^l W^j \\ &+ t^5 \frac{20}{5!} D_t\big|_{t=0} R_{iklj}(p) W^i V^k V^l W^j \\ \implies g_{ij}(tV) W^i W^j =& \delta_{ij} W^i W^j - t^2 \frac{1}{3} R_{iklj}(p) W^i V^k V^l W^j \\ &+ t^3 \frac{1}{6} \nabla_V R_{iklj}(p) W^i V^k V^l W^j + O(t^4) \\ =& \delta_{ij} W^i W^j - \frac{1}{3} R_{iklj}(p) W^i tV^k tV^l W^j \\ &+ \frac{1}{6} \nabla_{tV} R_{iklj}(p) W^i tV^k tV^l W^j + O(t^4). \end{aligned}$$

By replacing $tV^i$ with $x^i$, and since $|x|$ grows like $t$, we obtain our result. □

This expansion of the metric also gives us an expansion of the volume form in terms of the Ricci curvature.

**Proposition 2.31.** *The volume form $dV = \sqrt{\det g}$ on a Riemannian manifold can be expanded in local coordinates $(U, \{x^i\})$ as*

$$dV_g = \left[1 - \frac{1}{6} R_{kl} x^k x^l + O(|x|^3)\right] dV_E$$

*where $R_{ij}$ are the coefficients of the Ricci curvature tensor and $dV_E$ is the Euclidean volume form.*

*Proof.* For an invertible matrix-valued function $A(t)$, we have the identity

$$\frac{\partial}{\partial t} \log \det A = Tr\left(A^{-1} \frac{\partial}{\partial t} A\right).$$

Let $(U, \{x^i\})$ be local coordinates around $p$. Then, for a measurable set $B$ with $p \in B \subset U$, we have

$$V_g(B) = \int_B dV_g = \int_B \sqrt{\det g} dV_E.$$

We can Taylor expand the integrand at the point $x = (tV^1, \ldots, tV^n) \in B$ around $p$ as

$$\sqrt{\det g(x)} = \sqrt{\det g(p)} + \partial_k \sqrt{\det g(p)} x^k + \frac{1}{2} \partial_k \partial_l \sqrt{\det g(p)} x^k x^l + O(|x|^3).$$

Now, we compute the first and second derivatives of $\sqrt{\det g}$ along the geodesic $tV$.

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0} \sqrt{\det g} =& \frac{\partial}{\partial t}\Big|_{t=0} \exp\left(\frac{1}{2} \log \det g\right) \\ =& \frac{1}{2} \sqrt{det g(p)} \frac{\partial}{\partial t}\Big|_{t=0} \log \det g \\ =& \frac{1}{2} Tr\left(g^{-1}(p) \frac{\partial}{\partial t}\Big|_{t=0} g\right) \\ =& 0. \end{aligned}$$

Also,

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\Big|_{t=0}\sqrt{\det g} =& \frac{\partial^2}{\partial t^2}\Big|_{t=0}\exp\left(\frac{1}{2}\log\det g\right) \\
=& \frac{1}{2}\frac{\partial}{\partial t}\Big|_{t=0}\sqrt{\det g}Tr\left(g^{-1}\frac{\partial}{\partial t}\Big|_{t=0}g\right) \\
=& \frac{1}{4}\sqrt{\det g(p)}Tr\left(g^{-1}(p)\frac{\partial}{\partial t}\Big|_{t=0}g\right)^2 + \frac{1}{2}\sqrt{\det g(p)}Tr\left(\frac{\partial}{\partial t}\Big|_{t=0}g^{-1}\frac{\partial}{\partial t}\Big|_{t=0}g\right) \\
&+ \frac{1}{2}\sqrt{\det g(p)}Tr\left(g^{-1}(p)\frac{\partial^2}{\partial t^2}\Big|_{t=0}g\right) \\
=& \frac{1}{2}Tr\left(\frac{\partial^2}{\partial t^2}\Big|_{t=0}g\right).
\end{aligned}$$

But, from the Taylor expansion of the metric, we know

$$\frac{\partial^2}{\partial t^2}\Big|_{t=0}g_{ij} = -\frac{1}{3}R_{iklj},$$

so

$$\frac{\partial^2}{\partial t^2}\Big|_{t=0}\sqrt{\det g} = -\frac{1}{6}R_{kl},$$

and we have our result. □

Finally, we have an expansion of the volume of a geodesic ball in terms of the scalar curvature.

**Proposition 2.32.** *Let $B_t^g(p)$ be a geodesic ball of radius $t$ around $p \in M$, where $(M, g)$ is a Riemannian manifold of dimension $n$. Let $B_t(0)$ be a ball of radius $t$ around the origin in $\mathbb{R}^n$, and let $R^2$ be the scalar curvature at $p$. Then,*

$$V_g\left(B_t^g(p)\right) = \left[1 - \frac{R^2}{6(n+2)}t^2 + O(t^4)\right] V_E\left(B_t(0)\right),$$

*where $V_g$ is the Riemannian volume form, and $V_E$ is the Euclidean volume form.*

2.6. **The Gauss-Bonnet Theorem.** One result that will motivate us in the coming pages is the Gauss-Bonnet Theorem, which tells us that the sum of the exterior angles of a geodesic triangle on a Riemannian 2-manifold relates to the integral of the scalar curvature over the triangle. First, however, we will need a more general formula, called the Gauss-Bonnet Formula. This formula is concerned not only with the integral of the scalar curvature over a piecewise smooth polygon, but also with the integral of the signed curvature over its edges, and with the exterior angles at its vertices.

**Definition 2.33.** Let $\gamma : [a, b] \to M$ be a piecewise-smooth curve in a Riemannian 2-manifold $M$ such that $\gamma(a) = \gamma(b)$. Also, assume that there are a finite number of points $\{a_i\}_{i=1}^N$ in $[a, b]$, indexed by magnitude, $a \neq a_i \neq b$, at which $\gamma$ is not smooth. Then the **exterior angle** of $\gamma$ at $a_i$ is the angle $\epsilon_i \in (-\pi, \pi)$ between the tangent vector from the left at $a_i$, $\dot{\gamma}\left(a_i^-\right)$, and the tangent vector from the right at $a_i$, $\dot{\gamma}\left(a_i^+\right)$, which has the same sign as $dA\left(\dot{\gamma}\left(a_i^-\right), \dot{\gamma}\left(a_i^+\right)\right)$ (in other words, is positive if $\dot{\gamma}\left(a_i^-\right)$ is clockwise from $\dot{\gamma}\left(a_i^+\right)$, and negative otherwise).

**Definition 2.34.** Let $\gamma : I \to M$ be a smooth curve in a Riemannian 2-manifold $M$. Let $N(t)$ be the unit vector orthogonal to $\dot{\gamma}(t)$ in $T_{\gamma(t)}M$ so that $dA\left(\dot{\gamma}(t), N(t)\right) > 0$ (in other words, $\dot{\gamma}(t)$ and $N(t)$ form a positively oriented basis for $T_{\gamma(t)}M$). Then the **signed curvature** of $\gamma$ is the function

$$\kappa_N : I \longrightarrow \mathbb{R} : t \longmapsto \kappa_N(t) = g\left(D_t\dot{\gamma}(t), N(t)\right).$$

Now we can state the Gauss-Bonnet Formula.

**Theorem 2.35** (The Gauss-Bonnet Formula)**.** *Suppose $\gamma$ is a smooth polygon on a Riemannian 2-manifold $(M, g)$, bounding an open set $\Omega$ with compact closure. Then,*

$$\int_\Omega K dA + \int_\gamma \kappa_N ds + \sum_i \epsilon_i = 2\pi,$$

*where $K$ is the sectional curvature of $g$, $dA$ is the Riemannian area form on $M$, $\kappa_N$ is the signed curvature, and $\epsilon_i$ are the exterior angles of $\gamma$.*

This formula simplifies nicely in the case of a geodesic triangle.

**Corollary 2.36.** *Let $\triangle$ be a geodesic triangle on a Riemannian 2-manifold $M$, and let $\epsilon_i$ be its exterior angles. Then,*

$$\int_\triangle K dA = 2\pi - \sum_{i=1}^3 \epsilon_i.$$

The Gauss-Bonnet Formula can be used to prove the celebrated Gauss-Bonnet Theorem.

**Theorem 2.37** (The Gauss-Bonnet Theorem)**.** *If $M$ is a triangulated, compact, oriented, Riemannian 2-manifold, then*

$$\int_M K dA = 2\pi\chi(M),$$

*where $\chi(M)$ is the Euler characteristic of $M$.*

## 3. Laws of Cosines

In the sequel, we will use two laws of cosines. The first, the familiar Euclidean law of cosines, is stated below without proof. The second, the spherical law of cosines, will be proven for sectional curvature $K = 1$.

