Factorization in Unitary Loop Groups and Reduced Words in Affine Weyl Groups

by

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Dedication

For my family, the largest constant positive force in my life.

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Abstract

The purpose of this dissertation is to elaborate, with specific examples and calculations, on a new refinement of triangular factorization for the loop group of a simple, compact Lie group K, first appearing in [34]. This new factorization allows us to write a smooth map from S^1 into K (having a triangular factorization) as a triply infinite product of loops, each of which depends on a single complex parameter. These parameters give a set of coordinates on the loop group of K.

The order of the factors in this refinement is determined by an infinite sequence of simple generators in the affine Weyl group associated to K, having certain properties. The major results of this dissertation are examples of such sequences for all the classical Weyl groups.

We also produce a variation of this refinement which allows us to write smooth maps from S^1 into $SU(n)$ as products of $2n+1$ infinite products. By analogy with the semisimple analog of our factorization, we suggest that this variation of the refinement has simpler combinatorics than that appearing in [34].

Chapter 1
Introduction

The purpose of this dissertation is to elaborate, with specific examples and calculations, on a new factorization for the group of smooth functions from the unit circle S^1 into a simple, compact Lie group K, such as $SU(n)$. Such functions will be called "loops", or "loops in K". This factorization, a kind of nonabelian Fourier transform, makes its first appearance in the joint paper [34]. It induces a parametrization, by sequence spaces, of the generic subset of loops having a triangular factorization, a notion we will explain below.

The factorization appearing in [34] is a natural generalization of a factorization for compact, simple Lie groups, which has topological, measure-theoretic and Poisson-geometric significance [4, 37, 26, 6]. In this introduction, we will explain both of these factorizations, and the results of this dissertation, in the case of the group

$$SU(n) = \left\{g \in SL(n, \mathbb{C}) \;\middle|\; gg^* = I\right\} \subset SL(n, \mathbb{C}) = \left\{g \in GL(n, \mathbb{C}) \;\middle|\; \det(g) = 1\right\}$$

and its Lie algebra

$$\mathfrak{su}(n) = \left\{X \in \mathfrak{sl}(n, \mathbb{C}) \;\middle|\; X = -X^*\right\} \subset \mathfrak{sl}(n, \mathbb{C}) = \left\{X \in M_{n\times n} \;\middle|\; \mathrm{tr}(X) = 0\right\}.$$

Throughout, we will use the following notational conventions. For integers $i < j$we will use the notation $[i] = \{1, 2, \ldots, i\}$ and $[i, j] = \{i, i+1, \ldots, j\}$. For a real number r, $\lfloor r \rfloor$ denotes the largest integer less than r, while $\lceil r \rceil$ denotes the smallest integer greater than or equal to r. We will use the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. In general, for a space X, its loop space $C^\infty(S^1, X)$ will be denoted LX.

1.1 The Finite-Dimensional Case $SU(n)$

Any $n\times n$ traceless matrix $X = (X_{ij})$ can be written uniquely as a sum $X = L+D+U$ of strictly lower-triangular, diagonal, and strictly upper-triangular matrices. In terms of Lie theory, this corresponds to the triangular decomposition

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+,$$

where $\mathfrak{n}^\pm$ are the nilpotent subalgebras of strictly upper/lower-triangular matrices, respectively, and $\mathfrak{h}$ is the maximal abelian (Cartan) subalgebra of traceless diagonal matrices.

At the Lie group level, the corresponding factorization for $g \in SL(n, \mathbb{C})$ is the familiar factorization $g = LDU$, where L and U are unipotent lower- and upper-triangular matrices, and D is a diagonal matrix. One way to obtain this factorization, called the *LDU* or triangular factorization, is to apply Gaussian elimination to g. For sets of indices $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$, we define the minor

$$M^I_J(g) = \begin{pmatrix} g_{i_1 j_1} & \cdots & g_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ g_{i_k j_1} & \cdots & g_{i_k j_k} \end{pmatrix}.$$

If they exist, the entries of L, D, and U are given by the formulas

$$D_{ii} = \frac{\det\left(M^{[i]}_{[i]}(g)\right)}{\det\left(M^{[i-1]}_{[i-1]}(g)\right)}, \qquad L_{ij} = \frac{\det\left(M^{[j-1]\cup\{i\}}_{[j]}(g)\right)}{\det\left(M^{[j]}_{[j]}(g)\right)}, \qquad U_{ij} = \frac{\det\left(M^{[i]}_{[i-1]\cup\{j\}}(g)\right)}{\det\left(M^{[i]}_{[i]}(g)\right)}. \tag{1.1.1}$$

The condition for this factorization to exist is simply that $\det\left(M^{[i]}_{[i]}(g)\right) \neq 0$ for each $i \in [n]$. In other words, the principal minors $M^{[i]}_{[i]}(g)$ are invertible. This condition is satisfied on a set of full Haar measure.

Of course, we can go further and additively decompose X in terms of the basis elements e_{ij} for $\mathfrak{sl}(n, \mathbb{C})$, where e_{ij} has a one in position (i, j) and zeros elsewhere. The Lie algebra $\mathfrak{su}(n)$ has the basis

$$\left\{X_{ij} = e_{ij} - e_{ji},\ Y_{ij} = \sqrt{-1}\,(e_{ij} + e_{ji}) \;\middle|\; 1 \leq i < j \leq n\right\}$$
$$\cup \left\{H_i = \sqrt{-1}\left(e_{ii} - e_{(i+1)(i+1)}\right) \;\middle|\; i \in [n]\right\}$$

and a corresponding unique additive decomposition of $Z \in \mathfrak{su}(n)$

$$Z = \sum_{1 \leq i < j \leq n} (r_{ij} X_{ij} + s_{ij} Y_{ij}) + \sum_{i \in [n]} t_i H_i. \tag{1.1.2}$$

The factorization we are concerned with in this dissertation is the group analogue of this decomposition. We define the homomorphism

$$\iota_{ij} : SU(2) \longrightarrow SU(n) : \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \longmapsto \begin{pmatrix} I_{i-1} & & & & \\ & \alpha & & \beta & \\ & & I_{j-i-1} & & \\ & -\bar{\beta} & & \bar{\alpha} & \\ & & & & I_{n-j} \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix, and all other entries are zero. We also define the normalization factor $N(\zeta) = \frac{1}{\sqrt{1+|\zeta|^2}}$ (written $\mathbf{a}(\zeta)$ in [34]). The factorization we will discuss, when it exists, has the form

$$k = \prod_{1 \leq i < j \leq n} \iota_{ij}\left(N(\zeta_{ij}) \begin{pmatrix} 1 & -\bar{\zeta}_{ij} \\ \zeta_{ij} & 1 \end{pmatrix}\right) \prod_{l \in [n]} \exp(t_l H_l). \tag{1.1.3}$$

Of course, we must specify the order of the factors in the first product. The ordering we are interested in depends crucially on a reduced word for the longest permutation

$$w_0 : (1, 2, \ldots, n) \longmapsto (n, n-1, \ldots, 1)$$

in the permutation group S_n, the Weyl group of $SU(n)$ (but see Proposition 2.4.7). This is a decomposition of w_0 in terms of the adjacent transpositions $s_i = (i, i+1)$ which generate S_n. Such a decomposition will be denoted w_0:

$$\mathrm{w}_0 = s_{i_{\binom{n}{2}}} \ldots s_{i_1}.$$

Given w_0, define the sets of indices

$$I(j) = s_{i_1} \ldots s_{i_{j-1}} i_j, \qquad J(j) = s_{i_1} \ldots s_{i_{j-1}} (i_j + 1).$$

The following is basically due to Soibelman [37] (with input from Bott & Samelson [4], and in a formulation appearing in Caine & Pickrell [6]).

Theorem 1.1.4. *Given a reduced word* w_0 *and sets of indices* $\{I(m), J(m)\}_{m \in [\binom{n}{2}]}$ *as above, an element* $k \in SU(n)$ *has a triangular factorization if and only if* k *has a unique factorization*[1]

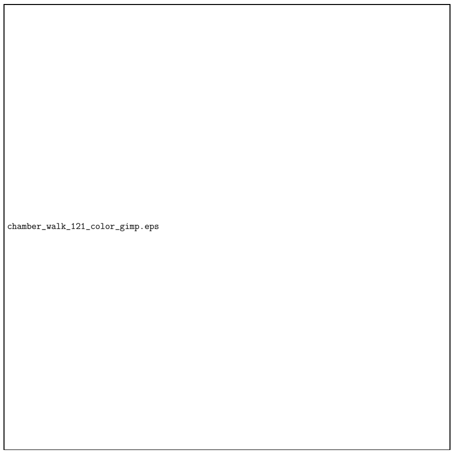
$$k = \overleftarrow{\prod_{m=1}^{\binom{n}{2}}} \iota_{I(m),J(m)} \left(N(\zeta_m) \begin{pmatrix} 1 & -\bar{\zeta}_m \\ \zeta_m & 1 \end{pmatrix} \right) \prod_{i \in [n-1]} \exp(t_i H_i).$$

For example, in $SU(3)$, there are two decompositions for $w_0 = (1, 3)$. The decomposition $\mathrm{w}_0 = s_1 s_2 s_1$ gives $I = (1, 1, 2)$ and $J = (2, 3, 3)$, leading to the factorization

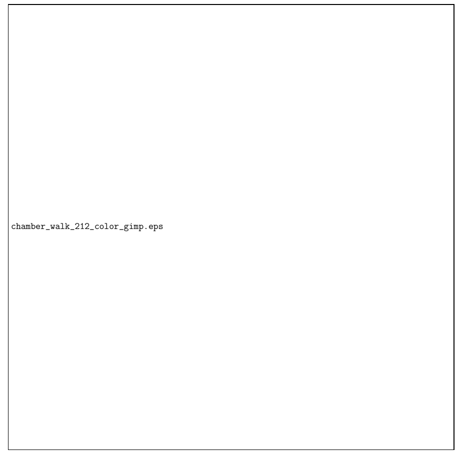
$$k = \begin{pmatrix} 1 & & \\ & N(\zeta_3) & -N(\zeta_3)\bar{\zeta}_3 \\ & N(\zeta_3)\zeta_3 & N(\zeta_3) \end{pmatrix} \begin{pmatrix} N(\zeta_2) & & -N(\zeta_2)\bar{\zeta}_2 \\ & 1 & \\ N(\zeta_2)\zeta_2 & & N(\zeta_2) \end{pmatrix} \times \begin{pmatrix} N(\zeta_1) & -N(\zeta_1)\bar{\zeta}_1 & \\ N(\zeta_1)\zeta_1 & N(\zeta_1) & \\ & & 1 \end{pmatrix} \begin{pmatrix} e^{it_1} & & \\ & e^{i(t_2 - t_1)} & \\ & & e^{-it_2} \end{pmatrix}.$$

[1] Here and throughout, we use $\overleftarrow{\prod_{i=1}^{n}}$ to mean a product "directed to the left", and $\overrightarrow{\prod_{i=1}^{n}}$ to mean a product "directed to the right", so that

$$\overleftarrow{\prod_{i=1}^{n}} f_i = f_n f_{n-1} \ldots f_1, \qquad \overrightarrow{\prod_{i=1}^{n}} f_i = f_1 f_2 \ldots f_n.$$



((a)) $\mathrm{w}_0 = s_1 s_2 s_1$.



((b)) $\mathrm{w}_0 = s_2 s_1 s_2$.

Figure 1.1: The chamber walks corresponding to reduced words $\mathrm{w}_0 = s_1 s_2 s_1$ (1.1(a)) and $\mathrm{w}_0 = s_1 s_2 s_1$ (1.1(b)).

The decomposition $w_0 = s_2 s_1 s_2$ gives $I = (2, 1, 1)$ and $J = (3, 3, 2)$, leading to the factorization

$$k = \begin{pmatrix} N(\zeta_3) & -N(\zeta_3)\bar{\zeta}_3 & \\ N(\zeta_3)\zeta_3 & N(\zeta_3) & \\ & & 1 \end{pmatrix} \begin{pmatrix} N(\zeta_2) & & -N(\zeta_2)\bar{\zeta}_2 \\ & 1 & \\ N(\zeta_2)\zeta_2 & & N(\zeta_2) \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & & \\ & N(\zeta_1) & -N(\zeta_1)\bar{\zeta}_1 \\ & N(\zeta_1)\zeta_1 & N(\zeta_1) \end{pmatrix} \begin{pmatrix} e^{it_1} & & \\ & e^{i(t_2-t_1)} & \\ & & e^{-it_2} \end{pmatrix}.$$

These factorizations have several interesting interpretations and applications, many of which are due to Lu [26]. For example the coordinates $\{\zeta_i\}_{i=1}^{\binom{n}{2}}$ diagonalize the so-called Evens-Lu Poisson structure on the top-dimensional isotypic part of K, they diagonalize Haar measure for K, they can be used to derive Harish-Chandra's famous formula for the diagonal distribution of K, and so on. We will discuss these applications in more depth in Chapter 2.

The reduced words for w_0 can be interpreted in terms of the action of S_n on $\mathbb{R}^n$, by permuting the standard basis vectors: $\sigma e_i = e_{\sigma(i)}$. This action identifies S_n with the group generated by reflections in the hyperplanes $H_{ij} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| x_i = x_j \right\}$, and the group of rigid transformations of the hyperplane arrangement

$$\mathcal{H}_0 = \bigcup_{1 \le i < j \le n} H_{ij}.$$

The connected components of $\mathbb{R}^n \backslash \mathcal{H}_0$, called chambers, are permuted by this action. Each reduced word $w_0 = s_{i_{\binom{n}{2}}} \dots s_{i_1}$ corresponds to a walk through $\binom{n}{2}$ adjacent chambers, beginning with the fundamental chamber

$$C_0 = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, x_1 > x_2 > \ldots > x_n > 0 \right\}$$

and ending with $-C_0$. Figure 1.1 depicts the chamber walks corresponding to the reduced words $w_0 = s_1 s_2 s_1$ and $w_0 = s_2 s_1 s_2$ for $SU(3)$. The figure shows the intersections of the chambers with the hyperplane $\mathbf{1}^\perp$ perpendicular to the vector $\mathbf{1} = (1, 1, 1)$; both are fixed by the action of S_3 on $\mathbb{R}^3$.

A result of Stanley [38] gives the number of decompositions of w_0 as

$$\frac{\binom{n}{2}!}{1^{n-1} 3^{n-2} \ldots (2n-3)^1}. \tag{1.1.5}$$

Stanley obtained this result by creating certain generating functions, now called Stanley symmetric functions. Extending Stanley's result to arbitrary Weyl groups turns out to be a deep and difficult question (Chapter 7, [3]), and I don't know of a closed formula for the number of reduced words for the longest element of a general Weyl group. It is interesting to ask whether (1.1.5) has interpretations in terms of the geometry of the action of S_n on $\mathbb{R}^n$ or $\mathbf{1}^\perp$, or in terms of the braid group associated to S_n.

The sets of indices I and J can also be interpreted in terms of the action of S_n on $\mathbb{R}^n$. The vectors $\alpha_{ij} = \pm(e_i - e_j)$, where $1 \le i < j \le m$, are permuted by this action. In terms of Lie theory, the collection of such vectors is the root system of $SU(n)$. For $1 \le i < j \le m$, the roots $e_i - e_j$ are called positive, and the roots $-(e_i - e_j)$ are called negative. The indices $I(m)$ and $J(m)$ have the following meaning: $s_{i_k} \ldots s_{i_1} e_{I(m),J(m)}$ is positive for $k < m$, but negative for $k \ge m$, so that $e_{I(m),J(m)}$ is the m^{th} root flipped from positive to negative by the successive application of the reflections s_{i_j} in the decomposition $w_0 = s_{i_{\binom{n}{2}}} \ldots s_{i_1}$. Figure 1.2 depicts the images of the positive roots for $SU(3)$ under the permutations s_1, $s_2 s_1$, and $s_1 s_2 s_1$, projected onto $\mathbf{1}^\perp$.

The lexicographically minimal reduced word

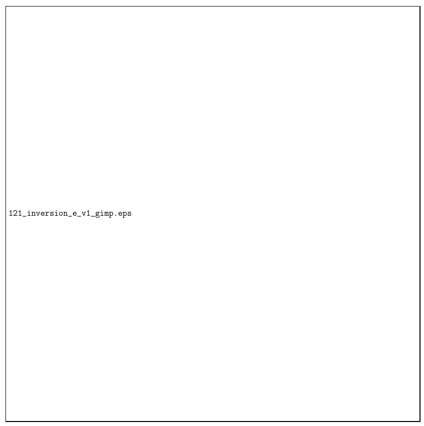
$$w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \ldots s_{n-1} s_{n-2} \ldots s_1$$

has been singled out in a similar context [20, 1] for its nice combinatorics; we single it out again in Chapter 2. It has the property that among reduced words $w_0 = s_{i_{\binom{n}{2}}} \ldots s_{i_1}$, it minimizes $\sum_{j \in [\binom{n}{2}]} i_j$, and it produces the indices

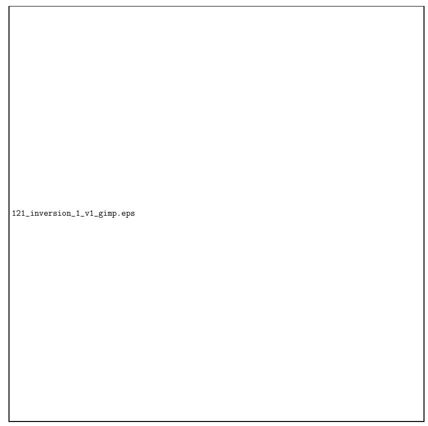
$$I = (1, \ldots, 1; 2, \ldots, 2; \ldots; n-2, n-2; n-1), \tag{1.1.6}$$

$$J = (2, 3, \ldots, n; 3, 4, \ldots, n; \ldots; n-1, n; n). \tag{1.1.7}$$

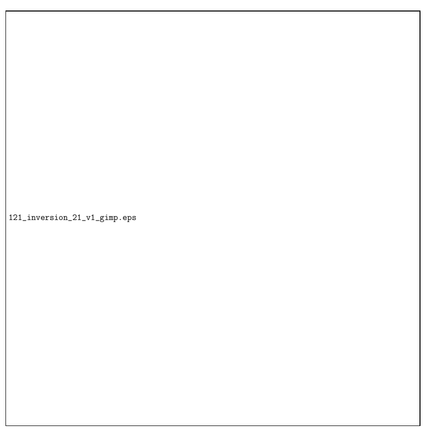
(Here and henceforth, semicolons appear in the same positions in both vectors.)