**Theorem 3.1.** *Let $T$ be a Euclidean triangle with side lengths $a, b$ and $c$. Let the angles opposite these sides be labeled $\alpha, \beta$ and $\gamma$ respectively. Then,*

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

**Theorem 3.2.** *Let $T$ be a geodesic triangle on a sphere of constant sectional curvature $K$, with vertices $A, B,$ and $C$. Let the sides opposite these vertices have lengths $a, b,$ and $c$, respectively, and let the angles at these vertices be $\alpha, \beta,$ and $\gamma$, respectively. Then,*

$$\cos\gamma = \frac{\cos\sqrt{K}c - \cos\sqrt{K}a\cos\sqrt{K}b}{\sin\sqrt{K}a\sin\sqrt{K}b}.$$

*Proof.* I will prove this theorem in the case that $K = 1$.

$$\cos\gamma = \frac{t_a^* t_b}{\|t_a\|\|t_b\|},$$

where $t_a$ is the tangent vector at $c$ in the direction of $a$, and $t_b$ is the tangent vector at $c$ in the direction of $b$. Let $u, v$, and $w$ be the (unit) vectors from the center of the sphere to $A, B$, and $C$, respectively. Then, $t_a$ is the vector perpendicular to $w$ in the $uw$-plane, which can be obtained by subtracting the projection of $u$ onto $w$ from $u$:

$$\begin{aligned} t_a =& \left(1 - ww^*\right) u \\ =& u - \left(w^* u\right) w. \end{aligned}$$

Also, we have that

$$v^* w = \cos a, \qquad u^* w = \cos b, \qquad u^* v = \cos c.$$

Thus,

$$t_a = u - (\cos b)\, w,$$

and

$$\begin{aligned} \|t_a\|^2 =& u^* u + \left(\cos^2 b\right) w^* w - 2\left(\cos b\right) u^* w \\ =& 1 - \cos^2 b \\ =& \sin^2 b. \end{aligned}$$

Likewise,

$$t_b = v - (\cos a)\, w, \qquad \|t_b\|^2 = \sin^2 a.$$

Thus

$$\begin{aligned} \cos\gamma =& \frac{u^* v + (\cos a \cos b)\, w^* w - (\cos b)\, v^* w - (\cos a)\, u^* w}{\sin a \sin b} \\ =& \frac{\cos c + \cos a \cos b - 2\cos a \cos b}{\sin a \sin b} \\ =& \frac{\cos c - \cos a \cos b}{\sin a \sin b}. \end{aligned}$$

The case $0 < K \neq 1$ is the rescaling given in the statement of the proof; if $K < 0$, then the expression is equivalent to the following hyperbolic law of cosines:

$$\cos\gamma = \frac{\cosh\sqrt{-K}a \cosh\sqrt{-K}b - \cosh\sqrt{-K}c}{\sinh\sqrt{K}a \sinh\sqrt{K}b}.$$

□

## 4. Simplicial Complexes

We will be dealing throughout this paper with simplicial complexes.

**Definition 4.1.** A **finite simplicial complex** $K$ is a finite set of elements called **vertices** and a set of finite, nonempty subsets of vertices called **simplices** that satisfies

(1) Any set containing a single vertex is a simplex.
(2) Any nonempty subset of a simplex is also a simplex.

A simplex containing $j+1$ vertices is said to have dimension $j$, is called a $j$-simplex, and is denoted by $\sigma^j$. The dimension of $K$ is $\sup_{\sigma \subset K} \dim \sigma$. A verctex has dimension 0, 1-simplices are called edges, $n-1$-simplices are called faces, and $n-2$-simplices are called bones.

It is often useful to think of simplices as subsets of $\mathbb{R}^n$.

**Definition 4.2.** Let $\{x_i\}$ be a set of $m+1$ vectors in $\mathbb{R}^n$, $n > m$, lying in no $m-1$-dimensional affine subspace of $\mathbb{R}^n$. Then the convex hull of $\{x_i\}$ (the intersection of all convex sets containing $\{x_i\}$) is called a **closed linear simplex**. Its interior is called an **open linear simplex**.

Coordinates for a closed linear simplex of dimension $m$ take the form $(t_1, \ldots, t_{m+1})$, with the condition $t_1 + \ldots + t_{m+1} = 1$, corresponding to the point in $\mathbb{R}^n$

$$t_1 x_1 + \ldots + t_{m+1} x_{m+1}.$$

Furthermore, we can associate every simplicial complex of dimension $n$ with a subset of $\mathbb{R}^n$ as follows. Let $\{e_i\}$ be the standard basis vectors in $\mathbb{R}^n$, and associate to each basis vector the corresponding vertex of $K$, $\sigma_i$. Then, to each simplex

$$\sigma^k = \{\sigma^0_{i_1}, \ldots, \sigma^0_{i_k}\}$$

associate the convex hull of the corresponding basis vectors,

$$\{e_{i_1}, \ldots, e_{i_k}\}.$$

This **geometric realization** of $K$ inherits a metric from $\mathbb{R}^n$, which is piecewise flat on all simplices. However, it can be equipped with any metric which is piecewise flat on each simplex - in other words, any metric so that each simplex is isometric to some linear simplex. Such a metric is completely determined by the edge lengths of the geometric realization of $K$.

**Definition 4.3.** A **piecewise flat space** is a geometric realization of a simplicial complex $K$, along with a set of edge lengths $L$ defining a piecewise flat metric on the simplices of $K$. The **mesh** of $(K, L)$ is $\eta = \sup L$.

We approximate manifolds with piecewise flat spaces as follows.

**Definition 4.4.** A **smooth triangulation** of a Riemannian manifold $(M, g)$ is a pair $(K, f)$ of a simplicial complex $K$ and a map $f : K \to M$ so that $f$ is smooth on each open simplex $\sigma \subset K$. The edge lengths which make $K$ into a piecewise flat space are the geodesic distances between vertices,

$$l_{ij} = \rho\left(f\left(\sigma^0_i\right), f\left(\sigma^0_j\right)\right),$$

where $\rho$ is the geodesic distance in $M$ and $\sigma^0$ is a vertex of $K$.

It is known that any compact smooth manifold admits a smooth triangulation (where $f$ above is required only to be smooth on simplices). Also, in [2] an argument is given that, for any piecewise linear space, it is possible to find a series of subdivisions with $\eta$ approaching zero. Finally, in dimension 2, if the triangulation is fine enough, it is possible to replace the edges of a smooth triangulation with minimizing geodesic segments, forming a **geodesic triangulation**

4.1. **Interior and Exterior Angles.** Given $\sigma^l \subset \sigma^k$ in a piecewise flat space, there is a set of vectors perpendicular to $\sigma^l$ pointing into $\sigma^k$, called the **normal cone** $C^{\perp}\left(\sigma^l, \sigma^k\right)$. Intersecting the normal cone with $S^{k-l}$ gives us a sector of $S^{k-l}$, the **link** $L\left(\sigma^l, \sigma^k\right)$.

**Definition 4.5.** The **internal angle** of $\sigma^l$ with $\sigma^k$ is

$$\left(\sigma^l, \sigma^k\right) = \frac{Vol\left(L\left(\sigma^l, \sigma^k\right)\right)}{Vol\left(S^{k-l}\right)}.$$

Now, look at the set of all vectors making an angle of more than $\frac{\pi}{2}$ with $C^{\perp}\left(\sigma^l, \sigma^k\right)$. If we intersect this with $S^{k-l}$, we get a different sector $L^*\left(\sigma^l, \sigma^k\right)$.

**Definition 4.6.** The **exterior angle** of $\sigma^l$ with $\sigma^k$ is

$$\left(\sigma^l, \sigma^k\right)^* = \frac{Vol\left(L^*\left(\sigma^l, \sigma^k\right)\right)}{Vol\left(S^{k-l}\right)}.$$

Note that

$$\sum_{\sigma^l \in \sigma^k} \left(\sigma^l, \sigma^k\right)^* = 1$$

and that

$$\left(\sigma^l, \sigma^{l+2}\right) = 1 - \left(\sigma^l, \sigma^{l+2}\right)^*.$$

Finally, for future reference, we define

$$C^{\perp}\left(\sigma^l\right) = \amalg_{\sigma^n \supset \sigma^l} C^{\perp}\left(\sigma^l, \sigma^n\right),$$

where $dim K = n$, and $dim C^{\perp}\left(\sigma^l\right) = n - l$.

*Example.* I will illustrate the formulas for an internal angle in two dimensions. Let $v_i$ be a vertex incident on a triangle $\triangle_j$. The edges of $\triangle_j$ incident on $v_i$ form a basis for $\mathbb{R}^2$. The set of vectors pointing into $\triangle_j$ from $v$ forms a cone $C^{\perp}(v_i, \triangle_j)$ consisting of rays beginning at $v_i$ through points of $\triangle_j$. This cone intersects $S^1(v_i)$, the unit circle centered at $v_i$, in an arc $L(v_i, \triangle_j)$. Then, it is well known that the angle made by $\triangle_j$ at $v_i$ is

$$\alpha_i^j = \frac{l(L(v_i, \triangle_j))}{\pi} = \frac{l(L(v_i, \triangle_j))}{l(S^1(v_i))}.$$

□

## 5. Scalar Curvature on Simplicial Complexes

The Gauss-Bonnet theorem tells us there is a relationship between the angles of our triangulation and the curvature, but what is the precise relationship we are looking for? This is easiest to see in two dimensions. Imagine a surface $S$ and an associated piecewise flat approximation $T$. By construction, the curvature is zero at any point in the interior of a triangle of $T$. Likewise, the curvature is zero at any point in the interior of an edge of $T$. Indeed, the only places that the curvature can be nonzero are at the vertices of $T$. How shall we define the curvature at vertices? Well, the only information we have about a vertex $v$ are the angles that adjacent traingles make there. If our surface is flat at $v$, the sum over the triangles adjacent to $v$ of the interior angles will be $2\pi$. If $v$ is a point of positive curvature for our smooth surface, it is easy to see that sum of these angles for our approximating simplicial complex will be less than $2\pi$, while if $v$ is a point of negative curvature for our smooth surface, the sum of these angles will be larger than $2\pi$. Thus, for simplicial complexes of dimension two, we choose our measure of curvature to be a multiple of $2\pi - \sum_j \alpha^j$, where $j$ indexes the triangles incident to $v$, and $\alpha^j$ indicates the interior angle each triangle makes at $v$. No matter the dimension of our manifold, the fact that the curvature tensor is determined by the sectional curvature tells us that curvature somehow lives on submanifolds of dimension 2. This suggests that in higher dimensions, we calculate the angle defect around simplices of codimension two, or bones.

A more rigorous argument is the following. Scalar curvature for a simplicial complex should scale the same way with length as classical scalar curvature does, so if length is multiplied everywhere by $c$, then $R^2_\eta \to c^{n-2} R^2_\eta$. The easiest way to obtain this is to include the volume of $n-2$ dimensional simplices in our simplicial scalar curvature measure.

Now, if $p \in \sigma^{n-2}$, then some neighborhood of $p$ looks like the product $U \times C^\perp\left(\sigma^{n-2}\right)$, where $U \subset \sigma^{n-2}$ and the normal cone $C^\perp\left(\sigma^{n-2}\right)$ has dimension 2. Now, $U$ is flat, being an open subset of a totally geodesic submanifold, but the volume of $\sigma^{n-2}$ scales in the correct way. If we want $R^2_\eta$ to be a locally computable invariant which scales correctly, then **??** claims that we must have something like

$$R^2_\eta\left(\sigma^{n-2}\right) = \phi\left(C^\perp\left(\sigma^{n-2}\right)\right) Vol\left(\sigma^{n-2}\right),$$

where $\phi$ is a locally computable invariant of $C^\perp\left(\sigma^{n-2}\right)$. Compare this with the smooth scalar curvature measure on a metric product between flat space and a manifold $M^2$,

$$R^2(M^2 \times E^{n-2}) = P_\chi(M^2) \times \omega_{E^{n-2}},$$

where $P_\chi = Pfaff(\Omega)$ is the Chern-Gauss-Bonnet form, for the symmetric matrix $\Omega$ whose entries are the curvature two-forms. Thus, the function $\phi$ that we seek is the analog of $P_\chi$ for the 2-dimensional piecewise-flat cone $C^\perp\left(\sigma^{n-2}\right)$.

In the smooth case, the Chern-Gauss-Bonnet Theorem tells us that

$$P_\chi(M) = \int_M Pfaff(\Omega) = 2^n \pi^n \chi(M),$$

where $\chi(M)$ is the Euler characteristic of $M$. If $M$ has dimension $n$ and is triangulated by a simplicial complex with simplices $\{\sigma^i\}$ having dimension $i$, then

$$\chi(M) = \sum_{i=1}^{n} (-1)^i \# \left(\sigma^i\right).$$

For any simplex $\sigma^i$, the sum of its exterior angles is 1:

$$\sum_{\sigma^0 \in \sigma^i} \left(\sigma^0, \sigma^i\right)^* = 1.$$

Thus, we can write $\chi(M)$ as

$$\begin{aligned}\chi(M) &= \sum_{i=1}^{n}(-1)^i \sum_{\sigma^i\in M}\sum_{\sigma^0\in\sigma^i}\left(\sigma^0,\sigma^i\right)^* \\ &= \sum_{\sigma^i\in M}\sum_{\sigma^0\in\sigma^i}(-1)^i\left(\sigma^0,\sigma^i\right)^* \\ &= \sum_{\sigma^0\in M}\sum_{\sigma^i\ni\sigma^0}(-1)^i\left(\sigma^0,\sigma^i\right)^*\end{aligned}$$

where the summation is understood to be over all possible $i$. Likewise, we have

$$\chi\left(C^\perp\left(\sigma^l,\sigma^k\right)\right) = \sum_{i=l+1}^{k}\sum_{\sigma^i\ni\sigma^l}(-1)^{i-l}\left(\sigma^l,\sigma^i\right)^*,$$

since we can think of the normal cone as the full space "modded out" by the $l$ dimensions of $\sigma^l$; the $\sigma^l$ becomes a vertex, the faces $\sigma^i$ become like faces of dimension $i-l$, and so on. This is the relation we take to define our scalar curvature measure. That is, we let

$$\phi\left(C^\perp\left(\sigma^l,\sigma^k\right)\right) = \sum_{i=l+1}^{k}\sum_{\sigma^i\ni\sigma^l}(-1)^{i-l}\left(\sigma^l,\sigma^i\right)^*.$$

Then, we have

$$\begin{aligned}R^2_\eta\left(\sigma^{n-2}\right) &= \sum_{\sigma^i\ni\sigma^{n-2}}(-1)^{i-n-2}\left(\sigma^{n-2},\sigma^i\right)^* Vol\left(\sigma^{n-2}\right) \\ &= \left[\sum_{\sigma^n\ni\sigma^{n-2}}(\sigma^{n-2},\sigma^n)^* - \sum_{\sigma^{n-1}\ni\sigma^{n-2}}(\sigma^{n-2},\sigma^{n-1})^*\right] Vol\left(\sigma^{n-2}\right) \\ &= \sum_{\sigma^n\ni\sigma^{n-2}}(\sigma^{n-2},\sigma^n)^* Vol\left(\sigma^{n-2}\right).\end{aligned}$$

In other words, the piecewise-linear scalar curvature measure is the sum of the exterior angles that $\sigma^{n-2}$ makes in $\sigma^n$, weighted by the volume $\sigma^{n-2}$. We'd like to express this in terms of *interior* angles, which is easy enough since

$$(\sigma^{n-2},\sigma^n)^* = 1-(\sigma^{n-2},\sigma^n).$$

Thus,

$$R^2_\eta\left(\sigma^{n-2}\right) = \sum_{\sigma^n\ni\sigma^{n-2}}\left[1-(\sigma^{n-2},\sigma^n)\right] Vol\left(\sigma^{n-2}\right).$$

To find $R^2_\eta$ on an open neighborhood $U$, we merely sum over $n-2$-dimensional faces:

$$R^2_\eta(U) = \sum_{\sigma^{n-2}\in U}\sum_{\sigma^n\ni\sigma^{n-2}}\left[1-(\sigma^{n-2},\sigma^n)\right] Vol\left(\sigma^{n-2}\right).$$

## 6. Convergence In Dimension 2

In dimension 2, I have been able to work out the details of the proof that our simplicial scalar curvature measure converges to the smooth scalar curvature measure. I follow the argument in [2], which relies only on local calculations. This makes the proof considerably simpler than the $n$-dimensional case.