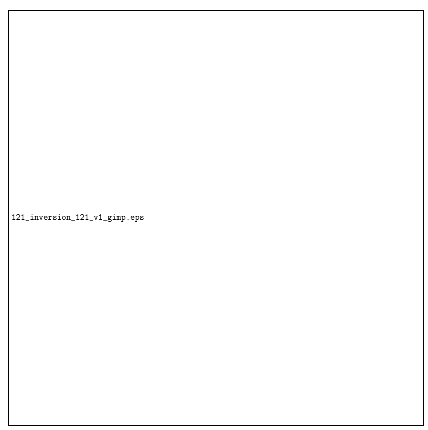
((a)) The positive roots.



((b)) Their s_1 images.



((c)) Their $s_2 s_1$ images.



((d)) Their $s_1 s_2 s_1$ images.

Figure 1.2: The images of the positive roots (1.2(a)) under successive application of s_1 (1.2(b)), s_2 (1.2(c)), and s_1 (1.2(d)).

1.2 The Infinite-Dimensional Case $LSU(n)$

1.2.1 The $LSU(2)$ Case

We now want to illustrate the factorization for the simplest loop group, corresponding to $K = SU(2)$, which appears in [32]. This is the group of smooth functions $k(z)$ from S^1 into $SU(2)$ (called "loops" or "loops in K"), with pointwise multiplication: $(k_1 k_2)(z) = k_1(z) k_2(z)$. We write an element of $LSU(2)$

$$k(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ -\beta^*(z) & \alpha^*(z) \end{pmatrix},$$

where $\alpha(z)$ and $\beta(z)$ are smooth $\mathbb{C}$-valued functions on S^1, and if $f(z) = \sum_{i\in\mathbb{Z}} f_i z^i$, then $f^*(z) = \sum_{i\in\mathbb{Z}} \bar{f}_{-i} z^i$.

In this context, the concept of triangular factorization is best understood in terms of the multiplication operator M_k corresponding to $k = \sum_{i\in\mathbb{Z}} k_i z^i \in LSU(2)$, where k_i is a 2×2 matrix. For $f(z) \in L^2(S^1, \mathbb{C}^2)$, $(M_k f)(z) = k(z) f(z)$, and in terms of the Fourier basis $\{z^i e_j\}_{i\in\mathbb{Z}, j\in[2]}$ for $L^2(S^1, \mathbb{C}^2)$, we have the matrix representation

$$M_k = \begin{pmatrix} \ddots & & & & \iddots \\ & k_0 & k_1 & k_2 & \\ & k_{-1} & k_0 & k_1 & \\ & k_{-2} & k_{-1} & k_0 & \\ \iddots & & & & \ddots \end{pmatrix}. \tag{1.2.1}$$

A triangular factorization $k(z) = l(z) d u(z)$ for k in terms of maps l, d, and u from $\mathbb{C}$ to $SL(2,\mathbb{C})$ corresponds to a factorization $M_k = LDU$, where the infinite matrices $L = M_l$, $D = M_d$, and $U = M_u$ are unipotent lower-triangular, diagonal, and unipotent upper-triangular, respectively. Letting Δ and Δ^* be the open unit disks at zero and infinity, this can be restated as follows: l is holomorphic on Δ^* and $l(\infty)$ is unipotent lower triangular, d is constant and diagonal, and u is holomorphic on Δ with $u(0)$ unipotent upper triangular. Explicitly,

$$l = \begin{pmatrix} 1 + \sum_{i\geq 1} l_{11;i} z^{-i} & \sum_{i\geq 1} l_{12;i} z^{-i} \\ l_{21;0} + \sum_{i\geq 1} l_{21;i} z^{-i} & 1 + \sum_{i\geq 1} l_{22;i} z^{-i} \end{pmatrix},$$

$$u = \begin{pmatrix} 1 + \sum_{i\geq 1} u_{11;i} z^{i} & u_{12;0} + \sum_{i\geq 1} u_{12;i} z^{-i} \\ \sum_{i\geq 1} u_{21;i} z^{-i} & 1 + \sum_{i\geq 1} u_{22;i} z^{-i} \end{pmatrix}.$$

Triangular factorization is essentially equivalent to Birkhoff (or Wiener-Hopf, or Riemann-Hilbert) factorization; see [32] or [35] for details.

The condition a loop must satisfy to have a triangular factorization is also stated in terms of M_k. From (1.2.1), we can see that there are essentially two "principal minors" of M_k. Block-diagonalizing M_k,

$$M_k = \left(\begin{array}{cccc|cccc} \ddots & & & & & & & \iddots \\ & k_0 & k_1 & & k_2 & k_3 & & \\ & k_{-1} & k_0 & & k_1 & k_2 & & \\ \hline & k_{-2} & k_{-1} & & k_0 & k_1 & & \\ & k_{-3} & k_{-2} & & k_{-1} & k_0 & & \\ \iddots & & & & & & & \ddots \end{array}\right) = \begin{pmatrix} A(k) & B(k) \\ C(k) & D(k) \end{pmatrix},$$

we obtain the first, the Toeplitz operator $A(k)$ associated to k. Shifting our block diagonalization by one row and one column, we obtain the shifted Toeplitz operator $A_1(k)$. A loop k has a triangular factorization if and only if $A(k)$ and $A_1(k)$ are invertible, which is equivalent to having

$$\det\left(A(k)^* A(k)\right) =: \det\left(|A(k)|^2\right) \neq 0 \neq \det\left(A_1(k)^* A_1(k)\right) =: \det\left(|A_1(k)|^2\right).$$

With this in mind, we have the following [32].

Theorem 1.2.2. *A loop $k \in LSU(2)$ has a triangular factorization if and only if it also has a factorization*

$$k(z) = k_1^*(z)\lambda(z)k_2(z), \tag{1.2.3}$$

where

$$\lambda = \begin{pmatrix} \exp\left(i\chi(z)\right) & \\ & \exp\left(-i\chi(z)\right) \end{pmatrix}$$

for $\chi = \sum_{k\in\mathbb{Z}} \chi_k z^k \in C^\infty\left(S^1, \mathbb{R}\right)$, and k_1 and k_2 satisfy the following equivalent conditions.

1. *There are factorizations*

$$k_1 = \lim_{M\to\infty} \overleftarrow{\prod_{i=0}^{M}} N\left(\eta_i\right) \begin{pmatrix} 1 & -\bar{\eta}_i z^i \\ \eta_i z^{-i} & 1 \end{pmatrix}, \qquad k_2 = \lim_{M\to\infty} \overleftarrow{\prod_{i=1}^{M}} N\left(\zeta_i\right) \begin{pmatrix} 1 & \zeta_i z^{-i} \\ -\bar{\zeta}_i z^i & 1 \end{pmatrix} \tag{1.2.4}$$

   for rapidly decreasing sequences $\{\eta_i\}$ and $\{\zeta_i\}$.

2. *There are triangular factorizations*

$$k_1 = \begin{pmatrix} 1 & \\ y^*(z) & 1 \end{pmatrix} \begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix} u_1(z), \qquad k_2 = \begin{pmatrix} 1 & x^*(z) \\ & 1 \end{pmatrix} \begin{pmatrix} a_2 & \\ & a_2^{-1} \end{pmatrix} u_2(z)$$

   where x^* and y^* are holomorphic on Δ^*, and a_1 and a_2 are real, positive constants.

3. The forms of k_1 and k_2 are

$$k_1 = \begin{pmatrix} \alpha_1(z) & \beta_1(z) \\ -\beta_1^*(z) & \alpha_1^*(z) \end{pmatrix}, \qquad k_2 = \begin{pmatrix} \alpha_2^*(z) & -\beta_2^*(z) \\ \beta_2(z) & \alpha_2(z) \end{pmatrix}$$

for functions α_i and β_i which are holomorphic on Δ, such that $\alpha_i(0) > 0$, $\beta_2(0) = 0$, and α_i and β_i do not simultaneously vanish on Δ.

Furthermore, in terms of the coordinates $\{\eta, \chi, \zeta\}$, we have

$$a_1 = \lim_{M\to\infty} \prod_{i=0}^{M} N(\eta_i), \qquad a_2 = \lim_{M\to\infty} \prod_{i=0}^{M} N(\zeta_i),$$

$$\det\left(|A(k)|^2\right) = \lim_{M\to\infty} \prod_{i=1}^{M} \left(1+|\eta_i|^2\right)^{-i} \exp\left(-\sum_{k=-M}^{M} k|\chi_k|^2\right) \prod_{j=1}^{M} \left(1+|\zeta_j|^2\right)^{-j}.$$

We can place Theorem 1.2.2 into the framework of Theorem 1.1.4 by showing that the orders of the factors in (1.2.4) also depend on words in a reflection group. The reflection group of interest is the infinite dihedral group D_∞. It is the group of rigid transformations of the integers. Its generators are $s_1(n) = -n$ and $s_0(n) = 2-n$, which are, respectively, reflection in the origin and reflection in 1. It is an infinite group, so it has no longest element. However, there are two infinite reduced sequences of generators, or "infinite reduced words", $s_0, s_1, s_0, s_1, \ldots$ and $s_1, s_0, s_1, s_0, \ldots$. Each of these gives us a sequence of integers, by the rule $I(j) = s_{i_1} \ldots s_{i_{j-1}}(1-i_j)$. The sequence $s_0, s_1, s_0, s_1, \ldots$ gives $I = (1,2,3,4,\ldots)$, while the sequence $s_1, s_0, s_1, s_0, \ldots$ gives $I = (0,-1,-2,-3,\ldots)$. For integers $p \geq 0$ and $q > 0$, we define homomorphisms ι_p and ι_q from $SU(2)$ into $LSU(2)$ by

$$\iota_{-p}\left(\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}\right) = \begin{pmatrix} \alpha & \beta z^p \\ -\bar\beta z^{-p} & \bar\alpha \end{pmatrix}, \qquad \iota_q\left(\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}\right) = \begin{pmatrix} \alpha & -\bar\beta z^{-q} \\ \beta z^q & \bar\alpha \end{pmatrix}.$$

The products

$$\lim_{M\to\infty} \prod_{m=1}^{M} \iota_{I(m)}\left(N(\zeta_i)\begin{pmatrix} 1 & -\bar\zeta_m \\ \zeta_m & 1 \end{pmatrix}\right)$$

then give us loops of types k_2 and k_1.

We can also describe these sequences as walks on the fundamental domains for D_∞. Removing the integers from the real line leaves behind the "chambers" for the infinite dihedral group, namely the open intervals $(i, i+1)$. The two infinite reduced words correspond to two infinite walks on intervals, beginning with the interval $(0,1)$. The sequence $s_0, s_1, s_0, s_1, \ldots$ corresponds to an interval walk towards ∞, while the sequence $s_1, s_0, s_1, s_0, \ldots$ corresponds to an interval walk towards $-\infty$.

1.2.2 Periodic Infinite Reduced Words

Theorem 1.2.2 is generalized to arbitrary simply-connected compact Lie groups in [34]. In this case, the factorization (1.2.3) depends on the choice of an "affine periodic infinite reduced word" in the affine Weyl group of K, an infinite reflection group. For $SU(n)$, the affine Weyl group is the affine permutation group $\tilde{S}_n$, the group of permutations of $\mathbb{Z}$ which commute with translation by n, $t_n(i) = i + n$. This group is generated by the transpositions s_i, which swap the indices $(i + mn, i + 1 + mn)$ for all $m \in \mathbb{Z}$, and for $i = 0, 1, \ldots, n - 1$.

An infinite reduced sequence or word is a sequence of transpositions $\left\{s_{i_j}\right\}_{j\in\mathbb{N}}$ such that for every $j \in \mathbb{N}$, the word $s_{i_j} s_{i_{j-1}} \ldots s_{i_1}$ cannot be made shorter by applying the relations defining $\tilde{S}_n$. Such a word is called periodic if there exists some integer p so that $i_{p+q} = i_q$, for all integers q. An infinite reduced word in $\tilde{S}_n$ can be associated with sequences of indices I, J, and P such that $1 \leq I(m) < J(m) \leq n$ for all m, and $P(m) \in \mathbb{Z}$. This is done in a way similar to the examples we have already seen, but we will not go into it here.

Focusing on the factor $k_2(z)$ in (1.2.3), which, in the context of the group $LSU(n)$, has a factorization of the form

$$k_2 = \lim_{M\to\infty} \prod_{p\in[M]} \prod_{1\leq i<j\leq n} \iota_{ij} \left(N\left(\zeta_{p;ij}\right) \begin{pmatrix} 1 & \zeta_{p;ij} z^{-p} \\ -\bar{\zeta}_{p;ij} z^p & 1 \end{pmatrix} \right),$$

we seek periodic infinite reduced words that produce sequences of indices I, J, and P which order the above product. (The sequences I and J specify a homomorphism $\iota_{I(m),J(m)}$, and the sequence P specifies a power of z appearing in the element of $LSU(2)$ mapped into $LSU(n)$ by $\iota_{I(m),J(m)}$.)

In the case of $K = SU(3)$ it is possible to write down examples of such sequences in a fairly simple way. Two such affine periodic reduced sequences are

$$s_0, s_1, s_2, s_1, s_0, s_1, s_2, s_1, s_0, \ldots, \qquad s_0, s_2, s_1, s_2, s_0, s_2, s_1, s_2, s_0, \ldots.$$

The first of these produces

$$I = (1, 2, 1, 1, 1, 2, 1, 1, 1, \ldots), \qquad J = (3, 3, 3, 2, 3, 3, 3, 2, \ldots),$$

$$P = (1, 1, 2, 1, 3, 2, 4, 2, \ldots),$$

and the factorization

$$k_2(z) = \dots \begin{pmatrix} 1 & & \\ & a(\zeta_6) & a(\zeta_6)\zeta_6 z^{-2} \\ & -a(\zeta_6)\bar{\zeta}_6 z^2 & a(\zeta_6) \end{pmatrix} \begin{pmatrix} a(\zeta_5) & & a(\zeta_5)\zeta_5 z^{-3} \\ & 1 & \\ -a(\zeta_5)\bar{\zeta}_5 z^3 & & a(\zeta_5) \end{pmatrix}$$
$$\times \begin{pmatrix} a(\zeta_4) & a(\zeta_4)\zeta_4 z^{-1} & \\ -a(\zeta_4)\bar{\zeta}_4 z & a(\zeta_4) & \\ & & 1 \end{pmatrix} \begin{pmatrix} a(\zeta_3) & & a(\zeta_3)\zeta_3 z^{-2} \\ & 1 & \\ -a(\zeta_3)\bar{\zeta}_3 z^2 & & a(\zeta_3) \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & & \\ & a(\zeta_2) & a(\zeta_2)\zeta_2 z^{-1} \\ & -a(\zeta_2)\bar{\zeta}_2 z & a(\zeta_2) \end{pmatrix} \begin{pmatrix} a(\zeta_1) & & a(\zeta_1)\zeta_1 z^{-1} \\ & 1 & \\ -a(\zeta_1)\bar{\zeta}_1 z & & a(\zeta_1) \end{pmatrix}.$$

You may notice that both the given affine periodic sequences for $K = SU(3)$ have the form $s_0 \mathrm{w}_0 s_0 \mathrm{w}_0 \dots$, for some reduced word w_0 for w_0. The same is true for the sequence $s_0 s_1 s_0 s_1 \dots$, since s_1 is the longest element of the Weyl group of $SU(2)$, a cyclic group of order 2. It is surprising that this pattern does not continue for higher rank groups, where it is more difficult to produce examples of affine periodic sequences. This is one of the main points addressed in this dissertation.

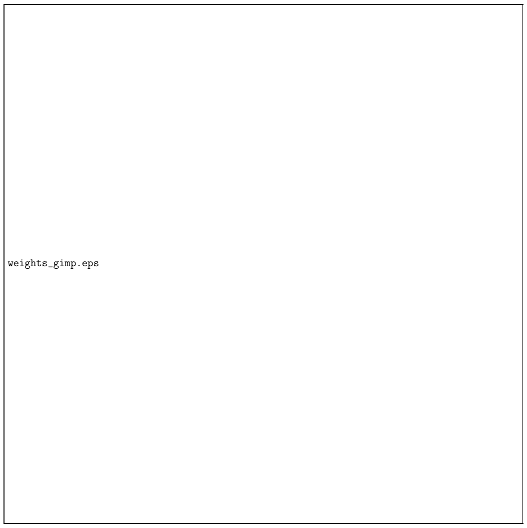
The major new results in this document are examples of affine periodic reduced sequences producing factorizations of the form (1.2.3) for all the classical Lie groups. Such sequences have been studied in several contexts [8, 24] by investigators including Cellini & Papi, Ito, Shi, and Lam & Pylyavskyy, a fact we discovered late in our investigations. The overlap between the present work and previous research is still being explored, but will be discussed in Chapter 3.

Our approach relies on a geometric realization of the affine Weyl group, and a closer analysis of the situation for $SU(3)$. The affine symmetric group, like the symmetric group, acts on $\mathbb{R}^n$. It is the group of reflections in the hyperplanes $H_{p;ij} = \left\{\boldsymbol{x} \in \mathbb{R}^n \middle| x_i - x_j = p\right\}$, or the group of rigid transformations of the hyperplane arrangement

$$\mathcal{H} = \bigcup_{\substack{p \in \mathbb{Z} \\ 1 \le i < j \le n}} H_{p;ij}.$$

For $i \in [n-1]$, s_i is the reflection in the hyperplane $H_{0;i(i+1)}$, while s_0 is the reflection in the hyperplane $H_{1;1n}$. (Note that the action of D_∞ on $\mathbb{R}$ can be obtained from the action of $\tilde{S}_2$ on $\mathbb{R}^2$ by projecting the latter space onto the line $e_1 + e_2 = 0$, which is $\mathbf{1}^\perp$ in this case.) The connected components of $\mathbb{R}^n \backslash \mathcal{H}$ are called alcoves, and the alcoves are permuted by the action of $\tilde{S}_n$. Figure 1.3(a) shows the intersection of the alcove diagram for $SU(3)$ with $\mathbf{1}^\perp$.

Infinite reduced words correspond to infinite alcove walks which are "minimal" in the sense that all their initial subwalks are as short as possible. Periodic infinite reduced words correspond to so-called affine periodic infinite minimal alcove walks.



((a)) The alcove diagram for $SU(3)$.

((b)) The minimal alcove walk corresponding to $s_0, s_1, s_2, s_1, s_0, s_1, s_2, s_1, \ldots$.

Figure 1.3: Alcoves for $SU(3)$.