Also in dimension 2, we have the Gauss-Bonnet theorem, which motivates us to define a second simplicial scalar curvature measure. We will see that it is an easy consequence of the Gauss-Bonnet theorem that this Gauss-Bonnet measure converges to the smooth scalar curvature measure, but this measure has less practical use than the one defined above, which we call the angle-defect measure.

### 6.1. Scalar Curvature Measures.

We deal with a Riemannian 2-manifold $M$ and a geodesic triangulation $T$ of $M$. The vertices of $T$ are the set $\{v_i\}$, while the triangles of $T$ are the set $\{\triangle_j\}$. Also, we redefine the interior angle $\alpha_i^j$ that $\triangle_j$ makes at $v_i$ to be unnormalized, so that if $M$ is flat at $v_i$,

$$\sum_{\triangle_j \ni v_i} \alpha_i^j = 2\pi,$$

rather than 1.

We must compare our simplicial scalar curvature measures to another measure. We can transform the scalar curvature into a measure simply by integrating it over a measurable set.

**Definition 6.1.** The scalar curvature measure for a measurable set $U$ on a Riemannian 2-manifold is

$$R^2(U) := \int_U R^2(x) dA,$$

where $dA$ is the Riemannian area element, and $R^2(x)$ is the scalar curvature function.

We rescale the simplicial scalar curvature measure defined in section 4 above to obtain the angle-defect measure.

**Definition 6.2.** The angle-defect scalar curvature measure for a measurable set $U$ on a Riemannian 2-manifold with geodesic triangulation $T$ is

$$R_\eta^2(U) := \sum_{v_i \in T \cap U} R_\eta^2(v_i),$$

where

$$R_\eta^2(v_i) := 4\pi - 2 \sum_{\triangle_j \ni v_i} \tilde{\alpha}_i^j,$$

where $\tilde{\alpha}_i^j$ is the angle at $v_i$ of the Euclidean triangle with the same edge lengths as $\triangle_j$.

We also have another scalar curvature measure, motivated by the Gauss-Bonnet Formula.

**Definition 6.3.** The Gauss-Bonnet scalar curvature measure for a measurable set $U$ on a Riemannian 2-manifold with geodesic triangulation $T$ is

$$\hat{R}^2_\eta(U) := \sum_{\triangle_j \in T\cap U} \hat{R}^2_\eta(\triangle_j),$$

where

$$\hat{R}^2_\eta(\triangle_j) := 2\sum_{v_i\in\triangle_j} \alpha_i^j - 2\pi,$$

where $\alpha_i^j$ is the angle of the geodesic triangle at vertex $v_i$.

The qualitative difference between these two measures is the following. In the angle-defect measure, we sum the differences between Euclidean and geodesic angles around each vertex; the sum of the geodesic angles around a vertex is always $2\pi$. In the Gauss-Bonnet measure, we sum the differences between Euclidean and geodesic angles for all the vertices in each triangle; the sum of the interior angles of a Euclidean triangle is always $\pi$. Only the angle-defect measure can be generalized to more general simplicial approximations and higher dimensions.

6.2. **The Gauss-Bonnet Measure.** The proof that the Gauss-Bonnet measure converges to the scalar curvature measure is a fairly straightforward application of the Gauss-Bonnet Formula.

**Theorem 6.4.** *There exist $c = c(\|R\|)$ and $\eta_0 = \eta_0(\|R\|)$, so that*

$$\left|\hat{R}^2_\eta(U) - R^2(U)\right| \le cA(\partial_\eta U))$$

*whenever $\eta < \eta_0$. $A(\partial_\eta U)$ is the Riemannian area of the set of points a geodesic distance less than $\eta$ from $\partial U$.*

*Proof.* For each $\triangle_j \in U\cap T$, the Gauss-Bonnet Formula gives us

$$2\pi - \sum_{i=1}^{3} \epsilon_i^j = \int_{\triangle_j} K dA,$$

where $\epsilon_i^j$ is the exterior angle of $\triangle_j$ at vertex $v_i$ and $K$ is the sectional curvature. Now, $\epsilon_i^j = \pi - \alpha_i^j$, so we have

$$\sum_{i=1}^{3}\alpha_i^j - \pi = \frac{1}{2}\int_{\triangle_j} R^2 dA \qquad \Longrightarrow \qquad 2\sum_{i=1}^{3}\alpha_i^j - 2\pi = \int_{\triangle_j} R^2 dA.$$

Summing over the triangles in $U$, we have

$$\begin{aligned}\hat{R}^2_\eta(U) &= R^2\left(\bigcup_{\triangle_j\in U}\triangle_j\right)\\ &= R^2(U) - R^2\left(U\setminus\bigcup_{\triangle_j\in U}\triangle_j\right)\\ &\le R^2(U) - c(\|R\|)A(\partial_\eta(U)),\end{aligned}$$

which completes our proof. ∎

6.3. **The Angle-Defect Measure.** Proving that the angle-defect measure converges to the scalar curvature measure is quite a bit more technical. The crucial step in the proof is estimating the angles of the geodesic triangulation with the corresponding angles of the piecewise linear triangulation - in other words, with the corresponding angles of Euclidean triangles with the same side lengths. The error we incur includes a function of the scalar curvature.

The Euclidean and constant-curvature laws of cosines give the value of an angle in terms of the lengths of the sides of a triangle, and they will be our main weapon. We'd like to wield them on 2-manifolds of nonconstant curvature; luckily, they hold in this case, but only up to third order. In dealing with laws of cosines, we will treat only the case of positive sectional curvature; the case of negative sectional curvature is similar.

Once we have expressed the difference between geodesic and Euclidean angles as a function of the scalar curvature plus an error term, we sum this difference for each of the triangles incident on a vertex. In this way we obtain the angle defect of the vertex as a function of the scalar curvature at the vertex plus an error term. By summing over the vertices in $U$, we obtain a relation between $R^2_\eta(U)$ and a sum of scalar curvatures over the vertices in $U$. Our final step is to relate this sum of scalar curvatures to the scalar curvature measure; we take care of this in a lemma.

6.3.1. *Preliminary Lemmas.*

**Lemma 6.5.** *Let $\triangle$ be a geodesic triangle on a Riemannian 2-manifold $M$ of nonconstant but bounded curvature. Let the covariant derivative of the curvature tensor be bounded, and let $\|R\|$, $\|\nabla R\|$ indicate the suprema of the operator norms of these tensors. Let $d$ be the diameter of the smallest geodesic ball containing $\triangle$, and assume that no conjugate points lie within this geodesic ball. Then the hyperbolic or spherical law of cosines holds for $T$ up to $O\left(d^3\right)$. More precisely,*

$$\cos(\alpha_i) = \frac{\cos(\sqrt{K}l_i) - \cos(\sqrt{K}l_j)\cos(\sqrt{K}l_k)}{\sin(\sqrt{K}l_j)\sin(\sqrt{K}l_k)} + O\left(d^3\right),$$

*where $\alpha_i$ is the angle at the $i^{th}$ vertex $v_i$, $l_m$ is the length of the side opposite $v_m$, $K$ is the scalar curvature at $v_i$, and the constant in $O\left(d^3\right)$ depends on $\|\nabla R\|$.*

*Proof.* Let $\gamma_1, \gamma_2, \gamma_3$ be the sides of $\triangle$, having lengths $l_1, l_2, l_3$. Let $x = (x^i)$ be normal coordinates centered at $v_1$, and let $K$ be the sectional curvature at $v_1$. Now consider a surface $S$ of constant sectional curvature $K$ with constant-curvature metric $g_K$ which is "tangent" to $M$ at $v_1$, in the following sense. Choose a point $v_1^K \in S$, and choose normal coordinates centered at $v_1^K$. These normal coordinates give an orthonormal basis for $T_{v_1^K}S$, just as the coordinate $x$ gives an orthonormal basis for $T_{v_1}M$. By identifying the two bases, we obtain a linear isomorphism which allows us to identify the two tangent spaces; we also identify $v_1$ with $v_1^K$.

Now, let $\gamma_2^K$ and $\gamma_3^K$ be the $g_K$-geodesics of length $l_2$ and $l_3$, respectively, starting at $v_1$ and heading in the direction of $\dot\gamma_2$ and $\dot\gamma_3$. Let $\gamma_1^K$ be the geodesic connecting the endpoint of $\gamma_2^K$ to the endpoint of $\gamma_3^K$. Let $\triangle^K$ be the triangle whose sides are $\gamma_1^K, \gamma_2^K, \gamma_3^K$, with respective lengths $l_1^K, l_2^K, l_3^K$.

Then $\triangle$ and $\triangle^K$ are both geodesic triangles having angle $\alpha_1$ at $v_1$. By construction $l_3 = l_3^K$ and $l_2 = l_2^K$.