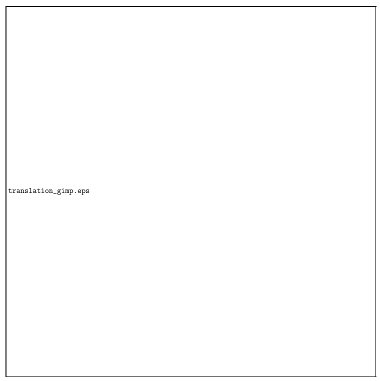
These walks have the property that for some $p \in \mathbb{Z}$, the walk from p^{th} alcove on is a translation of the original walk. The periodic reduced sequences we seek correspond to affine periodic infinite minimal walks which begin with the fundamental alcove

$$A_0 = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_1 > x_2 > \ldots > x_n,\ x_1 - x_n < 1 \right\}$$

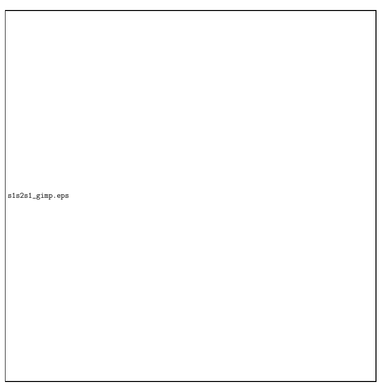
and proceed infinitely far into the interior of the fundamental Weyl chamber C_0. The alcove walk corresponding to the sequence $s_0, s_1, s_2, s_1, s_0, s_1, s_2, s_1, s_0, \ldots$ appears in Figure 1.3(b).

The action of $\tilde{S}_n$ on $\mathbb{R}^n$ allows it to be realized as the semidirect product of S_n with the set of translations by the lattice $\check{T} = \mathbb{Z}^n \cap \mathbf{1}^{\perp}$. One way to construct a reduced word corresponding to an affine periodic minimal walk is to choose a translation t, choose a reduced word t for that translation, and then repeat t (Figure 1.4). In $SU(3)$, we have chosen a translation by $h^* = e_1 - e_3$. To explain this, note that the closure $\bar{C}_0$ of C_0 is the convex hull of the lines spanned by the vectors $\Theta_1 = e_1$, $\Theta_2 = e_1 + e_2$, and $e_1 + e_2 + e_3$, and the sum of these vectors, which is clearly contained in C_0, projects to $e_1 - e_3$ in $\mathbf{1}^{\perp}$.

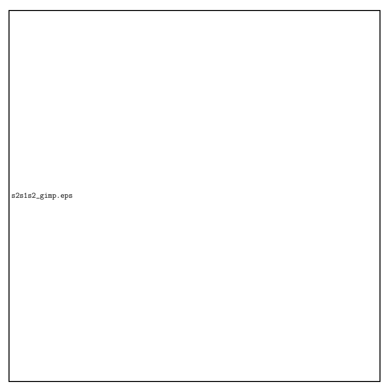
Each alcove walk is equivalent to a walk on the "dual graph" obtained by placing a vertex at the center of each alcove, and connecting the vertices of alcoves which share a boundary. The dual graph for the alcove diagram of $SU(3)$ is a tiling of the plane by hexagons. The two alcove walks from A_0 to $A_0 + h^*$ correspond to the two ways of traveling around a hexagon. Indeed, any walk from A_0 into C_0 can be reduced to a set of successive choices of whether to go right or left at a given vertex of the dual



((a)) Repeated translation by h^*.



((b)) The reduced word $\mathrm{t}_{\mathrm{h}^*} = s_1 s_2 s_1 s_0$.



((c)) The reduced word $\mathrm{t}_{\mathrm{h}^*} = s_2 s_1 s_2 s_0$.

Figure 1.4: Constructing infinite reduced words in $SU(3)$.

graph. Going right is equivalent, on the dual graph, to a translation by Θ_1, and going left is equivalent to a translation by Θ_2. This discussion leads us to the following.

Proposition 1.2.5. *Let A be an alcove in the alcove diagram of $SU(3)$ such that*

$$\text{center}(A) - \text{center}(A_0) = a\Theta_1 + b\Theta_2.$$

Then the number of minimal alcove walks from A_0 to A is $\binom{a+b}{a}$.

Furthermore, we can express the reduced words corresponding to these minimal alcove walks in terms of the transpositions s_0, s_1, and s_2, and we can characterize the sets of indices I, J, and P they produce.

Let $\mathrm{W}_1 = s_0 s_2$, let $\mathrm{W}_2 = s_0 s_1$, and let ${\mathrm{W}_2}^{(k)} = s_{0+k} s_{1+k}$, where the indices are taken mod n. Also, denote a choice of k integers from $[n]$ by $c \in \{a, b\}^n$, where $c_i = a$ indicates that i has been chosen, and denote the set of all such choices by $\left[{n \atop k}\right] \subset \{a, b\}^n$, not to be confused with $[\binom{n}{k}] = \{1, \ldots, \binom{n}{k}\}$.

Proposition 1.2.6. *1. For any alcove A in the alcove diagram of $SU(3)$, the minimal path from A_0 to A corresponding to $\sigma \in \left[{a+b \atop a}\right] \subset \{1, 2\}^{a+b}$ corresponds to the reduced word*

$$\mathrm{W}(\sigma) = {\mathrm{W}_{\sigma_{\mathrm{a+b}}}}^{\left(\sum_{i=1}^{a+b-1} \sigma_i\right)} \ldots {\mathrm{W}_{\sigma_3}}^{(\sigma_1+\sigma_2)} {\mathrm{W}_{\sigma_2}}^{\sigma_1} \mathrm{W}_{\sigma_1}.$$

2. The reduced word $\mathrm{W}(\sigma)$ produces sets of indices I, J, and P, where

$$I(m) = \begin{cases} 1 & m \text{ odd}, \\ \sigma_{m/2} & m \text{ even}, \end{cases} \qquad J(m) = \begin{cases} 3 & m \text{ odd}, \\ \sigma_{m/2} + 1 & m \text{ even}, \end{cases}$$

$$P(m) = \begin{cases} \left\lfloor \frac{m}{2} \right\rfloor & m \text{ odd}, \\ \delta_{1,\sigma_{m/2}} \left(\sum_{l<m/2} \delta_{1,\sigma_l} + 1\right) + \delta_{2,\sigma_{m/2}} \left(\sum_{l<m/2} \delta_{2,\sigma_l} + 1\right) & m \text{ even}. \end{cases}$$

In Chapter 3, we provide generalizations of Proposition 1.2.6 for all the classical Lie groups. Unfortunately, we have found no such generalizations of Proposition 1.2.5. In this affine case, so-called affine Stanley symmetric functions have been developed, and these can be used to enumerate the number of reduced words of a given element. However, I know of neither closed-form expressions, nor of geometric interpretations, for these numbers.

The reduced words W_l appearing in the version of Proposition 1.2.6 for $SU(n)$ are

$$\mathrm{W}_l = \overleftarrow{\prod_{k=0}^{l-1}} s_{l+1+k} s_{l+2+k} \ldots s_{n-1+k} s_k.$$

The above word produces the indices

$$P = (1, \ldots, 1), \qquad I = (1, \ldots, 1; 2, \ldots, 2; \ldots; l, \ldots, l),$$

$$J = (n, n-1, \ldots, l+1; n, n-1, \ldots, l+1; \ldots; n, n-1, \ldots, l+1).$$

This leads us to propose the following affine analogue of the lexicographically minimal reduced word,

$$\mathrm{W}_0 = \mathrm{W_{n-1}}^{(\binom{n-1}{2})} \ldots \mathrm{W}_3{}^{(3)} \mathrm{W}_2{}^{(1)} \mathrm{W}_1.$$

Why is this an affine analog of the lexicographically minimal reduced word? Its length is minimal among reduced words for which the indices I and J run through all possible pairs i, j with $1 \leq i < j \leq n$, and we have

$$\begin{aligned} I =& (1, \ldots, 1; 1, \ldots, 1, 2, \ldots, 2; \ldots; 1, 2, \ldots, n-1), \\ J =& (n, n-1, \ldots, 2; n, n-1, \ldots, 3, n, n-1, \ldots, 3; \ldots; n, n, \ldots, n), \end{aligned}$$

which we invite the reader to compare with (1.1.6). The indices P are given by

$$P = (1, \ldots, 1; 2, \ldots, 2, 1, \ldots, 1; \ldots; n-1, n-2, \ldots, 1).$$

For arbitrary n, C_0 is the convex hull of lines spanned by the vectors $\Theta_l = e_1 + \ldots + e_l$, where $l \in [n]$. The vector

$$\frac{1}{2}\check{\rho} = \sum_{l \in [n]} \Theta_l = ne_1 + \ldots + e_n$$

projects onto an element of $\check{T}$ only if n is odd. Thus, we take

$$h^* = \begin{cases} \frac{1}{2}\check{\rho} & n \text{ odd}, \\ \check{\rho} & n \text{ even}. \end{cases}$$

When n is odd, W_0 is a reduced word for t_{h^*}; when n is even, $\mathrm{W}_0{}^{(\binom{n}{2})}\mathrm{W}_0$ is a reduced word for this translation.

1.2.3 The $LSU(n)$ Case

In general, the Lie algebra $L\mathfrak{su}(n)$ of $LSU(n)$ has the basis

$$\left\{ X_{ij}z^p,\ Y_{ij}z^p,\ H_l z^p \,\middle|\, 1 \leq i < j \leq n,\ l \in [n],\ p \in \mathbb{Z} \right\},$$

and combining the Fourier transform with (1.1.2), we have the additive decomposition

$$Z(z) = \sum_{p \in \mathbb{Z}} \left(\sum_{1 \leq i < j \leq n} (r_{p;ij} X_{ij} + s_{p;ij} Y_{ij}) + \sum_{i \in [n]} t_{p;i} H_i \right) z^p$$

for $Z \in L\mathfrak{su}(n)$.

The corresponding factorization at the group level is given in the following analogue of Theorem 1.2.3. Let $\wedge^l \mathbb{C}^n \cong \mathbb{C}^{\binom{n}{l}}$ denote the space of alternating l-linear forms on $\mathbb{C}^n$, having basis

$$\left\{ e_{i_1} \wedge \ldots \wedge e_{i_l} \,\middle|\, \boldsymbol{i} \in [n]^l,\ i_j \neq i_k \right\}.$$

For a set of indices $\boldsymbol{i} \in [n]^l$ with $i_j \neq i_k$, we let $\phi^{i_1} \wedge \ldots \wedge \phi^{i_l} k$ denote the vector in $\wedge^l \mathbb{C}^n$ whose entries are the determinants of all $l \times l$ minors lying in rows $i_1, \ldots, i_l$ of k.

Theorem 1.2.7. *A loop $k \in LSU(n)$ has a triangular factorization if and only if it also has a factorization*

$$k(z) = k_1^*(z)\lambda(z)k_2(z),$$

where

$$\lambda = \begin{pmatrix} \exp\left(i\chi_1(z)\right) & & & \\ & \exp\left(i\left(\chi_2(z) - \chi_1(z)\right)\right) & & \\ & & \ddots & \\ & & & \exp\left(-i\chi_{n-1}(z)\right) \end{pmatrix},$$

for functions $\chi_i \in C^\infty\left(S^1, \mathbb{R}\right)$, and k_1 and k_2 satisfy the following equivalent pairs conditions.

1. *For the sequences I, J, and P produced by the sequence ${\mathrm{W}_0}^\infty$, and the sequences I', J', and P' produced by the sequence ${\mathrm{w}_0}^{-1}{\mathrm{W}_0}^\infty$, where w_0 is a reduced word for the longest element of S_n, there are factorizations*

$$k_1 = \lim_{M\to\infty} \prod_{m=0}^{M} \iota_{I(m),J(m)} \left(N\left(\eta_m\right) \begin{pmatrix} 1 & -\bar{\eta}_p z^{P(m)} \\ \eta_m z^{-P(m)} & 1 \end{pmatrix} \right),$$

$$k_2 = \lim_{M\to\infty} \prod_{m=1}^{M} \iota_{I'(m),J'(m)} \left(N\left(\zeta_m\right) \begin{pmatrix} 1 & \zeta_m z^{-P'(m)} \\ -\bar{\zeta}_m z^{P'(m)} & 1 \end{pmatrix} \right)$$

   for rapidly decreasing sequences $\{\eta_m\}$ and $\{\zeta_m\}$.

2. *There are triangular factorizations*

$$k_1 = l_1(z)a_1u_1(z), \qquad k_2 = l_2(z)a_2u_2(z),$$

   where l_1 is unipotent lower-triangular and $l_1(0) = I$, l_2 is unipotent upper triangular, and a_1 and a_2 are real, positive diagonal matrices with determinant one.

3. *For each* $i \in [n-1]$ *and* $j \in [2,n]$*, the functions*

$$\phi^1 \wedge \ldots \wedge \phi^i k_1 : S^1 \longrightarrow \wedge^i \mathbb{C}^n, \qquad \phi^j \wedge \ldots \wedge \phi^n k_2 : S^1 \longrightarrow \wedge^{n-i+1} \mathbb{C}^n$$

are holomorphic and nonvanishing on Δ*, and there are real, positive numbers* r_1 *and* r_2 *such that*

$$\phi^1 \wedge \ldots \wedge \phi^i k_1(0) = r_1 e_1 \wedge \ldots \wedge e_i, \qquad \phi^j \wedge \ldots \wedge \phi^n k_2(0) = r_2 e_j \wedge \ldots \wedge e_n.$$

Furthermore, in terms of the coordinates $\{\eta, \chi, \zeta\}$*, we have*

$$a_1 = \lim_{M\to\infty} \prod_{m=0}^{M} \iota_{I'(m),J'(m)} \left(\begin{pmatrix} N(\eta_m) & \\ & N(\eta_m)^{-1} \end{pmatrix} \right),$$
$$a_2 = \lim_{M\to\infty} \prod_{m=0}^{M} \iota_{I(m),J(m)} \left(\begin{pmatrix} N(\zeta_m) & \\ & N(\zeta_m)^{-1} \end{pmatrix} \right),$$
$$\det\left(|A(k)|^2\right) = \lim_{M\to\infty} \prod_{i=\binom{n}{2}+1}^{M} \left(1+|\eta_i|^2\right)^{-P(i)} \exp\left(-n\sum_{-M}^{M} k|\chi_k|^2\right) \prod_{j=1}^{M} \left(1+|\zeta_j|^2\right)^{-P(j)}.$$

We will now illustrate this theorem in the cases $n = 4, 5$, extending §5 of [34]. When $n = 4$, we have

$$\mathrm{W}_1 = s_2 s_3 s_0, \qquad \mathrm{W}_2 = s_0 s_1 s_3 s_0, \qquad \mathrm{W}_3 = s_2 s_1 s_0,$$

and

$$\mathrm{W}_0 = s_1 s_0 s_3 s_1 s_2 s_0 s_1 s_2 s_3 s_0,$$
$$\mathrm{t}_{\mathrm{h}^*} = s_3 s_2 s_1 s_3 s_0 s_2 s_3 s_0 s_1 s_2 s_1 s_0 s_3 s_1 s_2 s_0 s_1 s_2 s_3 s_0.$$

The word W_0 produces the indices

$$I = (1,1,1,1,1,2,2,1,2,3), \qquad J = (4,3,2,4,3,4,3,4,4,4),$$

$$P = (1,1,1,2,2,1,1,3,2,1).$$

Thus, a product of type k_2 begins

$$
\begin{aligned}
k_2 = \dots & \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & * & * \\ & & * & * \end{pmatrix} \begin{pmatrix} 1 & & & \\ & * & & * \\ & & 1 & \\ & * & & * \end{pmatrix} \begin{pmatrix} * & & & * \\ & 1 & & \\ & & 1 & \\ * & & & * \end{pmatrix} \\
\times & \begin{pmatrix} 1 & & & \\ & * & * & \\ & * & * & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & * & & * \\ & & 1 & \\ & * & & * \end{pmatrix} \begin{pmatrix} * & & * & \\ & 1 & & \\ * & & * & \\ & & & 1 \end{pmatrix} \begin{pmatrix} * & & & * \\ & 1 & & \\ & & 1 & \\ * & & & * \end{pmatrix} \\
\times & \begin{pmatrix} * & * & & \\ * & * & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} * & & * & \\ & 1 & & \\ * & & * & \\ & & & 1 \end{pmatrix} \begin{pmatrix} * & & & * \\ & 1 & & \\ & & 1 & \\ * & & & * \end{pmatrix} .
\end{aligned}
$$

We have the triangular factorization $k_2 = l_2 a_2 u_2$, where $l_2 = \exp(L_2)$ for

$$
L_2 = \begin{pmatrix} 0 & \bar{x}_3 z^{-1} + \dots & \bar{x}_2 z^{-1} + \bar{x}_5 z^{-2} + \dots & \bar{x}_1 z^{-1} + \bar{x}_4 z^{-2} + \bar{x}_8 z^{-3} + \dots \\ & 0 & \bar{x}_7 z^{-1} + \dots & \bar{x}_6 z^{-1} + \bar{x}_9 z^{-2} + \dots \\ & & 0 & \bar{x}_{10} z^{-1} + \dots \\ & & & 0 \end{pmatrix},
$$

and $a_2 = \operatorname{diag}\left(a_{21}, a_{22} a_{21}^{-1}, a_{23} a_{22}^{-1}, a_{23}^{-1}\right)$, for

$$
a_{21} = \prod_{i \neq 6,7,9,10,\dots} N(\zeta_i), \qquad a_{22} = \prod_{i \neq 3,10,\dots} N(\zeta_i), \qquad a_{23} = \prod_{i \neq 2,3,5,7,\dots} N(\zeta_i)^{-1}.
$$

Finally, for $k = k_1^* \lambda k_2$ as in the theorem, we have

$$
\det\left(|A(k)|^2\right) = \prod \left(1 + |\eta_i|^2\right)^{-P(i)} \exp\left(-4 \sum k |\chi_k|^2\right) \prod \left(1 + |\zeta_j|^2\right)^{P(j)},
$$

where the infinite sequence P begins

$$
P = (1,1,1,2,2,1,1,3,2,1,4,3,2,5,4,3,2,6,4,2,7,5,3,8,6,5,3,9,6,3,\dots).
$$

For $SU(5)$, we have

$$
\mathrm{W}_1 = s_2 s_3 s_4 s_0, \quad \mathrm{W}_2 = s_4 s_0 s_1 s_3 s_4 s_0, \quad \mathrm{W}_3 = s_1 s_2 s_0 s_1 s_4 s_0, \quad \mathrm{W}_4 = s_3 s_2 s_1 s_0,
$$

and

$$
\mathrm{W}_0 = \mathrm{t}_{\mathfrak{h}^*} = s_4 s_3 s_2 s_1 s_4 s_0 s_3 s_4 s_2 s_3 s_0 s_1 s_2 s_4 s_0 s_1 s_2 s_3 s_4 s_0,
$$

which produces the indices

$$
\begin{aligned}
I =& (1,1,1,1,1,1,1,2,2,2,1,1,2,2,3,3,1,2,3,4), \\
J =& (5,4,3,2,5,4,3,5,4,3,5,4,5,4,5,4,5,5,5,5), \\
P =& (1,1,1,1,2,2,2,1,1,1,3,3,2,2,1,1,4,3,2,1).
\end{aligned}
$$

Thus, a product of type k_2 begins

$$
\begin{aligned}
k_2 = \dots &\begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& * & * \\ &&& * & * \end{pmatrix}
\begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && * && * \\ &&& 1 & \\ && * && * \end{pmatrix}
\begin{pmatrix} 1 &&&& \\ & * &&& * \\ && 1 && \\ &&& 1 & \\ & * &&& * \end{pmatrix}
\begin{pmatrix} * &&&& * \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ * &&&& * \end{pmatrix} \\
\times &\begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && * & * & \\ && * & * & \\ &&&& 1 \end{pmatrix}
\begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && * && * \\ &&& 1 & \\ && * && * \end{pmatrix}
\begin{pmatrix} 1 &&&& \\ & * && * & \\ && 1 && \\ & * && * & \\ &&&& 1 \end{pmatrix}
\begin{pmatrix} 1 &&&& \\ & * &&& * \\ && 1 && \\ &&& 1 & \\ & * &&& * \end{pmatrix} \\
\times &\begin{pmatrix} * &&& * & \\ & 1 &&& \\ && 1 && \\ * &&& * & \\ &&&& 1 \end{pmatrix}
\begin{pmatrix} * &&&& * \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ * &&&& * \end{pmatrix}
\begin{pmatrix} 1 &&&& \\ & * & * && \\ & * & * && \\ &&& 1 & \\ &&&& 1 \end{pmatrix}
\begin{pmatrix} 1 &&&& \\ & * && * & \\ && 1 && \\ & * && * & \\ &&&& 1 \end{pmatrix} \\
\times &\begin{pmatrix} 1 &&&& \\ & * &&& * \\ && 1 && \\ &&& 1 & \\ & * &&& * \end{pmatrix}
\begin{pmatrix} * && * && \\ & 1 &&& \\ * && * && \\ &&& 1 & \\ &&&& 1 \end{pmatrix}
\begin{pmatrix} * &&& * & \\ & 1 &&& \\ && 1 && \\ * &&& * & \\ &&&& 1 \end{pmatrix}
\begin{pmatrix} * &&&& * \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ * &&&& * \end{pmatrix} \\
\times &\begin{pmatrix} * & * &&& \\ * & * &&& \\ && 1 && \\ &&& 1 & \\ &&&& 1 \end{pmatrix}
\begin{pmatrix} * && * && \\ & 1 &&& \\ * && * && \\ &&& 1 & \\ &&&& 1 \end{pmatrix}
\begin{pmatrix} * &&& * & \\ & 1 &&& \\ && 1 && \\ * &&& * & \\ &&&& 1 \end{pmatrix}
\begin{pmatrix} * &&&& * \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ * &&&& * \end{pmatrix}.
\end{aligned}
$$

We have the triangular factorization $k_2 = l_2 a_2 u_2$, where $l_2 = \exp(L_2)$ for

$$
L_2 = \begin{pmatrix}
0 & \sum_{\substack{i=4,\dots \\ j=1,\dots}} \bar{x}_i z^{-i} & \sum_{\substack{i=3,7,\dots \\ j=1,2,\dots}} \bar{x}_i z^{-j} & \sum_{\substack{i=2,6,12,\dots \\ j=1,2,3,\dots}} \bar{x}_i z^{-j} & \sum_{\substack{i=1,5,11,17,\dots \\ j=1,2,3,4,\dots}} \bar{x}_i z^{-j} \\
& 0 & \sum_{\substack{i=10,\dots \\ j=1,\dots}} \bar{x}_i z^{-j} & \sum_{\substack{i=9,14,\dots \\ j=1,2,\dots}} \bar{x}_i z^{-j} & \sum_{\substack{i=8,13,18,\dots \\ j=1,2,3,\dots}} \bar{x}_i z^{-j} \\
&& 0 & \sum_{\substack{i=16,\dots \\ j=1,\dots}} \bar{x}_i z^{-j} & \sum_{\substack{i=15,19,\dots \\ j=1,2,\dots}} \bar{x}_i z^{-j} \\
&&& 0 & \sum_{\substack{i=20,\dots \\ j=1,\dots}} \bar{x}_i z^{-j} \\
&&&& 0
\end{pmatrix},
$$

and $a_2 = \operatorname{diag}\left(a_{21}, a_{22}a_{21}^{-1}, a_{23}a_{22}^{-1}, a_{24}a_{23}^{-1}, a_{24}^{-1}\right)$, for

$$a_{21} = \prod_{i \neq 8-10,13-16,18-20,\ldots} N(\zeta_i), \qquad a_{22} = \prod_{i \neq 4,15,16,19,20,\ldots} N(\zeta_i),$$
$$a_{23} = \prod_{i \neq 3,4,7,10,15,20,\ldots} N(\zeta_i), \qquad a_{24} = \prod_{i \neq 2-4,6,7,9,10,12,14,16,\ldots} N(\zeta_i)^{-1},$$

where by $8-10$ we mean $8, 9, 10$, and so on. Finally, for $k = k_1^* \lambda k_2$ as in the theorem, we have

$$\det\left(|A(k)|^2\right) = \prod \left(1 + |\eta_i|^2\right)^{-P(i)} \exp\left(-5 \sum k |\chi_k|^2\right) \prod \left(1 + |\zeta_j|^2\right)^{P(j)},$$

where the infinite sequence P begins

$$\begin{aligned} P =& (1, 1, 1, 1, 2, 2, 2, 1, 1, 1, 3, 3, 2, 2, 1, 1, 4, 3, 2, 1, \\ & 5, 4, 3, 2, 6, 5, 4, 4, 3, 2, 7, 6, 5, 4, 3, 2, 8, 6, 4, 2, \ldots). \end{aligned}$$

1.3 Plan of this Dissertation

We now describe the contents of the rest of this document, indicating the dependence of the different chapters on each other. Notation which we refer to here will be introduced more thoroughly at the beginning of each chapter.

1.3.1 Chapter 2

In this mostly expository digression from the rest of the dissertation, we seek to put the finite-dimensional factorization appearing in Theorem 1.1.4 in a broader context. We also attempt to point out some of the unanswered questions regarding this factorization even in finite dimensions, and to motivate further results and questions about the infinite-dimensional version of this factorization.

The factorization and parametrization in which we are interested have a general form which appears multiple times in Lie theory. To a reduced word $s_{i_{l(w)}} \ldots s_{i_1}$ for an element w of the Weyl group W of G, we associate an ordered set $\left\{\beta_1, \ldots, \beta_{l(w)}\right\}$ of (positive) roots of G, and a product

$$g = g\left(\beta_{l(w)}, x_{l(w)}\right) \ldots g\left(\beta_1, x_1\right), \tag{1.3.1}$$

where $g(\beta_j, x_j) \in G$ and the parameters x_j take values in a parameter space X. The resulting map

$$X^{l(w)} \longrightarrow G : \boldsymbol{x} \longmapsto g$$

parametrizes some subset G_w which corresponds to w. In this chapter, after fixing notation for Lie algebras and groups, we discuss several such factorizations, and illustrate them with examples. Our focus is on reviewing and comparing parametrizations of the Birkhoff strata and Bruhat cells of a simple compact group K, studied in [26, 5, 6]. These include the generalization of Theorem 1.1.4 to arbitrary K.

We first review parametrizations for which $\beta_j = \alpha_{i_j}$, the simple root corresponding to the simple reflection s_{i_j}. When $X = \mathbb{C}$ and

$$g(\beta_j, x_j) = \iota_{\beta_j}\left(N(x_j)\begin{pmatrix} ix_j & 1 \\ 1 & -i\bar{x}_j \end{pmatrix}\right), \tag{1.3.2}$$

where ι_{β_j} denotes the root homomorphism corresponding to β_j, (1.3.1) defines a diffeomorphism from $\mathbb{C}^{l(w)}$ to the Schubert cell C_w of the flag manifold K/T. We explain the relationship of this parametrization to the topology and geometry of the flag manifold, and give a related formula for Haar measure, and an induced parametrization of the Bruhat cell C_w^K of K. We then briefly discuss the case $w = w_0$, $X = \mathbb{R}$, and

$$g(\beta_j, x_j) = \exp\left(x_j e_{\beta_j}\right), \tag{1.3.3}$$

where e_{β_j} denotes the positive root vector corresponding to β_j. This case yields a parametrization of the space of totally positive matrices. Finally, we study parametrizations of unipotent subgroups of $G = K^{\mathbb{C}}$ obtained by taking (1.3.3) and $X = \mathbb{C}$, and examine the heuristic relationship between these coordinates and the coordinates on Schubert cells.

Second, we give examples of parametrizations for which $\beta_j = s_{i_1} \dots s_{i_{j-1}}\alpha_{i_j}$, starting with coordinates on the Birkhoff stratum Σ_w of K/T obtained by taking $X = \mathbb{C}$ and

$$g(\beta_j, x_j) = \iota_{\beta_j}\left(N(x_j)\begin{pmatrix} 1 & x_j \\ -\bar{x}_j & 1 \end{pmatrix}\right).$$

Following Caine & Pickrell, we use the relationship between C_w and Σ_w to derive these coordinates and their properties; Theorem 1.1.4 is a special case of Proposition 2.4.1. We then put coordinates on certain unipotent subgroups of G by taking $X = \mathbb{C}$ and (1.3.3), and (heuristically) push these forward to Σ_w using the Iwasawa decomposition. Finally, we embed C_w into the top-dimensional stratum, Σ_1.

1.3.2 Chapter 3

In Chapter 3, we provide generalizations of Proposition 1.2.6 for affine Weyl groups of types A_{n-1}, B_n, C_n, and D_n, which correspond, respectively, to the groups $SU(n)$, $SO(2n+1)$, $Sp(n)$, and $SO(2n)$. This chapter is largely independent of the rest of the dissertation, as well as being more technical. We do, however, use the notation for affine Weyl groups introduced here in Chapter 4

Formulas for the words W_1 in terms of the generators s_i for the classical groups are given in the following table, where for a reduced word $\mathrm{w} = s_{i_1} \dots s_{i_{l(w)}}$ and a permutation σ of the integers, we use the notation

$$\mathrm{w}^{(k)} = s_{i_1+k}s_{i_2+k}\dots s_{i_l+k}, \qquad \mathrm{w}^{\sigma} = s_{\sigma(i_1)}\dots s_{\sigma\left(i_{l(w)}\right)}, \tag{1.3.4}$$

where, in the case of $SU(n)$, indices are to be read mod n, and in all other cases, indices are to be read mod $n+1$.

Type	$\mathrm{W_l}$
A_{n-1}	$\mathrm{W_l} = \prod_{i=0}^{\overleftarrow{l-1}} \mathrm{c_l} \qquad \mathrm{c_l} = \mathrm{s_l s_{l+1}\dots s_{n-1}s_n}$
B_n	$\mathrm{W_l} = \begin{cases} \prod_{i=1}^{\overleftarrow{l}} \mathrm{c_l}^{\sigma_0^i} & l \in [2,n] \\ s_0\mathrm{c_2} & l=1 \end{cases} \qquad \mathrm{c_l} = s_l s_{l+1}\dots s_{n-1}s_n s_{n-1}\dots s_3 s_2 s_0$
C_n	$\mathrm{W_l} = \begin{cases} (\mathrm{c_l})^l & l \in [1,n-1] \\ \prod_{i=0}^{n-1} \mathrm{d_i} & l=n \end{cases} \qquad \begin{matrix} \mathrm{c_l}{=}s_l s_{l+1}\dots s_{n-1}s_n s_{n-1}\dots s_1 s_0 \\ \mathrm{d_i}{=}s_i s_{i-1}\dots s_1 s_0 \end{matrix}$
D_n	$\mathrm{W_l} = \begin{cases} \prod_{i=1}^{\overleftarrow{l}} \mathrm{c_l}^{\sigma_0^i} & l \in [2,n] \\ s_0\mathrm{c_2} & l=1 \\ \prod_{i=2-(n \bmod 2)}^{\overrightarrow{\lfloor \frac{n-2}{2} \rfloor}} \mathrm{d_{2i}} & l=n \\ \prod_{i=2-(n \bmod 2)}^{\overrightarrow{\lfloor \frac{n-2}{2} \rfloor}} \mathrm{d_{2i}}^{\sigma_{n-1}} & l=n-1 \end{cases} \qquad \begin{matrix} \mathrm{c_l}{=}s_l s_{l+1}\dots s_{n-2}s_n s_{n-1}\dots s_3 s_2 s_0 \\ \mathrm{d_i}{=}s_{i-1}s_{i-2}\dots s_1 s_i s_{i-1}\dots s_2 s_0 \end{matrix}$

By combining the reduced words $\mathrm{W_l}$, we obtain reduced words $\mathrm{t_{h^*}}$ for translations t_{h^*} by the shortest element $h^* \in \check{T} \cap C_0$.

Type	h^*
A_n	$h^* = \begin{cases} \Theta_1 + \dots + \Theta_n = \check{\rho} & n \text{ even} \\ 2\Theta_1 + \dots + 2\Theta_n = 2\check{\rho} & n \text{ odd} \end{cases}$
B_n	$h^* = \begin{cases} \Theta_1 + \dots + \Theta_n = \check{\rho} & n = 0,3 \bmod 4 \\ 2\Theta_1 + \Theta_2 + \dots + \Theta_n = \check{\rho} + \Theta_1 & n = 1,2 \bmod 4 \end{cases}$
C_n	$h^* = \Theta_1 + \dots + \Theta_{n-1} + 2\Theta_n = \check{\rho} + \Theta_n$
D_n	$h^* = \begin{cases} \Theta_1 + \dots + \Theta_n = \check{\rho} & n = 0,1 \bmod 4 \\ 2\Theta_1 + \Theta_2 + \dots + \Theta_n = \check{\rho} + \Theta_1 & n = 2,3 \bmod 4 \end{cases}$

For type A_{n-1}, the proof is by induction. For the other types, our main tools in obtaining these results are permutation representations of the corresponding affine Weyl groups, homomorphisms mapping these groups into $\tilde{S}_n$ [2, 3, 11]. After fixing notation and giving some basic facts about affine Weyl groups, we present proofs for type A_{n-1}. Next, we present the permutation representations of the classical affine Weyl groups, and use them to prove results for types B_n, C_n, and D_n.

When possible, we connect our results to previous work on reduced words in affine Weyl groups. In 3.2, we discuss infinite elements and reduced words [8, 16], their geometric interpretation [36], and their relation to periodic infinite reduced words [8, 16]; these results are due to a number of researchers, including Cellini & Papi, Ito, and Shi. In 3.6, we discuss a method of Lam & Pylyavskyy [24] for finding reduced words for certain translations in $\tilde{S}_n$. We use the aforementioned permutation representations to extend this method to the other classical affine Weyl groups, and compare their method to ours.

1.3.3 Chapter 4

In the final chapter, we review the generalization to arbitrary K of Theorem 1.2.7. As we've noted, this material was recently published in [34], and our presentation will follow the presentation there closely. Aside from notation for Lie groups and Lie algebras, and affine Weyl groups, introduced in the first two chapters, this chapter is independent from the other two.

First, we establish notation for affine Kac-Moody Lie algebras, and for loop groups and their central extensions. Our notation for Kac-Moody algebras and loop groups differs slightly from that appearing in [34], and we point out these differences.

Second, we study finite and infinite products of the form

$$k_2 = \prod_{\alpha \in \mathbb{Z}\delta - \Phi^+} \iota_\alpha \left(N\left(\zeta_\alpha\right) \begin{pmatrix} 1 & -\bar{\zeta}_j \\ \zeta_j & 1 \end{pmatrix} \right),$$

where the factors are ordered by a total reflection ordering (a choice of reduced word for) the infinite inversion set $\mathbb{Z}\delta - \Phi^+$. In Theorem 4.3.2, we prove the equivalence of the three conditions characterizing loops of this type. We generalize condition (3) in Theorem 1.2.7 by characterizing these products in terms of their actions in representations of the finite-dimensional Lie group.

In Theorem 4.4.1, we prove that a loop $k \in LK$ has a triangular factorization if and only if it has a factorization $k = k_1^* \lambda k_2$. Finally, we indicate what variations of this factorization might look like, with a Proposition regarding $LSU(n)$.

Chapter 2

Factorization in Finite-Dimensional Lie Groups

In our notation for Lie groups and Lie algebras, we try to follow Carter [7] as closely as possible. For an elementary introduction to these topics, we can recommend Hall's excellent book [14]. More comprehensive expositions are given in Carter, as well as Knapp's classic [19].

2.1 Lie Algebras

Throughout, we define $\mathfrak{k}$ to be a simple, finite-dimensional Lie algebra of compact type with bracket $[\cdot,\cdot]$, $\mathfrak{g} := \mathfrak{k}^{\mathbb{C}}$ to be its complexification, and $x \to -x^*$ to be the anticomplex involution fixing $\mathfrak{k}$. We choose a maximal abelian (Cartan) subalgebra $\mathfrak{t}$ of $\mathfrak{k}$; $\mathfrak{h} := \mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra for $\mathfrak{g}$, and we define $\mathfrak{h}_{\mathbb{R}} := i\mathfrak{t}$. The complex dimension of $\mathfrak{h}$ is equal to the real dimension of $\mathfrak{t}$ and $\mathfrak{h}_{\mathbb{R}}$. This number is called the rank of $\mathfrak{g}$ or $\mathfrak{k}$ and denoted $r := \mathrm{rk}\mathfrak{g} = \mathrm{rk}\mathfrak{k}$.