We can apply the constant-curvature law of cosines to $\triangle^K$, and we'd like to apply it (up to some error) to $\triangle$. One way to do this is to replace $l_1^K$ in the constant-curvature cosine law by $l_1$, but in order to do this we must compare $l_1$ with $l_1^K$. In turn, we can only do this by pulling back the two geodesic triangles into the tangent space at $v_1$.

Since we've identified the normal coordinates for the two manifolds, $\hat{\gamma}_2 = \exp_g^{-1}(\gamma_2) = \exp_{g_K}^{-1}(\gamma_2^K)$, and likewise for $\hat{\gamma}_3$; both are straight lines. However, $\hat{\gamma}_1 = \exp_g^{-1}(\gamma_1)$ and $\hat{\gamma}_1^K = \exp_{g_K}^{-1}(\gamma_1^K)$ are not straight lines, and are not the same. Luckily they will have the same endpoints, $\hat{v}_2 = \exp^{-1}(v_2)$ and $\hat{v}_3 = \exp^{-1}(v_3)$. We want to compute

$$\begin{aligned} l_1 = l(\hat{\gamma}_1) &= \int_{\hat{\gamma}_1} \sqrt{\exp_{v_1}^* g(\dot{\hat{\gamma}}_1(t), \dot{\hat{\gamma}}_1(t))}dt, \\ l_1^K = l(\hat{\gamma}_1^K) &= \int_{\hat{\gamma}_1^K} \sqrt{\exp_{v_1}^* g_K(\dot{\hat{\gamma}}_1^K(t), \dot{\hat{\gamma}}_1^K(t))}dt, \end{aligned}$$

where the exponential maps are those generated by the respective metrics. But

$$\exp_{v_1}^* g(\dot{\hat{\gamma}}_1(t), \dot{\hat{\gamma}}_1(t)) = g((\exp_{v_1})_*(\dot{\hat{\gamma}}_1(t)), (\exp_{v_1})_*(\dot{\hat{\gamma}}_1(t))),$$

and

$$\begin{aligned} (\exp_{v_1})_*(\dot{\hat{\gamma}}_1(t)) &= (\exp_{v_1})_* \frac{d}{ds}(\exp_{v_1}^{-1} \gamma_1)|_{s=t}(s) \\ &= (\exp_{v_1})_*(\exp_{v_1}^{-1})_* \dot{\gamma}_1(t) \\ &= (\exp_{v_1} \exp_{v_1}^{-1})_* \dot{\gamma}_1(t) \\ &= \dot{\gamma}_1(t), \end{aligned}$$

so that

$$\exp_{v_1}^* g(\dot{\hat{\gamma}}_1(t), \dot{\hat{\gamma}}_1(t)) = g(\dot{\gamma}_1(t), \dot{\gamma}_1(t)).$$

The same is true for the second integrand, where the only dependence on $g_K$ is in the exponential map. Thus we have that

$$\begin{aligned} l_1 &= \int_{\hat{\gamma}_1} \sqrt{g(\dot{\gamma}_1(t), \dot{\gamma}_1(t))}dt, \\ l_1^K &= \int_{\hat{\gamma}_1^K} \sqrt{g_K(\dot{\gamma}_1^K(t), \dot{\gamma}_1^K(t))}dt, \end{aligned}$$

and we can remove the discrepancy in the "limits" of integration by reparameterizing both $\hat{\gamma}_1$ and $\hat{\gamma}_1^K$ so that they are images of the unit interval. Since we are computing length, this rescaling does not affect the value of the integral. Now, $\hat{\gamma}_1$ is a minimizing geodesic for $\exp^* g$, and $\hat{\gamma}_1^K$ is a minimizing geodesic for $\exp^* g_K$. Thus we have the following inequalities:

$$\begin{aligned} \int_{[0,1]} \sqrt{g(\dot{\gamma}_1(t), \dot{\gamma}_1(t))}dt &\leq \int_{[0,1]} \sqrt{g(\dot{\gamma}_1^K(t), \dot{\gamma}_1^K(t))}dt, \\ \int_{[0,1]} \sqrt{g_K(\dot{\gamma}_1^K(t), \dot{\gamma}_1^K(t))}dt &\leq \int_{[0,1]} \sqrt{g_K(\dot{\gamma}_1(t), \dot{\gamma}_1(t))}dt. \end{aligned}$$

In other words,

$$|l_1 - l_1^K| \le \max\{\int_{[0,1]} \Big| \sqrt{g(\dot\gamma_1(t), \dot\gamma_1(t))} - \sqrt{g_K(\dot\gamma_1(t), \dot\gamma_1(t))} \Big| dt, \\ \int_{[0,1]} \Big| \sqrt{g(\dot\gamma_1^K(t), \dot\gamma_1^K(t))} - \sqrt{g_K(\dot\gamma_1^K(t), \dot\gamma_1^K(t))} \Big| dt\}$$

We will assume the first integral is larger for definiteness; all the arguments are analogous if the second is larger.

We proceed by using Taylor expansions to simplify the integrand. Both $g$ and $g_K$ can be Taylor expanded in terms of $x$:

$$g = \delta_{ij} - \frac{1}{3} R_{iklj}(0) x^k x^l + \nabla_m R_{iklj}(0) x^k x^l x^m + O(|x|^4),$$
$$g_K = \delta_{ij} - \frac{1}{3} (R_K)_{iklj}(0) x^k x^l + \nabla_m (R_K)_{iklj}(0) x^k x^l x^m + O(|x|^4).$$

Let $x = \gamma_1(t)$, and $v = \dot\gamma_1(t) \in T_{\gamma_1(t)} M$. Since we are in normal coordinates, the vector $exp_{v_1}^{-1}(x) \in T_{v_1} M$ has the same coordinates as $x$; we will refer to it as $x$ also. Then,

$$g(v,v) = \|v\|_E^2 - \frac{1}{3} R(0)(v,x,x,v) + \frac{1}{6} \nabla_x R(0)(v,x,x,v) + O(|x|^4 |v|^2),$$
$$g_K(v,v) = \|v\|_E^2 - \frac{1}{3} R_K(0)(v,x,x,v) + \frac{1}{6} \nabla_x^K R_K(0)(v,x,x,v) + O(|x|^4 |v|^2),$$

where $\|v\|_E$ is the Euclidean norm of $v$. Taylor expanding the square root around $\|v\|_E^2$, we obtain

$$\int_{[0,1]} \Big| \sqrt{g(\dot\gamma_1(t), \dot\gamma_1(t))} - \sqrt{g_K(\dot\gamma_1(t), \dot\gamma_1(t))} \Big| dt$$
$$= \int_{[0,1]} \Big| \frac{1}{6\|v\|_E} \big[ R(0)(v,x,x,v) - R_K(0)(v,x,x,v) \\ - \frac{1}{2} \nabla_x R(0)(v,x,x,v) + \frac{1}{2} \nabla_x^K R_K(0)(v,x,x,v) + O(|x|^4) \big] \Big| dt.$$

Now, by uniqueness of geodesics, $v$ is not parallel to the radial vector $\frac{\partial}{\partial r}$ at $x$, so the components of $v$ and $x$ are linearly independent as vectors in $\mathbb{R}^2$. Since we are on a 2-dimensional manifold, the sectional curvature of any two linearly independent vectors in $T_{v_1} M$ is the same. Thus,

$$R(0)(v,x,x,v) = K \left( g(0)(x,x) g(0)(v,v) - g(0)(x,v)^2 \right) \\ = K \left( \|x\|_E^2 \|v\|_E^2 - g_E(x,v)^2 \right),$$
$$R_K(0)(v,x,x,v) = K \left( g_K(0)(x,x) g_K(0)(v,v) - g_K(0)(x,v)^2 \right) \\ = K \left( \|x\|_E^2 \|v\|_E^2 - g_E(x,v)^2 \right),$$

where $g_E$ is the flat metric. As a result,

$$R(0)(v,x,x,v) - R_K(0)(v,x,x,v) = 0,$$

as we might expect, and

$$|l_1 - l_1^K| \leq \int_{[0,1]} \left| \frac{1}{12\|v\|_E} \left[\nabla_x R(0)(v,x,x,v) - \nabla_x^K R_K(0)(v,x,x,v)\right] \right| + \int_{[0,1]} O(|x|^4|v|).$$

But,

$$\begin{aligned} &|\nabla_x R(0)(v,x,x,v) - \nabla_x^K R_K(0)(v,x,x,v)| \\ &\quad \leq |\nabla_x R(0)(v,x,x,v)| + |\nabla_x^K R_K(0)(v,x,x,v)| \\ &\quad \leq \|\nabla R\| \|x\|_E^3 \|v\|_E^2, \end{aligned}$$

since $\nabla^K R_K = 0$. Thus,

$$\begin{aligned} |l_1 - l_1^K| &\leq \int_{[0,1]} \left[ \frac{\|\nabla R\| \|x\|_E^3 \|v\|_E^2}{12\|v\|_E} + O(|x|^4|v|) \right] \\ &\leq c(\|\nabla R\|) d^4, \end{aligned}$$

since, thanks to our choice of parametrization, $\|v\|_E = l_1$.