2.1.1 Roots

For each $X, Y \in \mathfrak{g}$, we define $\mathrm{ad}_X(Y) = [X, Y]$. The roots α of $\mathfrak{g}$ or $\mathfrak{k}$ are elements of $\mathfrak{h}_{\mathbb{R}}^*$ such that there exists $X_\alpha \in \mathfrak{g}$ with $\mathrm{ad}_H(X_\alpha) = \alpha(H)X_\alpha$; the finite set of roots will be denoted Φ. We denote the set of simple positive roots by $\Delta = \left\{\alpha_i \,\middle|\, i \in [r]\right\}$, and these span $\mathfrak{h}_{\mathbb{R}}^*$. We denote the set of positive roots with respect to Δ by Φ^+, and the set of negative roots by Φ^-. The height of $\alpha = \sum_{i\in[r]} n_i\alpha_i \in \Phi$ is $\mathrm{ht}(\alpha) := \sum_{i\in[r]} n_i$; the unique highest root is denoted by θ. We will also make use of the vector

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha.$$

The root lattice of $\mathfrak{g}$ is

$$\hat{R} = \bigoplus_{i\in[r]} \mathbb{Z}\alpha_i.$$

2.1.2 Coroots

A negative definite bilinear form on $\mathfrak{h}_{\mathbb{R}}$, called the Killing form, is defined by $\kappa(\cdot,\cdot) := \mathrm{tr}\left(\mathrm{ad}_{(\cdot)}\,\mathrm{ad}_{(\cdot)}\right)$. We also denote the dual form on $\mathfrak{h}_{\mathbb{R}}^*$ by $\kappa(\cdot,\cdot)$. The dual Coxeter number is $g = \frac{1}{4}\kappa(\theta,\theta)$, and we let $\langle\cdot,\cdot\rangle := \frac{1}{2g}\kappa(\cdot,\cdot)$ denote this bilinear form on both $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$, so that $\langle\theta,\theta\rangle = 2$. The bilinear form $\langle\cdot,\cdot\rangle$ induces the following correspondences (as sets) between $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$:

$$\begin{aligned} h'_{(\cdot)}: \quad & \mathfrak{h}_{\mathbb{R}}^* \longrightarrow \mathfrak{h}_{\mathbb{R}} \\ & \lambda \longmapsto h'_\lambda: \quad \langle h'_\lambda, \cdot\rangle = \lambda(\cdot), \end{aligned} \qquad \begin{aligned} h_{(\cdot)}: \quad & \mathfrak{h}_{\mathbb{R}}^* \longrightarrow \mathfrak{h}_{\mathbb{R}} \\ & \lambda \longmapsto h_\lambda: \quad \langle h_\lambda, \cdot\rangle = \tfrac{2}{\langle\lambda,\lambda\rangle}\lambda(\cdot). \end{aligned}$$

Then $\left\{h_\alpha \,\middle|\, \alpha\in\Phi\right\}$ is the set of coroots, $\left\{h_\alpha \middle| \alpha\in\Phi^+\right\}$ is the set of positive coroots, the simple positive coroots are $\left\{h_i := h_{\alpha_i} \,\middle|\, i\in[r]\right\}$, and

$$\check{\rho} = \frac{1}{2}\sum_{\alpha\in\Phi^+} h_\alpha.$$

The coroot lattice of $\mathfrak{g}$ is

$$\check{T} = \bigoplus_{i\in[r]} \mathbb{Z}h_i.$$

2.1.3 Weights and Coweights

A weight of $\mathfrak{g}$ is any element λ of $\mathfrak{h}_{\mathbb{R}}^*$ having $\lambda(h_i)\in\mathbb{Z}$ for all $i\in[r]$. We define the set of fundamental weights of $\mathfrak{g}$ to be the set

$$\left\{\Lambda_i\in\mathfrak{h}_{\mathbb{R}}^* \,\middle|\, \Lambda_i(h_j) = \delta_{ij},\ i,j\in[r]\right\}.$$

The weight lattice of $\mathfrak{g}$ is

$$\hat{T} = \bigoplus_{i\in[r]} \mathbb{Z}\Lambda_i.$$

The set of fundamental coweights of $\mathfrak{g}$ is the set $\left\{\Theta_i\in\mathfrak{h}_{\mathbb{R}} \,\middle|\, \alpha_j(\Theta_i) = \delta_{ij},\ i,j\in[r]\right\}$, and

$$\Theta_i = \frac{\langle\alpha_i,\alpha_i\rangle}{2}h'_{\Lambda_i} = \frac{\langle\alpha_i,\alpha_i\rangle\langle\Lambda_i,\Lambda_i\rangle}{4}h_{\Lambda_i}.$$

The coweight lattice of $\mathfrak{g}$ is

$$\check{R} = \bigoplus_{i\in[r]} \mathbb{Z}\Theta_i.$$

The lattices $\hat{T}$ and $\check{T}$ are in duality, as are the lattices $\hat{R}$ and $\check{R}$. We will make use of the identities

$$\rho = \sum_{i\in[r]}\Lambda_i, \qquad \check{\rho} = \sum_{i\in[r]}\Theta_i.$$

2.1.4 Simply-Laced Groups

We say $\mathfrak{g}$ is simply-laced if every root α has the same length. In this case, $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Phi$, and for $i \in [r]$,

$$h_i = h'_{\alpha_i}, \qquad \Theta_i = h'_{\Lambda_i},$$

so that

$$h'_{\hat{T}} = \check{T}, \qquad h'_{\hat{R}} = \check{R}.$$

In other words, under the isomorphism of $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}^*_{\mathbb{R}}$ induced by $\langle \cdot, \cdot \rangle$, roots are identified with coroots, and weights are identified with coweights. In particular, $\check{\rho} = h'_\rho$.

2.1.5 The Weyl Group

The Weyl group W of $\mathfrak{g}$ is the finite Coxeter group generated by the reflections $\left\{ s_i \;\middle|\; i \in [r] \right\}$, which act on $\alpha \in \mathfrak{h}^*_{\mathbb{R}}$ by

$$s_i \alpha = \alpha - \alpha\left(h_i\right) \alpha_i,$$

and on $h \in \mathfrak{h}_{\mathbb{R}}$ by

$$s_i h = h - \alpha_i(h) h_i.$$

A word of length n for an element $w \in W$ is a sequence of simple reflections $s_{i_1}, \ldots, s_{i_n}$ such that $w = s_{i_n} \ldots s_{i_1}$. The minimum length of a word for w is called the length of w, and denoted $l(w)$. Any word of length $l(w)$ for w is called a reduced word for w. To distinguish a reduced word from the Weyl group element it "spells", we will use Roman typeface. Thus, when we write $\mathrm{w} = s_{i_{l(w)}} \ldots s_{i_1}$, w denotes the sequence $s_{i_1}, \ldots, s_{i_{l(w)}}$, which gives a reduced word for w. The longest element of W will be denoted w_0; it sends Φ^+ to Φ^-.

The inversion set of $w \in W$ is

$$\mathrm{Inv}(w) = \left\{ \alpha \in \Phi^+ \;\middle|\; w\alpha \in \Phi^- \right\} = w^{-1}\Phi^- \cap \Phi^+.$$

We have $\mathrm{Inv}(w_0) = \Phi^+$.

Given a reduced word $\mathrm{w} = s_{i_{l(w)}} \ldots s_{i_1}$, for $m \in [l(w)]$ define $\mathrm{w}_m = s_{i_m} \ldots s_{i_1}$, and

$$\tau_m(\mathrm{w}) = \mathrm{w}_{m-1}^{-1} \alpha_{i_m} = s_{i_1} \ldots s_{i_{m-1}} \alpha_{i_m}.$$

Lemma 2.1.1. *(Proposition 4.4.6, [3]) For $m \in [l(w)]$, $\tau_m(\mathrm{w}) \in \Phi^+$. For $j < m$, $\mathrm{w}_j \tau_m(\mathrm{w}) \in \Phi^+$, while for $j \geq m$, $\mathrm{w}_j \tau_m(\mathrm{w}) \in \Phi^-$. Thus,*

$$\left\{ \tau_m(\mathrm{w}) \right\}_{m=1}^{l(w)} = \mathrm{Inv}(w),$$

and $l(w) = \#\left(\mathrm{Inv}(w)\right)$.

Characterizations of inversion sets, and orderings $\{\tau_m(\mathrm{w})\}_{m=1}^{l(w)}$, were provided by Dyer [10].

Definition 2.1.2. A subset $I \subset \Phi^+$ is called biconvex if

1. whenever $\alpha, \beta \in I$, and $\alpha+\beta \in \Phi^+$, then $\alpha+\beta \in I$, and
2. whenever $\alpha, \beta \in \Phi^+$ and $\alpha+\beta \in I$, either $\alpha \in I$ or $\beta \in I$.

Definition 2.1.3. An order $\prec$ on a biconvex subset $I \subset \Phi^+$ is a total reflection order if

1. whenever $\alpha, \beta, \alpha+\beta \in I$, either $\alpha \prec \alpha+\beta \prec \beta$ or $\beta \prec \alpha+\beta \prec \alpha$, and
2. whenever $\alpha, \alpha+\beta \in I$ and $\beta \notin I$, then $\alpha \prec \alpha+\beta$.

Proposition 2.1.4. *The biconvex subsets of Φ^+ are in correspondence with the elements of W, such that $w \in W$ corresponds to* $\mathrm{Inv}(w)$*. The total reflection orders on* $\mathrm{Inv}(w)$ *are in correspondence with the reduced words for* w*. The order $\prec_{\mathrm{w}}$ corresponding to* w *is defined by setting $\tau_i(\mathrm{w}) \prec_{\mathrm{w}} \tau_j(\mathrm{w})$ whenever $i<j$.*

Given the central importance of inversion sets in this document, we provide a proof of this result.

Proof. First, we show that every set of the form $\mathrm{Inv}(w)$ is biconvex, and every ordering $\prec_{\mathrm{w}}$ is a total reflection ordering, by induction on $l(w)$. Certainly $\alpha_i = \mathrm{Inv}\left(s_i\right)$ is biconvex, and, vacuously, the "ordering" on α_i is a total reflection ordering. Assume now that $\mathrm{Inv}(w)$ is biconvex, and every ordering $\prec_{\mathrm{w}}$ is a total reflection ordering, for all $w \in W$ with $l(w) \leq k < l\left(w_0\right)$.

Let $l(w)=k$, and let ws_i be reduced. Then

$$\mathrm{Inv}\left(ws_i\right) = \{\alpha_i\} \sqcup s_i \mathrm{Inv}(w), \tag{2.1.5}$$

and $\alpha_i \notin \mathrm{Inv}(w)$. Since s_i permutes $\Phi^+ \setminus \{\alpha_i\}$ and s_i is linear, $s_i\mathrm{Inv}(w)$ is biconvex. Now, if $\beta \in \mathrm{Inv}(w)$ and $\alpha_i + s_i\beta \in \Phi^+$, then again because s_i permutes $\Phi^+ \setminus \{\alpha_i\}$, $\beta - \alpha_i \in \Phi^+$. Since $\alpha_i \notin \mathrm{Inv}(w)$, $w\alpha_i \in \Phi^+$ and $w\left(\beta-\alpha_i\right) = w\beta - w\alpha_i \in \Phi^-$, so that $\beta-\alpha_i \in \mathrm{Inv}(w)$ and $\alpha_i + s_i\beta \in \mathrm{Inv}\left(ws_i\right)$. Thus $\mathrm{Inv}\left(ws_i\right)$ is biconvex.

Next, for any reduced word w, $\tau_1\left(\mathrm{w}s_i\right) = \alpha_i$ and for $j \geq 2$, $\tau_j\left(\mathrm{w}s_i\right) = s_i\tau_{j-1}(\mathrm{w})$. Now, assume $\alpha, \alpha+\beta, \beta \in I$. First, assume $\alpha \neq \alpha_i \neq \beta$. Then, we have $\alpha = s_i\gamma$, $\beta = s_i\delta$ for $\gamma, \delta, \gamma+\delta \in \mathrm{Inv}(w)$, and $\gamma \prec_{\mathrm{w}} \gamma+\delta \prec_{\mathrm{w}} \delta$ implies $\alpha \prec_{\mathrm{w}s_i} \alpha+\beta \prec_{\mathrm{w}s_i} \beta$. Next, assume $\alpha = \alpha_i$. Then $\alpha_i \prec_{\mathrm{w}s_i} \alpha_i+\beta$ and $\alpha_i \prec_{\mathrm{w}} \beta$. Furthermore, $\beta = s_i\gamma$ for some $\gamma \in \mathrm{Inv}\left(w\right)$, and $s_i\left(\alpha_i+\beta\right) = \gamma-\alpha_i$, while $s_i\beta = \gamma$. Thus, $\gamma-\alpha_i, \gamma \in \mathrm{Inv}(w)$, while $\alpha_i \notin \mathrm{Inv}(w)$, so that $\gamma-\alpha_i \prec_{\mathrm{w}} \gamma$, and $\alpha+\beta \prec_{\mathrm{w}s_i} \beta$. Thus $\prec_{\mathrm{w}s_i}$ is a total reflection ordering.

Now, we show that every biconvex set is the inversion set of some $w \in W$. Indeed, (2.1.5) shows us how to construct a reduced word w for w inductively. If $\#(I)=1$, then the definition of biconvexity forces $I = \{\alpha_i\}$ for some simple root α_i, and so $I = \mathrm{Inv}\,(s_i)$. Now, assume that for all biconvex sets I with $\#(I) \leq k < l\,(w_0)$, there exists some unique $w_I \in W$ such that $I = \mathrm{Inv}\,(w_I)$. Let J be biconvex with $\#(J) = k+1$. The definition of biconvexity guarantees that J contains a simple root α_i, and we have

$$\begin{aligned} J &= \{\alpha_i\} \sqcup J \backslash \{\alpha_i\} = \{\alpha_i\} \sqcup s_i s_i \left(J \backslash \{\alpha_i\}\right) \\ &= \{\alpha_i\} \sqcup s_i \mathrm{Inv}\left(w_{s_i(J\backslash\{\alpha_i\})}\right) = \mathrm{Inv}\left(w_{s_i(J\backslash\{\alpha_i\})} s_i\right). \end{aligned}$$

This argument also basically shows that every total reflection order on an inversion set I corresponds to a reduced word for the corresponding Weyl group element. We simply point out that the first element in any total reflection ordering of I is necessarily some simple root $\alpha_i \in I$. Next, the "push forward" by s_i onto $s_i I \backslash \{\alpha_i\}$ of a total reflection ordering on I is also a total reflection ordering. □

2.1.6 Root Spaces and Triangular Decomposition

The root spaces are the eigenspaces of $\mathrm{ad}_{\mathfrak{h}_\mathbb{R}} = \left\{\mathrm{ad}_h \;\middle|\; h \in \mathfrak{h}_\mathbb{R}\right\}$,

$$\mathfrak{n}^{\pm}_{\alpha} = \left\{X \in \mathfrak{g} \middle| [h, X] = \pm\alpha(h) X \;\forall h \in \mathfrak{h}_\mathbb{R}\right\}.$$

We have

$$\left[\mathfrak{n}^{\pm}_{\alpha}, \mathfrak{n}^{\pm}_{\beta}\right] = \mathfrak{n}^{\pm}_{\alpha+\beta}. \tag{2.1.6}$$

The fact that $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$ implies that $\left(\mathfrak{n}^{\pm}_{\alpha}\right)^* = \mathfrak{n}^{\mp}_{\alpha}$.

The subalgebras of $\mathfrak{g}$

$$\mathfrak{n}^{\pm} := \bigoplus_{\alpha \in \Phi^+} \mathfrak{n}^{\pm}_{\alpha}$$

are nilpotent, and $\left(\mathfrak{n}^+\right)^* = \mathfrak{n}^-$. There is a decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

called the triangular decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

The formula (2.1.6) and the biconvexity of $\mathrm{Inv}(w)$ imply the following.

Proposition 2.1.7. *For $w \in W$, the spaces*

$$\mathfrak{n}^+_w = \bigoplus_{\alpha \in \mathrm{Inv}(w)} \mathfrak{n}^+_\alpha, \qquad \mathfrak{n}^-_w = \bigoplus_{\alpha \in \mathrm{Inv}(w)} \mathfrak{n}^-_\alpha$$

are subalgebras of $\mathfrak{n}^+$ and $\mathfrak{n}^-$.

2.1.7 Chevalley Generators

The Chevalley generators of $\mathfrak{g}$,

$$\left\{e_i, e_{-i} \in \mathfrak{g} \;\middle|\; i \in [r]\right\},$$

alware eigenvectors of Δ, so that for all $h \in \mathfrak{h}_{\mathbb{R}}$,

$$[h, e_i] = \alpha_i(h)e_i, \qquad [h, e_{-i}] = -\alpha_i(h)e_{-i}.$$

They satisfy the additional relation

$$[e_i, e_{-i}] = h_i.$$

Then $\left\{e_i, e_i, h_i \;\middle|\; i \in [r]\right\}$ generates $\mathfrak{g}$.

2.1.8 Root Isomorphisms

For each $i \in [r]$, we fix the root isomorphism $\iota_i : \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathfrak{g}$ determined by

$$\iota_i\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = e_{-i}, \qquad \iota_i\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = h_i, \qquad \iota_i\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = e_i.$$

2.2 Lie Groups

We denote the simply-connected groups corresponding to $\mathfrak{g}$ and $\mathfrak{k}$ by G and K, respectively. Let $N^{\pm} = \exp(\mathfrak{n}^{\pm})$, $H = \exp(\mathfrak{h})$, $T = \exp(\mathfrak{t})$, and $A = \exp(\mathfrak{h}_{\mathbb{R}})$. The Borel subgroups of G with respect to H are $B^{\pm} = HN^{\pm}$. For $i \in [r]$, we define $\iota_i : SL(2, \mathbb{C}) \longrightarrow G$ by $\iota_i(\exp(X)) = \exp(\iota_i(X))$ for each $X \in \mathfrak{sl}(2, \mathbb{C})$.

2.2.1 The Iwasawa Decomposition

The Iwasawa decomposition for G is

$$G = KAN^+,$$

so that $g \in G$ has a unique Iwasawa decomposition

$$g = \mathbf{k}(g)\mathbf{a}(g)\mathbf{n}(g).$$

Note that ι_α and $\mathbf{k}$ commute.

2.2.2 The Weyl Group

The Weyl group for $\mathfrak{g}$ is isomorphic to the quotient group $N_G(H)/H \simeq N_K(T)/T$. A representative for $w \in W$ will be denoted with bold type $\mathbf{w}$. Such a representative acts on $g \in G$ by conjugation, and on $X \in \mathfrak{g}$ by the derivative of conjugation at the identity, denoted Ad. Examples of representatives are the elements

$$\mathbf{s_i} = \iota_i\left(\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}\right).$$

A reduced word $\mathrm{w} = s_{i_{l(w)}} \ldots s_{i_1}$ for $w \in W$, along with a set of representatives $\mathbf{s_i}$ for the simple reflections, gives a set of representatives for w_m, namely $\mathbf{w_m} = \mathbf{s_{i_m}} \ldots \mathbf{s_{1_1}}$. Given $\mathbf{w} = \mathbf{s_{i_{l(w)}}} \ldots \mathbf{s_{i_1}}$, we define the root homomorphisms $\iota_{\tau_j(\mathbf{w})} : \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{g}$ by

$$\iota_{\tau_j(\mathbf{w})}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \mathrm{Ad}_{\mathbf{w_{j-1}}}\, e_{-i_j}, \qquad \iota_{\tau_j(\mathbf{w})}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \mathrm{Ad}_{\mathbf{w_{j-1}}}\, h_{i_j},$$

$$\iota_{\tau_j(\mathbf{w})}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \mathrm{Ad}_{\mathbf{w_{j-1}}}\, e_{i_j}.$$

We define $\iota_{\tau_j(\mathbf{w})} : SL(2,\mathbb{C}) \longrightarrow G$ by $\iota_{\tau_j(\mathbf{w})} \exp(X) = \exp\left(\iota_{\tau_j(\mathbf{w})}(X)\right)$ for each $X \in \mathfrak{sl}(2,\mathbb{C})$.

For $w \in W$, we define the subgroups

$$N_w^+ = \exp\left(\mathfrak{n}_w^+\right) \cong N^+ \cap \mathbf{w}^{-1} N^- \mathbf{w}, \qquad N_w^- = \exp\left(\mathfrak{n}_w^-\right) \cong N^- \cap \mathbf{w}^{-1} N^+ \mathbf{w},$$

as well as the "opposite" subgroups

$$\bar{N}_w^+ = N^+ \backslash N_w^+ \cong N^+ \cap \mathbf{w}^{-1} N^+ \mathbf{w}, \qquad \bar{N}_w^- = N^- \backslash N_w^- \cong N^- \cap \mathbf{w}^{-1} N^- \mathbf{w}.$$

These do not depend on the choice of representative $\mathbf{w}$. We have $N_e^\pm = \emptyset$, and $N_{w_0}^\pm = N^\pm$.

Example 2.2.1. For $SL(3,\mathbb{C})$, the Weyl group is the group of 3×3 permutation matrices. It has six elements, and two generators,

$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

which act by conjugation on elements of $SL(3,\mathbb{C})$. We also have

$$N^+_{s_1} = \left\{ \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \qquad N^+_{s_2} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\},$$
$$N^+_{s_1 s_2} = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \qquad N^+_{s_2 s_1} = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

2.2.3 Triangular Factorization, Bruhat and Birkhoff Decompositions

An element $g \in N^- H N^+$ has a unique triangular decomposition

$$g = \mathbf{l}(g)\mathbf{d}(g)\mathbf{u}(g), \quad \text{where} \quad \mathbf{d}(g) = \mathbf{m}(g)\mathbf{a}(g) = \prod_{j=1}^{r} \sigma_j(g)^{h_j},$$

and $\sigma_i(g) = \phi_{\Lambda_i}(\pi_{\Lambda_i}(g)v_{\Lambda_i})$ is the fundamental matrix coefficient for the highest weight vector corresponding to Λ_i.

More generally, the Birkhoff decomposition for G is

$$G = \bigsqcup_{w \in W} \Sigma^G_w, \qquad \Sigma^G_w := N^- \mathbf{w} H N^+.$$

The Bruhat decomposition for G is

$$G = \bigsqcup_{w \in W} C^G_w, \qquad C^G_w := N^+ \mathbf{w} H N^+.$$

Thus, $N^- H N^+ = \Sigma^G_1 = C^G_{w_0}$. Neither the Bruhat cells C^G_w nor the Birkhoff strata Σ^G_w depend on the choice of representative $\mathbf{w}$.

We can intersect both of these decompositions with K, obtaining Bruhat and Birkhoff decompositions for K:

$$K = \bigsqcup_{w \in W} C^K_w = \bigsqcup_{w \in \dot{W}} \Sigma^K_w,$$

$$C^K_w := N^+ \mathbf{w} H N^+ \cap K, \qquad \Sigma^K_w := N^- \mathbf{w} H N^+ \cap K.$$

In finite dimensions, the Bruhat and Birkhoff decompositions are equivalent. More specifically, we have the following.

Lemma 2.2.2. *C_w and $\Sigma_{w_0 w}$ are diffeomorphic, with the diffeomorphism given by left translation by $\mathbf{w_0}$.*

Proof. We have

$$\mathbf{w_0}C_w = \mathbf{w_0}N^+{\mathbf{w_0}}^{-1}\mathbf{w_0}\mathbf{w}HN^+ = N^-\mathbf{w_0}\mathbf{w}HN^+ = \Sigma_{w_0 w}.$$

□

Example 2.2.3. For $SL(3,\mathbb{C})$, we have

$$N^- = \left\{\begin{pmatrix}1 & 0 & 0\\ * & 1 & 0\\ * & * & 1\end{pmatrix}\right\}, \qquad H = \left\{\begin{pmatrix}* & & \\ & * & \\ & & *\end{pmatrix}\right\}, \qquad N^+ = \left\{\begin{pmatrix}1 & * & *\\ & 1 & *\\ & & 1\end{pmatrix}\right\}.$$

The Birkhoff strata are

$$\begin{aligned}
\Sigma_1 &= N^-HN^+ = \left\{\begin{pmatrix}* & * & *\\ * & * & *\\ * & * & *\end{pmatrix}\right\},\\
\Sigma_{s_1} &= N^-\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}HN^+ = \left\{\begin{pmatrix}0 & * & *\\ * & * & *\\ * & * & *\end{pmatrix}\right\},\\
\Sigma_{s_2} &= N^-\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}HN^+ = \left\{\begin{pmatrix}z & w & *\\ \mu z & \mu w & *\\ * & * & *\end{pmatrix}\right\},\\
\Sigma_{s_1s_2} &= N^-\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}HN^+ = \left\{\begin{pmatrix}0 & 0 & *\\ * & * & *\\ * & * & *\end{pmatrix}\right\},\\
\Sigma_{s_2s_1} &= N^-\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}HN^+ = \left\{\begin{pmatrix}0 & * & *\\ 0 & * & *\\ * & * & *\end{pmatrix}\right\},\\
\Sigma_{w_0} &= N^-\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}HN^+ = \left\{\begin{pmatrix}0 & 0 & *\\ 0 & * & *\\ * & * & *\end{pmatrix}\right\}.
\end{aligned}$$

The Bruhat cells are

$$C_1 = N^+HN^+ = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\},$$

$$C_{s_1} = N^+ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} HN^+ = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\},$$

$$C_{s_2} = N^+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} HN^+ = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\},$$

$$C_{s_1s_2} = N^+ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} HN^+ = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \right\},$$

$$C_{s_2s_1} = N^+ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} HN^+ = \left\{ \begin{pmatrix} * & * & * \\ z & w & * \\ \mu z & \mu w & * \end{pmatrix} \right\},$$

$$C_{w_0} = N^+ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} HN^+ = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \right\}.$$

2.2.4 The Flag Manifold G/B^+

The (generalized) flag manifold of G is the quotient space G/B^+. This space can also be presented as K/T. The diffeomorphism between the two quotient spaces is provided by the Iwasawa decomposition, so we identify gB^+ and $\mathbf{k}(g)T$.

By intersecting the Bruhat cells and Birkhoff strata with G/B^+, we obtain cell decompositions of the flag manifold:

$$G/B^+ = \bigsqcup_{w \in \dot{W}} C_w = \bigsqcup_{w \in \dot{W}} \Sigma_w,$$

$$C_w := N^+\mathbf{w}B^+/B^+, \qquad \Sigma_w := N^-\mathbf{w}B^+/B^+.$$

Alternatively, C_w and Σ_w are the orbits of the unipotent subgroups $N^\pm$ in G/B^+. The cells C_w are called the Schubert cells of the flag manifold, and their closures $\bar{C}_w$ are algebraic varieties representing the homology classes of G/B^+, denoted Schubert varieties.

The two presentations of the flag manifold lead to two presentations of the Schubert cells and the Birkhoff strata.

Lemma 2.2.4. *There are diffeomorphisms*

$$N^+_{w^{-1}} \to C_w \to N^+_{w^{-1}}\mathbf{w}AN^+ \cap K, \qquad \bar{N}^-_{w^{-1}} \to \Sigma_w \to \bar{N}^-_{w^{-1}}\mathbf{w}AN^+ \cap K.$$

Proof. The first pair of diffeomorphisms result from the fact that $N^\pm$ acts transitively on the coset space $N^\pm \mathbf{w}B^+/B^+$. The stabilizer of the point $\mathbf{w}B^+$ is $N^\pm \cap \mathbf{w}N^+\mathbf{w}^{-1}$. This leaves the quotient isomorphic to $N^\pm / (N^\pm \cap \mathbf{w}N^+\mathbf{w}^{-1})$, which in turn is isomorphic to $N^\pm \cap \mathbf{w}N^-\mathbf{w}^{-1}$. The second two diffeomorphisms result from the fact that N^+ is normalized by T. $\square$

We note for future use that these two presentations are related by diffeomorphisms

$$\begin{array}{ccc} N^+_{w^{-1}} & \longrightarrow & N^+_{w^{-1}}\mathbf{w}AN^+ \cap K \\ u & \longmapsto & \mathbf{k}\left(u\mathbf{w}\right), \end{array} \qquad \begin{array}{ccc} \bar{N}^-_{w^{-1}} & \longrightarrow & \bar{N}^-_{w^{-1}}\mathbf{w}AN^+ \cap K \\ l & \longmapsto & \mathbf{k}\left(l\mathbf{w}\right). \end{array}$$

Lemma 2.2.4 allows us to describe the Bruhat cells and Birkhoff strata in G more completely.

Lemma 2.2.5. *The multiplication map induces diffeomorphisms*

$$N^+_{w^{-1}} \times \{\mathbf{w}\} \times H \times N^+ \longrightarrow C^G_w, \qquad \bar{N}^-_{w^{-1}} \times \{\mathbf{w}\} \times H \times N^+ \longrightarrow \Sigma^G_w.$$

In particular, we have diffeomorphisms

$$N^- \longrightarrow \Sigma_1 \longrightarrow N^-AN^+ \cap K.$$

This can be seen directly. First, $gB^+ = \mathbf{l}(g)B^+$. Furthermore, if $l \in N^+$ has Iwasawa decomposition $l = \mathbf{k}(l)\mathbf{a}(l)\mathbf{n}(l)$, then $\mathbf{k}(l)$ has the triangular factorization

$$\mathbf{k}(l) = l\mathbf{n}(l)^{-1}\mathbf{a}(l)^{-1} = l\mathbf{a}(l)^{-1}\left(\mathbf{a}(l)\mathbf{n}(l)^{-1}\mathbf{a}(l)^{-1}\right),$$

so that $\mathbf{k}(l) \in N^-AN^+ \cap K$.

2.3 Parametrizations with Sequences of Simple Roots

A reduced word $\mathrm{w} = s_{i_{l(w)}} \dots s_{i_1}$ for an element $w \in W$ determines a sequence of simple roots $\left\{\alpha_{i_j}\right\}_{j=1}^{l(w)}$. Such sequences of simple roots play a role in parametrizations of several important objects in Lie theory and algebra, namely Schubert cells, Bruhat cells, and totally positive matrices.

2.3.1 Parametrizations of Schubert and Bruhat Cells

The first such factorization is incipient in the work of Bott & Samelson [4], who used the map

$$\iota_{i_{\#(\Phi^+)}}\left(SU(2)/S^1\right) \times \ldots \times \iota_{i_1}\left(SU(2)/S^1\right) \longrightarrow K/T,$$

where S^1 is identified with the diagonal subgroup of $SU(2)$, to study the homology of the flag manifold. The homology of K/T is generated by the two-cells appearing on the left-hand side of the product above, and in fact $H_2(K/T, \mathbb{Z}) = \mathbb{Z}^r$, where r is the rank of K. The first relations between the homology classes appear in dimension 4, and can be given in terms of the structure constants of $\mathfrak{g}$ [4].

Soibelman was the first to write down the implied parametrization of the Bruhat cell C_w [37], which was also used by Lu [26]. We parametrize C_w by parametrizing $\iota_{i_j}(SU(2)/S^1)$ by a single complex coordinate ζ_j. Defining the normalization factor $N(\zeta) = (1 + |\zeta|^2)^{-1/2}$, we have the following.

Theorem 2.3.1. *For each reduced decomposition* $\mathrm{w} = s_{i_{l(w)}} \ldots s_{i_1}$, *there is a diffeomorphism*

$$\mathbb{C}^{l(w)} \longrightarrow C_w : \boldsymbol{\zeta} \longmapsto \overleftarrow{\prod_{i=1}^{l(w)}} \iota_{i_j}\left(\sqrt{-1}N(\zeta_j)\begin{pmatrix}\zeta_j & 1 \\ 1 & -\bar{\zeta}_j\end{pmatrix}\right). \tag{2.3.2}$$

This parametrization has some topological significance. A beautiful fact about flag manifolds is that the Schubert cells form a basis for the homology of K/T. This is in line with the results of Bott & Samelson, since

$$C_{s_i} = \iota_i\left(SU(2)/S^1\right).$$

The inclusion of a cell $C_{w'}$ in the variety $\bar{C}_w$ occurs if and only if $w \geq w'$ in the Bruhat ordering on W (p. 29, [3]). Thus, the topology of K/T can be studied via the Bruhat ordering on W. The meaning of $w \geq w'$ is that we can obtain a reduced expression for w' by leaving out some of the simple reflections, say those with indices $I_1, \ldots, I_{l(w)-l(w')}$, in a reduced decomposition of w. In this case, we reach $C_{w'}$ by going to infinity along the negative imaginary axis in the coordinates $\zeta_{I_1}, \ldots, \zeta_{I_{l(w)-l(w')}}$ for C_w. Taking all the coordinates to infinity in this way, we reach the point corresponding to the identity. Conversely, the coordinate $\boldsymbol{\zeta} = 0$ corresponds to the point

$$\mathbf{w} = \iota_{i_{l(w)}}\left(\begin{pmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{pmatrix}\right) \ldots \iota_{i_1}\left(\begin{pmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{pmatrix}\right).$$

This coordinate system has several other nice properties. Pickrell [30] showed the following.

Theorem 2.3.3. *In the coordinates (2.3.2), the diagonal component of* $k \in C_w$ *is the product of the diagonal components of its factors,*

$$\mathbf{a}(k) = \overleftarrow{\prod_{i=1}^{l(w)}} \iota_{i_j} \left(\begin{pmatrix} N(\zeta_j)^{-1} & \\ & N(\zeta_j) \end{pmatrix} \right).$$

The coordinates also have significance to the Poisson geometry of K/T. A Poisson structure on a manifold M is a bilinear map

$$\{\cdot,\cdot\}_\Pi : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

which is antisymmetric, satisfies a Leibniz rule in each slot, and satisfies the Jacobi identity. It can be represented as a bivector field $\Pi \in \wedge^2 TM$, and this bivector field induces a map

$$\Pi^\# : T^*M \longrightarrow TM : \omega \longmapsto \Pi(\omega, \cdot).$$

The image of $\Pi^\#$ is an involutive distribution on M, and its integrable submanifolds are symplectic manifolds, called the symplectic leaves of Π.

The Lu-Weinstein Poisson structure or the standard Poisson Lie group structure is a Poisson structure on K denoted here by Π_{LW}^K [27]. Π_{LW}^K is invariant under right translation by T, so it descends to a Poisson structure on K/T, which we will also denote Π_{LW}. The symplectic leaves of Π_{LW} are exactly the Schubert cells C_w.

The forms dual to the cells C_w, which give a basis for the de Rham cohomology of K/T, were discovered by Kostant [21]. Lu used the above factorization to give a Poisson geometric interpretation to Kostant's harmonic forms [26] with respect to Π_{LW}. Using the multiplicativity of Π_{LW}, she showed the following.

Theorem 2.3.4. *The symplectic form* Ω_w *induced by* Π_{LW} *on* C_w *is diagonal in the coordinates (2.3.2), and given by the expression*

$$\Omega_w = \sum_{j=1}^{n} \frac{2\sqrt{-1}}{\langle \alpha_{i_j}, \alpha_{i_j} \rangle} \frac{1}{1+|\zeta_j|^2} d\zeta_j \wedge d\bar{\zeta}_j.$$

In the process, she showed that Haar measure μ_w on the subgroup N_w^+ is a product measure in the ζ coordinates.

Theorem 2.3.5. *The Haar measure on* N_w^+ *in the coordinates (2.3.2) is given by the expression*

$$d\mu_w = \prod_{j=1}^{l(w)} \frac{1}{2\pi\sqrt{-1}} \rho\left(h_{\tau_j(\mathrm{w})}\right) \left(1+|\zeta_j|^2\right)^{4\rho\left(h_{\tau_j(\mathrm{w})}\right)-1} d\zeta_j d\bar{\zeta}_j.$$

These results can be "lifted" to the Bruhat cells C_w^K of K, by pairing the ζ coordinates with coordinates on T. All the results about C_w still hold for C_w^K, more or less. The symplectic leaves of Π_{LW}^K foliate the cells C_w^K.

One result of these facts is a product expression for Haar measure μ_K on K, since C_{w_0} is a set of full measure with respect to μ_K. Let μ_{N^+} and μ_T be the Haar measures on N^+ and T, respectively, and let $\{\theta_i\}_{i=1}^r$ be a suitable set of coordinates on T. The following appears in [26].

Theorem 2.3.6. *We can express μ_K as*

$$d\mu_K(k) = \left(\prod_{\alpha\in\Phi^+} \frac{\pi}{\langle\rho,\alpha\rangle}\right) \mathbf{a}(k)^{4\rho} d\mu_{N^+}(\mathbf{n}(k)) d\mu_T(\mathbf{m}(k))$$
$$= \prod_{j=1}^{\#(\Phi^+)} \frac{1}{2\sqrt{-1}\langle\tau_j\left(\mathrm{w}\right), \tau_j\left(\mathrm{w}\right)\rangle} \left(1+|\zeta_j|^2\right)^{2\rho\left(h_{\tau_j(\mathrm{w})}\right)-1} d\zeta_j d\bar{\zeta}_j \prod_{i=1}^{r} \frac{1}{2\pi} d\theta_i.$$

2.3.2 Parametrizations of Totally Positive Matrices

Definition 2.3.7. A totally positive (nonnegative) matrix $A \in GL(n,\mathbb{R})$ is a matrix all of whose minors are > 0 (≥ 0). The space of totally positive (nonnegative) matrices will be denoted TP (TNN).

The study of nonnegative matrices has a long history. Perron discovered in 1907 [29] that if a matrix $A \in GL(n,\mathbb{R})$ has entries in $\mathbb{R}_{>0}$, then the eigenvalue of largest modulus is simple, real and positive. In the 30s, Gantmacher & Krein [13] observed that if the determinant of every minor of A is also real and positive, then all the eigenvalues of A are simple, real, and positive. The study of such matrices, and their relations to canonical bases for quantum groups, was pursued by Lusztig, Berenstein, Fomin, Zelevinsky, and many others [28, 1]. An introduction to the subject can be found in [28].

Since the entries of the triangular factorization of A can be written as ratios of minors of A (recall 1.1.1), if $A = LDU$, then L and U also have real, positive entries. Indeed, L and U are totally positive, in the following sense. Let a minor whose determinant is not identically zero on $N^{\pm}$ be denoted a minor for $N^{\pm}$.