Now, let $\alpha_1^K$ be the angle of the constant-curvature triangle at $v_1$. By construction, $\alpha_1^K = \alpha_1$. Let $l_1^K$, $l_2^K$, and $l_3^K$ be the constant-curvature lengths of the sides of the triangle. If $K > 0$, then, by the spherical law of cosines,

$$\begin{aligned} \cos(\alpha_1) = \cos(\alpha_1^K) &= \frac{\cos(\sqrt{K} l_1^K) - \cos(\sqrt{K} l_2^K)\cos(\sqrt{K} l_3^K)}{\sin(\sqrt{K} l_2^K)\sin(\sqrt{K} l_3^K)} \\ &= \frac{\cos(\sqrt{K}(l_1 + O(d^4))) - \cos(\sqrt{K} l_2)\cos(\sqrt{K} l_3)}{\sin(\sqrt{K} l_2)\sin(\sqrt{K} l_3)}. \end{aligned}$$

By Taylor expanding cosine, we can see that

$$\begin{aligned} \cos(\sqrt{K} l_1 + O(\sqrt{K} d^4)) &= 1 - \frac{1}{2}(\sqrt{K} l_1 + O(\sqrt{K} d^4))^2 \\ &\quad + \frac{1}{24}(\sqrt{K} l_1 + O(\sqrt{K} d^4))^4 + \dots \\ &= 1 - \frac{1}{2}(\sqrt{K} l_1)^2 + \frac{1}{24}(\sqrt{K} l_1)^4 + \dots + O(K d^5) \\ &= \cos(\sqrt{K} l_1) + O(K d^5). \end{aligned}$$

Substituting this into the above expression, and recalling that the constant in $O(Kd^5)$ depends on $\|\nabla R\|$, we obtain our result. □

**Proposition 6.6.** *Let $\triangle$ be a geodesic triangle on a Riemannian manifold of nonconstant but bounded curvature. Let the covariant derivative of the curvature tensor be bounded, and let $d$ be the diameter of the smallest geodesic ball containing $\triangle$. Then the angles of $\triangle$ can be expanded in terms of the angles of a Euclidean triangle $\hat{\triangle}$ with the same edge lengths. In particular,*

$$\alpha_i = \alpha_E + \frac{K}{3} A_E + O(d^3),$$

*where $\alpha_i$ is the angle of $\triangle$ at $v_i$, $\alpha_E$ is the corresponding angle of $\hat{\triangle}$, and $A_E$ is the area of $\hat{\triangle}$.*

*Proof.* Let $l_1, l_2$, and $l_3$ be the lengths of the sides of $\triangle$, with $l_i$ being opposite vertex $v_i$. Our strategy is to apply the nonconstant curvature law of cosines from Lemma 6.5, to Taylor expand this expression, and finally to simplify the resulting expression using the Euclidean law of cosines:

$$\begin{aligned}\cos(\alpha_E) &= \frac{(l_2)^2 + (l_3)^2 - (l_1)^2}{2l_2l_3},\\ \cos^2(\alpha_E) &= \frac{(l_1)^4 + (l_2)^4 + (l_3)^4 + 2(l_2)^2(l_3)^2 - 2(l_1)^2\left[(l_2)^2 + (l_3)^2\right]}{4(l_2l_3)^2}.\end{aligned}$$

Let $s_i = \sqrt(K)l_i$. Then, by Lemma 6.5,

$$\cos(\alpha_1) = [\cos s_1 - \cos s_2 \cos s_3] \, / \, [\sin s_2 \sin s_3] + O(d^3).$$

Taylor expanding, we find the numerator of the first term to be

$$\begin{aligned}&\left[1 - \frac{(s_1)^2}{2} + \frac{(s_1)^4}{24} + O((s_1)^6)\right]\\ &\quad - \left[1 - \frac{(s_2)^2}{2} + \frac{(s_2)^4}{24} + O((s_2)^6)\right]\left[1 - \frac{(s_3)^2}{2} + \frac{(s_3)^4}{24} + O((s_3)^6)\right]\\ &= 1 - \frac{(s_1)^2}{2} + \frac{(s_1)^4}{24}\\ &\quad - \left[1 - \frac{(s_2)^2}{2} - \frac{(s_3)^2}{2} + \frac{(s_2)^2(s_3)^2}{4} + \frac{(s_2)^4}{24} + \frac{(s_3)^4}{24}\right] + O(K^3d^6)\\ &= -\frac{1}{2}\left[(s_1)^2 - (s_2)^2 - (s_3)^2\right] - \frac{1}{4}(s_2s_3)^2\\ &\quad + \frac{1}{24}\left[(s_1)^4 - (s_2)^4 - (s_3)^4\right] + O(K^3d^6).\end{aligned}$$

The denominator looks like

$$\begin{aligned}&\left[s_2 - \frac{(s_2)^3}{6} + O((s_2)^5)\right]\left[s_3 - \frac{(s_3)^3}{6} + O((s_3)^5)\right]\\ &= s_2s_3 - \frac{s_2(s_3)^3 + s_3(s_2)^3}{6} + O(K^3d^6)\\ &= s_2s_3\left[1 - \frac{1}{6}\left[(s_2)^2 + (s_3)^2\right] + O(K^2d^4)\right].\end{aligned}$$

Expanding the denominator using the Taylor expansion for $(1-x)^{-1}$, we get

$$\begin{aligned}
\cos(\alpha_1) &= -\frac{1}{s_2 s_3} \\
&\quad \times \left[\frac{1}{2}\left[(s_1)^2-(s_2)^2-(s_3)^2\right]+\frac{1}{4}(s_2s_3)^2-\frac{1}{24}\left[(s_1)^4-(s_2)^4-(s_3)^4\right]+O(K^3d^6)\right] \\
&\quad \times \left[1+\frac{1}{6}\left[(s_2)^2+(s_3)^2\right]+O(Kd^4)\right]+O(d^3) \\
&= \frac{(s_2)^2+(s_3)^2-(s_1)^2}{2s_2s_3}-\frac{(s_2s_3)^2}{4s_2s_3}+\frac{(s_1)^4-(s_2)^4-(s_3)^4}{24s_2s_3} \\
&\quad -\frac{\left[(s_1)^2-(s_2)^2-(s_3)^2\right]\left[(s_2)^2+(s_3)^2\right]}{12s_2s_3}+O(Kd^4) \\
&= \cos(\alpha_E)+\frac{(s_1)^4-(s_2)^4-(s_3)^4}{24s_2s_3}-\frac{3(s_2s_3)^2}{12s_2s_3} \\
&\quad -\frac{\left[(s_1s_2)^2-(s_2)^4-(s_3s_2)^2+(s_1s_3)^2-(s_2s_3)^2-(s_3)^4\right]}{12s_2s_3}+O(d^3) \\
&= \cos(\alpha_E)+\frac{(s_1)^4-(s_2)^4-(s_3)^4}{24s_2s_3}+\frac{\left[(s_2)^4+(s_3)^4\right]}{12s_2s_3} \\
&\quad -\frac{\left[(s_1s_2)^2+(s_1s_3)^2+(s_2s_3)^2\right]}{12s_2s_3}+O(d^3) \\
&= cos(\alpha_E)+\frac{s_2s_3}{6}\left(\frac{(s_1)^2+(s_2)^2+(s_3)^2-2\{(s_1s_2)^2+(s_1s_3)^2+(s_2s_3)^2\}}{4(s_2s_3)^2}\right)+O(d^3) \\
&= cos(\alpha_E)+\frac{s_2s_3}{6}\left[cos^2(\alpha_E)-1\right]+O(d^3) \\
&= cos(\alpha_E)-\frac{s_2s_3}{6}sin^2(\alpha_E)+O(d^3).
\end{aligned}$$