Definition 2.3.8. A lower-triangular unipotent matrix $L \in GL(n,\mathbb{R})$ will be called totally positive (nonnegative) if all the minors for N^- are > 0 (≥ 0) for L. The space of totally positive (nonnegative) lower-triangular matrices will be denoted TP^- (TNN^-).

The same definitions can be made regarding upper-triangular matrices. We will call the space of all totally positive (nonnegative) unipotent upper-triangular matrices TP^+ (TNN^+).

The question of the parametrization of TP thus reduces to the parametrization of $TP^{\pm}$. The following interpretation of a result of Whitney [41] is due to Loewner [25].

Proposition 2.3.9. *Given a reduced word* $\mathrm{w}_0 = s_{i_{\binom{n}{2}}} \dots s_{i_1}$ *for the longest permutation* w_0 *in the permutation group* S_n*, there are diffeomorphisms*

$$f_{\mathrm{w}_0} : (\mathbb{R}_{\geq 0})^{\binom{n}{2}} \longrightarrow TNN^{\pm} : \boldsymbol{r} \longmapsto \overleftarrow{\prod_{j=1}^{\binom{n}{2}}} \exp\left(r_j e_{i_j}\right),$$

$$g_{\mathrm{w}_0} : (\mathbb{R}_{>0})^{\binom{n}{2}} \longrightarrow TP^{\pm} : \boldsymbol{r} \longmapsto \overleftarrow{\prod_{j=1}^{\binom{n}{2}}} \exp\left(r_j e_{i_j}\right).$$

Given two reduced words w_0 and w_0', the transition maps $R_{\mathrm{w}_0}^{\mathrm{w}_0'} = f_{\mathrm{w}_0'}^{-1} \circ f_{\mathrm{w}_0}$ have been studied in the paper [1]. These transition maps provide the connection between total positivity and canonical bases for quantum groups.

As in the case of the flag manifold, intersecting the Bruhat decomposition of $GL(n, \mathbb{C})$ with $TNN^{\pm}$ yields cell decompositions

$$TNN^{\pm} = \bigsqcup_{w \in W} TP_w^{\pm}.$$

Given a reduced decomposition $\mathrm{w} = s_{i_1} \dots s_{i_{l(w)}}$, each cell $TP_w^{\pm}$ is completely parametrized by $(\mathbb{R}_{>0})^{l(w)}$,

$$(\mathbb{R}_{>0})^{l(w)} \longrightarrow TP_w^{\pm} : \boldsymbol{r} \longmapsto \overleftarrow{\prod_{j=1}^{l(w)}} \exp\left(r_j e_{i_j}\right).$$

In fact, TP_w^{+} is the set of upper-triangular totally nonnegative matrices A such that $\mathbf{w}A$ is totally positive [28]. These concepts have been extended to more general groups, including recent extensions to loop groups by Lam & Pylyavskyy [23, 24].

2.3.3 Parametrizations of $N_w^{\pm}$

Of course, the parametrizations intermediate between these two, which we have not mentioned, are the parametrizations of $N_w^{\pm}$ by $\mathbb{C}^{l(w)}$. Indeed, we have the following.

Proposition 2.3.10. *Given a set of indices* $I = \{i_1, \dots, i_n\}$*, with each* $i_j \in [r]$*, the map*

$$f_I : \mathbb{C}^n \longrightarrow N^{\pm} : \boldsymbol{\zeta} \longmapsto \overleftarrow{\prod_{j=1}^{n}} \exp\left(\zeta_j e_{i_j}\right)$$

is injective if and only if $s_{i_n} \dots s_{i_1}$ *is a reduced word in* W*; in this case, its is a diffeomorphism onto* N_w^{+}*.*

Proof. We proceed by induction on n. For $n = 1$, we have $f_i : \zeta \longmapsto \exp(\zeta e_i)$, and f_i is clearly a diffeomorphism from $\mathbb{C}$ to $N^+_{s_i} = N^+ \cap \mathbf{s_i} N^- \mathbf{s_i}$.

Assume the proposition is true for $n < k$. Now, choose a set of indices $I = \{i_1, \ldots, i_k\}$, with each $i_j \in [r]$, and let $I' = \{i_1, \ldots, i_{k-1}\}$. Then, if $f_{I'}$ is not a diffeomorphism onto its image, neither is f_I. Thus, we consider only the case where $\mathrm{w} = s_{i_{k-1}} \ldots s_{i_1}$ is a reduced word for $w \in W$.

For each reduced word $\mathrm{w}' = s_{j_{k-1}} \ldots s_{j_1}$ for w, write $I(\mathrm{w}') = \{j_1, \ldots, j_{k-1}\}$. Given reduced words w' and w'' for w, the composition $R^{I(\mathrm{w}'')}_{I(\mathrm{w}')} = f^{-1}_{I(\mathrm{w}'')} \circ f_{I(\mathrm{w}')} : \mathbb{C}^{l(w)} \longrightarrow \mathbb{C}^{l(w)}$ is also a diffeomorphism, by our induction hypothesis.

Now, let $i_k = i$, and assume that $s_i w$ is not reduced. This means $l(s_i w) = k - 2$, and there is a reduced word $\mathrm{w}^* = s_i s_{j_{k-2}} \ldots s_{j_1}$ for w. Given the standard basis $\{\lambda_i\}$ for $\mathbb{C}^k$, let $\boldsymbol{\zeta} = \sum_{i \in [k]} \zeta_i \lambda_i$ and $\boldsymbol{\zeta}' = \sum_{i \in [k-1]} \zeta_i \lambda_i$. Then we have

$$f_I(\boldsymbol{\zeta}) = \exp(\zeta_k e_i) f_{I'}(\boldsymbol{\zeta}') = \exp(\zeta_k e_i) f_{I(\mathrm{w}^*)} \circ R^{I(\mathrm{w}^*)}_{I'}(\boldsymbol{\zeta}').$$

But then,

$$\begin{aligned} f_I(\boldsymbol{\zeta}) &= \exp(\zeta_k e_i) \exp\left(\left(R^{I(\mathrm{w}^*)}_{I'}(\boldsymbol{\zeta}')\right)_{k-1} e_i\right) n \\ &= \exp\left(\left(\zeta_k + \left(R^{I(\mathrm{w}^*)}_{I'}(\boldsymbol{\zeta}')\right)_{k-1}\right) e_i\right) n = f_{I(\mathrm{w}^*)}(\boldsymbol{\zeta}''), \end{aligned}$$

where $\boldsymbol{\zeta}'' = \zeta_k \lambda_{k-1} + R^{I(\mathrm{w}^*)}_{I'}(\boldsymbol{\zeta}')$. Thus f_I cannot be a diffeomorphism.

Now assume that $s_i w$ is reduced, and let $w'' = s_i s_{i_{k-1}} \ldots s_{i_2}$. The induction hypothesis gives us a diffeomorphism from $f_{I(\mathrm{w}'')} : \mathbb{C}^{k-1} \longrightarrow N^+_{w''}$. Then, $\mathrm{Inv}(w'' s_{i_1}) = \{\alpha_{i_1}\} \cup s_{i_1} \mathrm{Inv}(w'')$. Since $s_{i_1} \mathrm{Inv}(w) \subset \Phi^+$, we know $\alpha_{i_1} \notin \mathrm{Inv}(w)$. Thus, by the properties of biconvex sets and (2.1.6), the multiplication map gives a diffeomorphism

$$N^+_{w''} \times \exp(\mathbb{C} e_i) \longrightarrow N^+_{s_i w}.$$

Composing this with $f_{I(\mathrm{w}'')}$ yields a diffeomorphism $f_{\{i, i_{k-1}, \ldots, i_1\}} : \mathbb{C}^k \longrightarrow N^+_{s_i w}$. This proves the result. ◻

I have said that these parametrizations are intermediate between those for Schubert cells and totally positive matrices mentioned above. The connection between the parametrizations of $N^\pm_w$ and TP_w is simply a change in the parameter space from $\mathbb{C}$ to $\mathbb{R}_{>0}$. Is there a direct way to obtain the parametrization of $N^+ w A N^+ \cap K$ from this parametrization of N^+_w? The answer is yes, sort of. Recalling that the map $u \longmapsto \mathbf{k}(uw)$ is a diffeomorphism from $N^+_{w^{-1}}$ onto $N^+ w A N^+ \cap K$, the following heuristic manipulations take us from one parametrization to another:

$$\mathbf{k}\left(\left(\overleftarrow{\prod_{j=1}^{l(w)}} \exp\left(\zeta_j e_{i_j}\right)\right)\mathbf{w}\right) \rightsquigarrow \mathbf{k}\left(\overleftarrow{\prod_{j=1}^{l(w)}}\left(\exp\left(\zeta_j e_{i_j}\right)\mathbf{s_{i_j}}\right)\right)$$

$$\rightsquigarrow \overleftarrow{\prod_{j=1}^{l(w)}} \mathbf{k}\left(\exp\left(\zeta_j e_{i_j}\right)\mathbf{s_{i_j}}\right).$$

To see that the final product gives the parametrization we've written down in Theorem 2.3.1, note that

$$\begin{aligned}\mathbf{k}\left(\exp\left(\zeta_j e_{i_j}\right)\mathbf{s_{i_j}}\right) &= \iota_{i_j}\left(\mathbf{k}\left(\begin{pmatrix}1 & \zeta_j\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}\right)\right)\\ &= \iota_{i_j}\left(\sqrt{-1}N\left(\zeta_j\right)\begin{pmatrix}\zeta_j & 1\\ 1 & -\bar{\zeta_j}\end{pmatrix}\right).\end{aligned}$$

We might ask the following question. Given a set of n indices $i_j \in [r]$, when is the map

$$\mathbb{C}^n \longrightarrow K/T : \boldsymbol{\zeta} \longmapsto \overleftarrow{\prod_{i=1}^{n}} \iota_{i_j}\left(\sqrt{-1}N\left(\zeta_j\right)\begin{pmatrix}\zeta_j & 1\\ 1 & -\bar{\zeta_j}\end{pmatrix}\right)$$

a diffeomorphism? I don't know of any results that answer this question, but Proposition 2.3.10 seems to suggest that $s_{i_1}\dots s_{i_j}$ must be a reduced word. We will see below that for Birkhoff cells, a different answer is suggested.

Example 2.3.11. For $SU(3)$, for the set of indices $\{1,2,2\}$, $s_2s_2s_1 = s_1$ is not a reduced word. Abbreviating $N\left(\zeta_i\right)$ by N_i, we have the map $\mathbb{C}^3 \longrightarrow K/T$ given by

$$\begin{aligned}\boldsymbol{\zeta} \longmapsto &\begin{pmatrix}\sqrt{-1}N_3\zeta_3 & \sqrt{-1}N_3 & \\ \sqrt{-1}N_3 & -\sqrt{-1}N_3\bar{\zeta_3} & \\ & & 1\end{pmatrix}\begin{pmatrix}\sqrt{-1}N_2\zeta_2 & \sqrt{-1}N_2 & \\ \sqrt{-1}N_2 & -\sqrt{-1}N_2\bar{\zeta_2} & \\ & & 1\end{pmatrix}\\ &\times\begin{pmatrix}1 & & \\ & \sqrt{-1}N_1\zeta_1 & \sqrt{-1}N_1\\ & \sqrt{-1}N_1 & -\sqrt{-1}N_1\bar{\zeta_1}\end{pmatrix}\\ &= \begin{pmatrix}-N_2N_3\left(1+\zeta_2\zeta_3\right) & \sqrt{-1}N_1N_2N_3\zeta_1\left(\bar{\zeta_2}-\zeta_3\right) & \sqrt{-1}N_1N_2N_3\left(\bar{\zeta_2}-\zeta_3\right)\\ -N_2N_3\left(\zeta_2-\bar{\zeta_3}\right) & -\sqrt{-1}N_1N_2N_3\zeta_1\left(1+\bar{\zeta_2}\bar{\zeta_3}\right) & -\sqrt{-1}N_1N_2N_3\left(1+\bar{\zeta_2}\bar{\zeta_3}\right)\\ 0 & \sqrt{-1}N_1 & -\sqrt{-1}N_1\bar{\zeta_1}\end{pmatrix}.\end{aligned}$$

∎

Since the image of this map lies (at best) in $C_{s_1s_2}$, a manifold of complex dimension two, it cannot be a diffeomorphism.

2.4 Parametrizations with Inversion Sets

2.4.1 Parametrizations of Birkhoff Cells

Using the diffeomorphism between C_w and $\Sigma_{w_0 w}$, we can use (2.3.2) to parametrize the Birkhoff strata. The resulting parametrization utilizes the inversion set of ww_0.

Proposition 2.4.1. *For each reduced decomposition* $\mathrm{w} = s_{i_1} \dots s_{i_{l(w)}}$*, there is a diffeomorphism*

$$\mathbb{C}^{l(w_0 w)} \longrightarrow \Sigma_w : \boldsymbol{\zeta} \longmapsto \mathbf{w} \overleftarrow{\prod_{j=1}^{l(w_0 w)}} \iota_{\tau_j(\mathbf{w_0 w})} \left(N(\zeta_j) \begin{pmatrix} 1 & -\bar{\zeta}_j \\ \zeta_j & 1 \end{pmatrix} \right), \tag{2.4.2}$$

where we choose the representatives

$$\mathbf{s_{i_j}} = \iota_{i_j} \left(\begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \right), \qquad \mathbf{w} = \mathbf{s_{i_1}} \dots \mathbf{s_{i_{l(w)}}}.$$

Proof. Take the reduced decomposition

$$\mathrm{w}_0 = (\mathrm{w}_0 \mathrm{w})\, \mathrm{w}^{-1} = \left(s_{i_{\#(\Phi^+)}} \dots s_{i_{l(w)+1}} \right) \left(s_{i_1} \dots s_{i_{l(w)}} \right),$$

so we have $\mathrm{w}_0 \mathrm{w} = s_{i_{\#(\Phi^+)}} \dots s_{i_{l(w)+1}}$. The diffeomorphism $\mathbf{w_0}^{-1} C_{w_0 w} = \Sigma_w$ induces the parametrization

$$\mathbb{C}^{l(w_0 w)} \longrightarrow \Sigma_w : \boldsymbol{\zeta} \longmapsto \mathbf{w_0}^{-1} \overleftarrow{\prod_{j=l(w)+1}^{\#(\Phi^+)}} \iota_{i_j} \left(\sqrt{-1} N(\zeta_j) \begin{pmatrix} \zeta_j & 1 \\ 1 & -\bar{\zeta}_j \end{pmatrix} \right).$$

Then, we have

$$\begin{aligned}
&\mathbf{w_0}^{-1} \overleftarrow{\prod_{j=l(w)+1}^{\#(\Phi^+)}} \iota_{i_j} \left(\sqrt{-1} N(\zeta_j) \begin{pmatrix} \zeta_j & 1 \\ 1 & -\bar{\zeta}_j \end{pmatrix} \right) \\
&= \mathbf{w} \overleftarrow{\prod_{j=1}^{\#(\Phi^+)-l(w)}} (\mathbf{w_0 w})_{j-1}^{-1} \mathbf{s_{i_{l(w)+j}}} \iota_{i_{l(w)+j}} \left(\sqrt{-1} N(\zeta_j) \begin{pmatrix} \zeta_j & 1 \\ 1 & -\bar{\zeta}_j \end{pmatrix} \right) (\mathbf{w_0 w})_{j-1} \\
&= \mathbf{w} \overleftarrow{\prod_{j=1}^{l(w_0 w)}} (\mathbf{w_0 w})_{j-1}^{-1} \iota_{i_{l(w)+j}} \left(\begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \sqrt{-1} N(\zeta_j) \begin{pmatrix} \zeta_j & 1 \\ 1 & -\bar{\zeta}_j \end{pmatrix} \right) (\mathbf{w_0 w})_{j-1} \\
&= \mathbf{w} \overleftarrow{\prod_{j=1}^{l(w_0 w)}} \iota_{\tau_j(w_0 w)} \left(N(\zeta_j) \begin{pmatrix} 1 & -\bar{\zeta}_j \\ \zeta_j & 1 \end{pmatrix} \right).
\end{aligned}$$

□

In particular, for each reduced word w_0, we have the following parametrization of Σ_1:

$$\mathbb{C}^{\#\left(\Phi^+\right)} \longrightarrow \Sigma_1 : \boldsymbol{\zeta} \longmapsto \overleftarrow{\prod_{j=1}^{\#\left(\Phi^+\right)}} \iota_{\tau_j(\mathbf{w_0})} \left(N\left(\zeta_j\right) \begin{pmatrix} 1 & -\bar{\zeta}_j \\ \zeta_j & 1 \end{pmatrix} \right).$$

Pickrell's result (2.3.3) has the following corollary.

Corollary 2.4.3. *In the coordinates (2.4.2), the real diagonal part of $k \in \Sigma_w$ is*

$$\mathbf{a}(k) = \overleftarrow{\prod_{j=1}^{l(\mathbf{w_0 w})}} \iota_{\tau_j(\mathrm{w_0 w})} \left(\begin{pmatrix} N\left(\zeta_j\right) & 0 \\ 0 & N\left(\zeta_j\right)^{-1} \end{pmatrix} \right).$$

Recent results of Caine & Pickrell on homogeneous Poisson structures for symmetric spaces [6] give a Poisson-geometric interpretation to the Birkhoff strata of K, and to the above coordinates. This involves the canonical Poisson structure for Riemannian symmetric spaces constructed by Evens & Lu [12]. Consider $K = \left\{(k,k) \middle| k \in K\right\}$ as a homogeneous space for $K \times K$, which acts on itself by the action $(k_1, k_2) * (k_1', k_2') = \left(k_1 k_1' k_2^{-1}, k_1 k_2' k_2^{-1}\right)$. When K is identified with the symmetric space $K \times K/K$, the Evens-Lu construction produces a $K \times K$-homogeneous Poisson structure on K, which we will denote Π_{EL}^K [5].

Caine showed in [5] that the symplectic leaves of Π_{EL}^K foliate the Birkhoff strata Σ_w^K of K. The leaves contained in Σ_w^K are parametrized by representatives $\mathbf{w}$ of w in $N_K(T)$. Each such symplectic leaf is diffeomorphic to Σ_w, and admits a Hamiltonian action by a subtorus T_w of T; an action of $T \backslash T_w$ takes one symplectic leaf to another [6].

In fact, this is entirely isomorphic to the Bruhat picture. The following appears in §5 of [6].

Theorem 2.4.4. *Left multiplication by $\mathbf{w_0}$ is a Poisson map from $\left(C_w^K, \Pi_{LW}^K\right)$ to $\left(\Sigma_w^K, -\Pi_{EL}^K\right)$.*

Thus we also have the following.