Here $\alpha_E$ is the angle at $v_1$ of a Euclidean triangle with side lengths $s_1$, $s_2$, and $s_3$. Now we perform a final Taylor expansion, for $\cos^{-1}(x)$ around $\cos(\alpha_E)$. The expansion is

$$\cos^{-1}(x)=\cos^{-1}(a)-\frac{1}{\sqrt{1-a^2}}(x-a)+O(x-a)^2.$$

The result is that

$$\begin{aligned}
\alpha_1 &= \cos^{-1}(\cos(\alpha_1)) \\
&= \cos^{-1}(\cos(\alpha_E))-\frac{\left(-\frac{s_2s_3}{6}\right)\sin^2(\alpha_E)+O(d^3)}{\sqrt{1-\cos^2(\alpha_E)}}+O(K^2d^4) \\
&= \alpha_E+\frac{s_2s_3}{6}\sin(\alpha_E)+O(d^3) \\
&= \alpha_E+\frac{K}{3}\left(\frac{l_2l_3}{2}\right)\sin(\alpha_E)+O(d^3) \\
&= \alpha_E+\frac{K}{3}A_E+O(d^3)
\end{aligned}$$

where $A_E=\frac{l_2l_3}{2}\sin(\alpha_E)=O(Kd^2)$ is the area of the Euclidean triangle with side lengths $l_1$, $l_2$, and $l_3$. □

**Lemma 6.7.** *Let $U$ be a measurable set in a Riemannian 2-manifold. Let the curvature tensor and its covariant derivative be bounded on $U$, and let $T$ be a geodesic triangulation of $U$ with $\eta$ the mesh of $T$. Then,*

$$\left| R^2(U) - \frac{1}{3} \sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A(\triangle_j) \right| \leq c(\|R\|, \|\nabla R\|) \left[ A(U)\eta + A(\partial_\eta U) \right],$$

*where $v_i$ and $\triangle_j$ are the vertices and faces of $T$, $A$ denotes the Riemannian area measure, and $\partial_\eta U$ is the set of points a distance of less than $\eta$ from $\partial U$.*

*Proof.* This is a standard approximation of an integral by a finite sum.

By summing first over all the faces containing $v_i$, and then over all the vertices $v_i$ in $U$, we end up counting each face three times - once with each incident vertex. So we have

$$\sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A(\triangle_j) = \sum_{\triangle_j \in U} A(\triangle_j) \sum_{i=1}^{3} R^2(v_i^j),$$

where $v_i^j$ is the $i$th vertex of $\triangle_j$. Now, choosing normal coordinates at $v_1$, say, we can expand $R$ as

$$R^2(x) = R^2(v_1) + \nabla R^2(v_1)(x) + O(|x|^2).$$

Then, if $\triangle_j \ni v_1$ has diameter $d$,

$$\begin{aligned}
&R^2(v_1^j) + R^2(v_2^j) + R^2(v_3^j) \\
&= R^2(v_1^j) + \Big( R^2(v_1^j) + \nabla R^2(0)(v_2^j - v_1^j) + O(|v_2^j - v_1^j|^2) \Big) \\
&\quad + \Big( R^2(v_1^j) + \nabla R^2(0)(v_3^j - v_1^j) + O(|v_3^j - v_1^j|^2) \Big) \\
&= 3R^2(v_1^j) + \nabla R^2(0)(v_2^j - v_1^j) + \nabla R^2(0)(v_3^j - v_1^j) + O(d^2) \\
&\leq 3R^2(v_1^j) + 2\|\nabla R\| d + O(d^2).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{3} \sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A(\triangle_j) &\leq \sum_{\triangle_j \in U} A(\triangle_j) \left[ R^2(v_1^j) + \frac{2}{3} \|\nabla R\| \eta + O(\eta^2) \right] \\
&= \sum_{\triangle_j \in U} R^2(v_1^j) A(\triangle_j) + \frac{2}{3} \|\nabla R\| A(U) \eta + O(\eta^2),
\end{aligned}$$

so that

$$\left| \frac{1}{3} \sum_{v_i \in U} R^2(v_i) \sum_{\triangle_j \ni v_i} A(\triangle_j) - \sum_{\triangle_j \in U} R^2(v_1^j) A(\triangle_j) \right| = O(\eta),$$

with the constant depending on $\|\nabla R\|$ and $A(U)$. Now, let's estimate the integral of $R^2$ over a single geodesic triangle, $\triangle$. Choosing normal coordinates at vertex $v_1$, we can expand $R^2$ as

$$R^2(x) = R^2(v_1) + \nabla R^2(v_1)(x) + O(|x|^2).$$

Now, integrating both sides of this equation over $\triangle$, we have

$$\begin{aligned}\int_{\triangle} R^2(x)dA &= \int_{\triangle} \left[R^2(v_1) + \nabla R^2(v_1)(x) + O(|x|^2)\right] dA \\ &= R^2(v_1)A(\triangle) + \int_{\triangle} \nabla R^2(v_1)(x)dA + A(\triangle)O(d^2) \\ \implies R^2\left(\triangle\right) - R^2(v_1)A(\triangle) &\le \|\nabla R^2\|A(\triangle)d + O(d^2),\end{aligned}$$

where the constant in the $O(d^2)$ depends on $A(\triangle)$. Summing over all the triangles in $U$, we have

$$\begin{aligned}&\sum_{\triangle_j\in U} R^2\left(\triangle_j\right) - \sum_{\triangle_j\in U} R^2(v_1^j)A(\triangle_j) \\ &= R^2\left(\bigcup_{\triangle_j\in U}\triangle_j\right) - \frac{1}{3}\sum_{v_i\in U} R^2(v_i)\sum_{\triangle_j\ni v_i} A(\triangle_j) + O(\eta) \\ &\le \sum_{\triangle_j\in U}\left[\|\nabla R\|A(\triangle_j)d_j + O(d_j^2)\right] \\ &\le \|\nabla R\|\eta\sum_{\triangle_j\in U} A(\triangle_j) + O(\eta^2) \\ &= \|\nabla R\|A\left(\bigcup_{\triangle_j\in U}\triangle_j\right)\eta + O(\eta^2). \\ &\le \|\nabla R\|A(U)\eta + O(\eta^2).\end{aligned}$$

Now, we also have

$$\begin{aligned}R^2(U) - R^2\left(\bigcup_{\triangle_j\in U}\triangle_j\right) &= R^2\left(U\setminus\bigcup_{\triangle_j\in U}\triangle_j\right) \\ &= \int_{U\setminus\bigcup_{\triangle_j\in U}\triangle_j} R^2(x)dA \\ &\le \|R\|A\left(U\setminus\bigcup_{\triangle_j\in U}\triangle_j\right) \\ &\le \|R\|A\left(\partial_\eta U\right).\end{aligned}$$

Combining the previous two estimates, we obtain our result. □

6.3.2. *Proof of the Main Theorem.*

**Theorem 6.8.** *Let $U$ be a measurable subset of a Riemannian 2-manifold. There exist $c = c(\|R\|, \|\nabla R\|)$ so that*

$$\left|R_\eta^2(U) - R^2(U)\right| \le c\left[A(U)\eta + A(\partial_\eta U)\right].$$

*$A(U)$ is the Riemannian area of $U$, and $\partial_\eta U$ is the set of points a geodesic distance less than $\eta$ from $\partial U$.*

*Proof.* The proof proceeds as follows. We sum the relation in Proposition 6.6 over all angles at a vertex and over all vertices inside of $U$, thus obtaining a relation between $R^2_\eta(U)$ and a finite sum of (smooth) scalar curvatures over points in $U$. Then, using Lemma 6.7, and a few other facts, we obtain a relation between this finite sum of scalar curvatures and $R^2(U)$. Combining the two estimates, and estimating the difference between the areas of geodesic and Euclidean triangles, we obtain our result.

Proposition 6.6 tells us

$$\alpha_i^j - \hat{\alpha}_i^j - \frac{1}{3}K(v_i)A_E(\triangle_j) = O(d^3),$$

where $\alpha_i^j$ is the geodesic angle of the $j$th triangle at vertex $v_i$, $\hat{\alpha}_i^j$ is the corresponding angle of the Euclidean triangle with the same edge lengths, $A_E(\triangle_j)$ is the area of that triangle, $d$ is the diameter of the geodesic triangle, and $K$ is the sectional curvature of our manifold at $v_i$. Multiplying by 2 and adding up over all the triangles containing $v_i$, we have

$$\begin{aligned} &4\pi - 2\sum_{\triangle_j \ni v_i} \hat{\alpha}_i^j - \frac{2}{3}K(v_i)\sum_{\triangle_j \ni v_i} A_E(\triangle_j) \\ &= R^2_\eta(v_i) - \frac{1}{3}R^2(v_i)\sum_{\triangle_j \ni v_i} A_E(\triangle_j) = O(d^3), \end{aligned}$$

since $K(v_i) = \frac{1}{2}R^2(v_i)$. Now, summing over all the vertices in $U$ we have

$$\begin{aligned} &\sum_{v_i \in U} R^2_\eta(v_i) - \sum_{v_i \in U}\left[\frac{1}{3}R^2(v_i)\sum_{\triangle_j \ni v_i} A_E(\triangle_j)\right] \\ &= R^2_\eta(U) - \frac{1}{3}\sum_{v_i \in U} R^2(v_i)\sum_{\triangle_j \ni v_i} A_E(\triangle_j) = O(\eta^3). \end{aligned}$$

Now, we find

$$\begin{aligned} &\left|R^2_\eta(U) - R^2(U)\right| \\ &\leq \left|R^2_\eta(U) - \frac{1}{3}\sum_{v_i \in U} R^2(v_i)\sum_{\triangle_j \ni v_i} A_E(\triangle_j)\right| \\ &\quad + \left|\frac{1}{3}\sum_{v_i \in U} R^2(v_i)\sum_{\triangle_j \ni v_i} A_E(\triangle_j) - \frac{1}{3}\sum_{v_i \in U} R^2(v_i)\sum_{\triangle_j \ni v_i} A(\triangle_j)\right| \\ &\quad + \left|\frac{1}{3}\sum_{v_i \in U} R^2(v_i)\sum_{\triangle_j \ni v_i} A(\triangle_j) - R^2(U)\right| \\ &\leq \left|\frac{1}{3}\sum_{v_i \in U} R^2(v_i)\sum_{\triangle_j \ni v_i} A_E(\triangle_j) - \frac{1}{3}\sum_{v_i \in U} R^2(v_i)\sum_{\triangle_j \ni v_i} A(\triangle_j)\right| \\ &\quad + c\left(\|R\|, \|\nabla R\|\right)\left[A(U)\eta + A\left(\partial_\eta U\right)\right] + O(\eta^3). \end{aligned}$$

To estimate the remaining term, we use the expansion of the Riemannian volume element in terms of the Ricci curvature, writing

$$\begin{aligned} A(\triangle) &= \int_\triangle dV = \int_\triangle \left[1 - \frac{1}{6}R_{jk}x^j x^k + O(|x|^3)\right] dV_E \\ &= A_E(\triangle) - \frac{1}{6}\int_\triangle R_{ij}x^i x^j dA + O(d^3) \\ &\le A_E(\triangle)\left[1 + O(d^2)\right], \end{aligned}$$

where $d$ is the diameter of the triangle, so that

$$A(\triangle) - A_E(\triangle) = A_E(\triangle)O(d^2).$$

Thus we have

$$\begin{aligned} &\left|\frac{1}{3}\sum_{v_i\in U} R^2(v_i)\sum_{\triangle_j\ni v_i} A(\triangle_j) - \frac{1}{3}\sum_{v_i\in U} R^2(v_i)\sum_{\triangle_j\ni v_i} A_E(\triangle_j)\right| \\ &= \left|\frac{1}{3}\sum_{v_i\in U} R^2(v_i)\sum_{\triangle_j\ni v_i}\left[A(\triangle_j) - A_E(\triangle_j)\right]\right| \\ &\le \frac{1}{3}\sum_{v_i\in U}\left|R^2(v_i)\right|\sum_{\triangle_j\ni v_i} A_E(\triangle_j)O\left(\eta^2\right) \\ &\le \|R\| A_E(U) O\left(\eta^2\right) = O\left(\eta^2\right). \end{aligned}$$

Combining this with our above results, we have

$$\begin{aligned} \left|R^2_\eta(U) - R^2(U)\right| &\le c(\|R\|, \|\nabla R\|)\left[A(U)\eta + A(\partial_\eta U)\right] + O\left(\eta^2\right) + O\left(\eta^3\right) \\ &\le c(\|R\|, \|\nabla R\|)\left[A(U)\eta + A(\partial_\eta U)\right] \end{aligned}$$

up to leading order, as desired. □

## 7. Convergence In Dimension $n$

The theorem is the same as in dimension 2.

**Theorem 7.1.** *Let $U$ be a measurable subset of a Riemannian manifold $(M, g)$, and let $R^2$ and $R^2_\eta$ be defined as in Section 4. There is a constant $c = c(\| R \|, \| \nabla R \|)$ so that*

$$|R^2_\eta(U) - R^2(U)| \le c\left[V(U)\sqrt{\eta} + V(\partial_{\sqrt{\eta}}U)\right],$$

*where $V$ is the Riemannian volume measure, and $\partial_{\sqrt{\eta}}U$ is the set of points a geodesic distance less than $\sqrt{\eta}$ from $\partial U$.*

Unfortunately, the local approach used in dimension 2 cannot be extended to higher dimensions. The main problem is that given a geodesic triangulation of a higher-dimensional manifold, the interior angle an $n-2$ -dimensional simplex $\sigma^{n-2}$ makes with an $n$-dimensional simplex $\sigma^n$ may vary along its length. Let $\sigma_1^{n-1}$ and $\sigma_2^{n-1}$ be the faces of $\sigma^n$ with

$$\sigma^{n-2} = \sigma_1^{n-1} \cap \sigma_2^{n-1}.$$

The angle between the vectors normal to $\sigma^{n-2}$ pointing into $\sigma_1^{n-1}$ and $\sigma_2^{n-1}$ may vary as much as $O\left(\eta^2\right)$ as we move along $\sigma^{n-2}$. This is an untenable situation.

Instead, we find that

$$R^2_\eta(B_r(p)) \sim L_\eta(p)(r, T_\eta)Vol(B_r(p)),$$

where $L_\eta$ is a linear expression in the curvature at $p$, some point in $U$, whose coefficients are uniformly bounded, regardless of triangulation. The proof of convergence has two parts.

First, it is a theorem of Gilkey ([4]) that any linear function of curvature is equal to $R^2$ if it has the following three properties:

(1) It is invariant under orthogonal transformations of the tangent space;
(2) $L(M^1 \times E^{n-1}) = 0$, where $M^1$ is a one-dimensional manifold;
(3) $L(S^2 \times E^{n-2}) = *R^2(S^2 \times E^{n-2})$.

Above, $E^i$ denotes Euclidean space of dimension $i$. In [2], it is shown that $L_\eta$ satisfies these three properties. The second property is built into $L_\eta$, while the final property is shown by choosing a special approximating triangulation.

Secondly, it is shown that $L_\eta(p)(r, T_\eta)$ is independent of the particular triangulation $T_\eta$ in the limit of small $\eta$. Part of the proof of this involves constructing appproximating triangulations as follows. Given two triangulations $T_1$ and $T_2$ of a bounded region, two other triangulations $T_3$ and $T_4$ can be created, which agree on the boundary, and agree with $T_1$ and $T_2$, respectively, in the interior up to high order.

These ideas are still not entirely clear to me. I have some parts of these arguments worked out, but for the sake of clarity I've chosen to leave the above bare-bones sketch. The rest of the $n$-dimensional case is something to be worked out in the future.

## References

[1] Cheeger, J., Müller, W. & Schrader, R. "Lattice gravity or Riemannian structure on piecewise linear spaces," pp. 176-188, *Unified Theories of Elementary Particles: Critical Assessment and Prospects: Proceedings of the Heisenberg Symposium, held in Munich, July 16-21, 1981.* Springer, Berlin (1981).

[2] Cheeger, J., Müller, W. & Schrader, R. "On the curvature of piecewise flat spaces." Communications in Mathematical Physics 92, pp. 405-454 (1984).

[3] Cheeger, J., Müller, W. & Schrader, R. "Kinematic and tube formulas for piecewise linear spaces." Indiana University Mathematics Journal 35, pp.737-754 (1986).

[4] Gilkey, P.B. *The Index Theorem and the Heat Equation.* Publish or Perish, Boston (1974).

[5] Fröhlich, J. "Regge calculus and discretized gravitational integrals," pp. 523-545, *Non-perturbative Quantum Field Theory: Mathematical Aspects and Applications: Selected Papers of Jürg Fröhlich.* World Scientific, Singapore (1992).

[6] Lee, J.M. *Riemannian Manifolds: An Introduction to Curvature.* Springer, New York (1997).

[7] Regge, T. "General relativity without coordinates." Nuovo Cimento 19, no. 3, pp. 558-571 (1961).

Scalar Curvature on Simplicial Approximations