Corollary 2.4.5. *The symplectic form on Σ_w induced by Π_{EL}^K is given in the coordinates (2.4.2) by*

$$\Omega_w = \sum_{j=1}^{n} \frac{2\sqrt{-1}}{\langle \alpha_{i_j}, \alpha_{i_j} \rangle} \frac{1}{1 + |\zeta_j|^2} d\zeta_j \wedge d\bar{\zeta}_j.$$

Finally, we can rewrite all the topological facts about the Bruhat decomposition in terms of the Birkhoff decomposition, replacing w with $w_0 w$. (This has the effect of reversing the Bruhat order on W.) Unfortunately, showing the inclusion relations using our coordinate system is more complicated. We merely point out that $\boldsymbol{\zeta} =$

$(0,\ldots,0)$ again corresponds to $\mathbf{w}$, and if we take all the coordinates for Σ_w to infinity along the negative imaginary axis, we reach the point

$$\begin{aligned}
&\mathbf{w}\overleftarrow{\prod_{j=1}^{l(w_0w)}}\iota_{\tau_j(\mathrm{w_0w})}\left(\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}\right) = \mathbf{w}\overleftarrow{\prod_{j=1}^{l(w_0w)}}\left(\mathbf{w_0w}\right)_{j-1}^{-1}\mathbf{s_{i_j}}\left(\mathbf{w_0w}\right)_{j-1}\\
&= \mathbf{w}\mathbf{w}^{-1}\mathbf{w_0}^{-1}\left(\mathbf{w_0w}\right)_{l(w_0w)-1}\left(\mathbf{w_0w}\right)_{l(w_0w)-2}^{-1}\mathbf{s_{i_{l(w_0w)-1}}}\ldots\left(\mathbf{w_0w}\right)_2\left(\mathbf{w_0w}\right)_1^{-1}\mathbf{s_{i_1}}\\
&= \mathbf{w_0}^{-1}\mathbf{s}^2_{\mathbf{i_{l(w_0w)-1}}}\ldots\mathbf{s}^2_{\mathbf{i_1}} = \mathbf{w_0}^{-1},
\end{aligned}$$

which is also a representative for w_0.

Example 2.4.6. For $SU(3)$, choosing the reduced word $\mathrm{w}_0 = s_2s_1s_2$ yields the diffeomorphism $\mathbb{C}^3 \longrightarrow \Sigma_1$ given by

$$\begin{aligned}
&\boldsymbol{\zeta} \longmapsto k\left(\boldsymbol{\zeta}\right)\\
&= \begin{pmatrix} N\left(\zeta_3\right) & -N\left(\zeta_3\right)\bar{\zeta}_3 & \\ N\left(\zeta_3\right)\zeta_3 & N\left(\zeta_3\right) & \\ & & 1\end{pmatrix}\begin{pmatrix} N\left(\zeta_2\right) & & -N\left(\zeta_2\right)\bar{\zeta}_2\\ & 1 & \\ N\left(\zeta_2\right)\zeta_2 & & N\left(\zeta_2\right)\end{pmatrix}\begin{pmatrix}1 & & \\ & N\left(\zeta_1\right) & -N\left(\zeta_1\right)\bar{\zeta}_1\\ & N\left(\zeta_1\right)\zeta_1 & N\left(\zeta_1\right)\end{pmatrix}\\
&= \begin{pmatrix} N_2N_3 & -N_1N_3\bar{\zeta}_3 - N_1N_2N_3\zeta_1\bar{\zeta}_2 & N_1N_3\bar{\zeta}_1\bar{\zeta}_3 - N_1N_2N_3\bar{\zeta}_2\\ N_2N_3\zeta_3 & N_1N_3 - N_1N_2N_3\zeta_1\bar{\zeta}_2\zeta_3 & -N_1N_3\bar{\zeta}_1 - N_1N_2N_3\bar{\zeta}_2\zeta_3\\ N_2\zeta_2 & N_1N_2\zeta_1 & N_1N_2\end{pmatrix},
\end{aligned}$$

where in the second expression we have replaced $N\left(\zeta_i\right)$ with N_i to save space. The point $k\left(\boldsymbol{\zeta}\right)$ has a triangular factorization given by

$$\begin{aligned}
\mathbf{l}(k) &= \begin{pmatrix} 1 & & \\ \zeta_1 & 1 & \\ \frac{\zeta_3}{N(\zeta_1)} & N\left(\zeta_1\right)\left(\frac{\zeta_2}{N(\zeta_3)} + \bar{\zeta}_1\zeta_3\right) & 1\end{pmatrix}\\
\mathbf{a}(k) &= \begin{pmatrix} N\left(\zeta_1\right)N\left(\zeta_3\right) & & \\ & N\left(\zeta_1\right)N\left(\zeta_3\right)^{-1} & \\ & & \left(N\left(\zeta_1\right)N\left(\zeta_2\right)\right)^{-1}\end{pmatrix}.
\end{aligned}$$

It is easy to see that the map from $\boldsymbol{\zeta}$ to k is a diffeomorphism, since $\mathbf{l}(k)$ is a unique, arbitrary element of N^-. If $T \subset SU(3)$ is given coordinates $\{\theta_1,\theta_2\}$, the Haar measure for $SU(3)$ has the expression

$$\mu_{SU(3)} = \frac{\sqrt{-1}}{64}\left(1+|\zeta_1|^2\right)\left(1+|\zeta_2|^2\right)^3\left(1+|\zeta_3|^2\right)\prod_{i\in[3]}d\zeta_i d\bar{\zeta}_i\frac{1}{4\pi^2}d\theta_1 d\theta_2.$$

For $SU(4)$, Stanley's result [38] tells us that there are 16 reduced decompositions of w_0, and so 16 possible choices of coordinates. From the reduced decomposition

$\mathrm{w}_0 = s_2 s_1 s_2 s_3 s_2 s_1$ we obtain the diffeomorphism

$$
\begin{aligned}
\boldsymbol{\zeta} &\longmapsto k\left(\boldsymbol{\zeta}\right) \\
&= \begin{pmatrix} 1 & & & \\ & N\left(\zeta_6\right) & -N\left(\zeta_6\right)\bar{\zeta}_6 & \\ & N\left(\zeta_6\right)\zeta_6 & N\left(\zeta_6\right) & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & N\left(\zeta_5\right) & & -N\left(\zeta_5\right)\bar{\zeta}_5 \\ & & 1 & \\ & N\left(\zeta_5\right)\zeta_5 & & N\left(\zeta_5\right) \end{pmatrix} \\
&\times \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & N\left(\zeta_4\right) & -N\left(\zeta_4\right)\bar{\zeta}_4 \\ & & N\left(\zeta_4\right)\zeta_4 & N\left(\zeta_4\right) \end{pmatrix} \begin{pmatrix} N\left(\zeta_3\right) & & & -N\left(\zeta_3\right)\bar{\zeta}_3 \\ & 1 & & \\ & & 1 & \\ N\left(\zeta_3\right)\zeta_3 & & & N\left(\zeta_3\right) \end{pmatrix} \\
&\times \begin{pmatrix} N\left(\zeta_2\right) & & -N\left(\zeta_2\right)\bar{\zeta}_2 & \\ & 1 & & \\ N\left(\zeta_2\right)\zeta_2 & & N\left(\zeta_2\right) & \\ & & & 1 \end{pmatrix} \begin{pmatrix} N\left(\zeta_1\right) & -N\left(\zeta_1\right)\bar{\zeta}_1 & & \\ N\left(\zeta_1\right)\zeta_1 & N\left(\zeta_1\right) & & \\ & & 1 & \\ & & & 1 \end{pmatrix},
\end{aligned}
$$

where (replacing $N\left(\zeta_i\right)$ with N_i for compactness) k has a triangular factorization with

$$
\begin{aligned}
\mathbf{l}(k) &= \begin{pmatrix} 1 & & & \\ N_5\left(\frac{1}{N_3 N_2}\zeta_1 + N_4\bar{\zeta}_5\left(\frac{1}{N_3}\zeta_2\zeta_4 - \zeta_3\right)\right) & 1 & & \\ N_5\zeta_6\left(\frac{1}{N_3 N_2}\zeta_1 + N_4\bar{\zeta}_5\left(\frac{1}{N_3}\zeta_2\zeta_4 - \zeta_3\right)\right) + N_4\left(\frac{1}{N_3}\zeta_2 - \zeta_3\bar{\zeta}_4\right) & \zeta_6 & 1 & \\ N_5\left(\frac{1}{N_3 N_2}\zeta_1\zeta_5 + N_4\left(\zeta_3 + \frac{1}{N_3}\zeta_2\zeta_4\right)\right) & \zeta_5 & \frac{1}{N_5}\zeta_4 & 1 \end{pmatrix} \\
\mathbf{a}(k) &= \begin{pmatrix} N_1 N_2 N_3 & & & \\ & N_1^{-1} N_5 N_6 & & \\ & & N_2^{-1} N_4 N_6^{-1} & \\ & & & N_3 N_4 N_5 \end{pmatrix}.
\end{aligned}
$$

Given coordinates $\{\theta_1, \theta_2, \theta_3\}$ on $T \subset SU(4)$, the Haar measure for $SU(4)$ has the expression

$$
\mu_{SU(4)} = \frac{\sqrt{-1}}{4^5} \prod_{i=1,4,6} \left(1 + |\zeta_i|^2\right) \prod_{j=2,5} \left(1 + |\zeta_j|^2\right)^3 \left(1 + |\zeta_3|^2\right)^5 \prod_{k\in[6]} d\zeta_k d\bar{\zeta}_k \prod_{l\in[3]} \frac{1}{2\pi}\theta_l.
$$

2.4.2 Parametrizations of $N_w^{\pm}$ and Subgroups

We might also try to put coordinates on Σ_w by starting with coordinates on $N_w^{\pm}$. The easiest way to put coordinates on a Lie group near the identity is to use the inverse exponential map, which maps the Lie group to its (Euclidean) Lie algebra. This is especially convenient for $N_w^{\pm}$, since we have

$$
N_w^{\pm} = \exp\left(\mathfrak{n}_w^{\pm}\right) = \exp\left(\oplus_{\alpha\in\mathrm{Inv}(w)}\mathbb{C}e_{\pm\alpha}\right).
$$

For any indexing $\mathrm{Inv}(w) = \left\{\beta_1, \dots, \beta_{l(w)}\right\}$, there are diffeomorphisms

$$\mathbb{C}^{l(w)} \longrightarrow N_w^{\pm} : \boldsymbol{\zeta} \longmapsto \exp\left(\sum_{i=1}^{l(w)} \zeta_i e_{\pm\beta_i}\right),$$

which give us holomorphic coordinates $\{\zeta_i\}_{i\in[l(w)]}$ on $N_w^{\pm}$.

Another possibility is to multiply the factors $\exp\left(\zeta_i e_{\pm\beta_i}\right)$ together in $N^{\pm}$. In contrast to Proposition 2.3.10, we have the following.

Proposition 2.4.7. *For any $w \in W$, any indexing $\mathrm{Inv}(w) = \left\{\beta_1, \dots, \beta_{l(w)}\right\}$, and any choice of root vectors $e_{\pm\beta_i} \in \mathfrak{n}^{\pm}_{\beta_i}$, there is a diffeomorphism*

$$\mathbb{C}^{l(w)} \longrightarrow N_w^{\pm} : \boldsymbol{\zeta} \longmapsto \overleftarrow{\prod_{i=1}^{l(w)}} \exp\left(\zeta_i e_{\pm\beta_i}\right).$$

Proof. Composing the given smooth map with the logarithm on $N^{\pm}$ gives us a smooth map f from $\mathbb{C}^{l(w)}$ to itself. By the Baker-Campbell Hausdorff formula and (2.1.6), we see that

$$\prod_{\alpha\in\mathrm{Inv}(w)} \exp\left(z_\alpha e_{\pm\alpha}\right) = \exp\left(\sum_{\alpha\in\mathrm{Inv}(w)} u_\alpha e_{\pm\alpha}\right),$$

where for $\alpha, \beta \in \Phi^+$, u_α depends on z_β if and only if $\alpha = \beta + \gamma$ for some $\gamma \in \Phi^+$. Thus, for the simple roots, $u_{\alpha_i} = z_{\alpha_i}$, and we can solve for the roots of height k in terms of the roots of height $k-1$. Thus, f is one to one. $\square$

Example 2.4.8 ($SL(n, \mathbb{C})$)**.** Recall that the positive roots for $SL(n, \mathbb{C})$ can be identified with pairs of indices (i, j), where $1 \leq i < j \leq n$. The root α_{ij} corresponds to the vector $\lambda_i - \lambda_j \in \mathbb{C}^n \cong T$, where $\{\lambda_i\}_{i\in[n]}$ denotes the standard orthonormal basis for $\mathbb{C}^n$. The simple roots are denoted $\alpha_i = \alpha_{i(i+1)} = \lambda_i - \lambda_{(i+1)}$, and we have $\alpha_{ij} = \alpha_i + \ldots + \alpha_j$. There is a natural choice of root vector for α_{ij}, namely $e_{\alpha_{ij}} = e_{ij}$, where e_{ij} is a matrix having a one in the $(i, j)^{\text{th}}$ position and zeros elsewhere.

For $N^- \subset SL(4, \mathbb{C})$, if

$$\beta_1 = \alpha_1, \quad \beta_2 = \alpha_2, \quad \beta_3 = \alpha_3, \quad \beta_4 = \alpha_1 + \alpha_2, \quad \beta_5 = \alpha_2 + \alpha_3, \quad \beta_6 = \alpha_1 + \alpha_2 + \alpha_3,$$

we have the coordinates

$$\mathbb{C}^6 \ni \boldsymbol{\zeta} \longmapsto \begin{pmatrix} 1 & & & \\ \zeta_1 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & \zeta_2 & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & \zeta_3 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & & & \\ & 1 & & \\ \zeta_4 & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & \zeta_5 & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \zeta_6 & & & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & & & \\ \zeta_1 & 1 & & \\ \zeta_4 + \zeta_1\zeta_2 & \zeta_2 & 1 & \\ \zeta_6 + \zeta_1\zeta_5 + \zeta_1\zeta_2\zeta_3 & \zeta_5 + \zeta_2\zeta_3 & \zeta_3 & 1 \end{pmatrix}.$$

A nicer coordinate system is given by taking

$$\beta_1 = \alpha_1 + \alpha_2 + \alpha_3, \quad \beta_2 = \alpha_2 + \alpha_3, \quad \beta_3 = \alpha_3, \quad \beta_4 = \alpha_1 + \alpha_2, \quad \beta_5 = \alpha_2, \quad \beta_6 = \alpha_1,$$

in which case we have

$$\mathbb{C}^6 \ni \boldsymbol{\zeta} \longmapsto \begin{pmatrix} 1 & & & \\ \zeta_6 & 1 & & \\ \zeta_4 & \zeta_5 & 1 & \\ \zeta_1 & \zeta_2 & \zeta_3 & 1 \end{pmatrix}.$$

Indeed, this last coordinate system can be generalized. Every element $m \in N^+ \subset SL(n, \mathbb{C})$ is of the form

$$m = I_n + \sum_{1 \le i < j \le n} m_{ij} e_{ij},$$

where I_n is the $n \times n$ identity matrix. Then any ordering of the pairs $\{(i,j)\}_{1 \le i < j \le n}$ for which the index j is nondecreasing yields $\zeta_{ij} = m_{ij}$.

Proposition 2.4.9. *Let $\{\sigma_j\}_{j \in [2, n-1]}$ with $\sigma_l \in S_l$ be any set of permutations. Then we have the equality*

$$\overleftarrow{\prod_{j=2}^{n}} \overleftarrow{\prod_{i=1}^{j-1}} \exp\left(\zeta_{ij} e_{\sigma_{j-1}(i), j}\right) = I_n + \sum_{1 \le i < j \le n} \zeta_{ij} e_{\sigma_{j-1}(i), j},$$

for any $\boldsymbol{\zeta} = \{\zeta_{ij}\}_{1 \le i < j \le n} \in \mathbb{C}^{\binom{n}{2}}$.

Proof. For $i, k \in [j-1]$, since $\alpha_{ij} + \alpha_{kj} = \lambda_i + \lambda_k - 2\lambda_j \notin \Phi^+$, we have $\left[\mathfrak{n}^+_{\alpha_{ij}}, \mathfrak{n}^+_{\alpha_{kj}}\right] = 0$. Thus the subalgebra

$$\mathfrak{n}^+_{C,j} = \oplus_{i \in [j-1]} \mathfrak{n}^+_{\alpha_{ij}} \subset \mathfrak{n}^+,$$

which contains matrices with nonzero entries only in the superdiagonal part of the j^{th} column, is abelian. So is the subgroup $N_{C,j}^{+} = \exp\left(\mathfrak{n}_{C,j}^{+}\right)$. Thus, we have

$$\overleftarrow{\prod_{j=2}^{n}} \overleftarrow{\prod_{i=1}^{j-1}} \exp\left(\zeta_{ij} e_{\sigma_{j-1}(i),j}\right) = \overleftarrow{\prod_{j=2}^{n}} \exp\left(\sum_{i\in[j-1]} \zeta_{ij} e_{\sigma_{j-1}(i),j}\right) = \overleftarrow{\prod_{j=2}^{n}} \left(I_n + \sum_{i\in[j-1]} \zeta_{ij} e_{\sigma_{j-1}(i),j}\right). \tag{2.4.10}$$

Now, since $e_{ij}e_{kl} = \delta_{jk}e_{il}$, for $i \in [j]$ and $i' \in [j-1]$, $e_{i(j+1)}e_{i'j} = 0$. Thus, in expanding the right-hand side of (2.4.10), all the terms not of the form $I_n m$ where $m \in N_{C,j}^{+}$ vanish, and

$$\overleftarrow{\prod_{j=2}^{n}} \exp\left(\sum_{i\in[j-1]} \zeta_{i,j} e_{\sigma_{j-1}(i),j}\right) = I_n + \sum_{1\le i<j\le n} \zeta_{ij} e_{\sigma_{j-1}(i),j}.$$

□

There is a version of Proposition 2.4.9 for N^-. In that context, any ordering of the pairs $\{(i,j)\}_{1\le i<j\le n}$ for which j is nonincreasing yields $\zeta_{ij} = m_{ij}$.

To use Proposition 2.4.7 to put coordinates on Σ_w, we first put coordinates on $\bar{N}_w^-$. Observing that

$$\Phi^- \cap w^{-1}\Phi^- = \Phi^- \cap w^{-1}w_0^{-1}\Phi^+ = -\mathrm{Inv}\left(w_0 w\right),$$

so that

$$\bar{N}_w^- = \bigoplus_{\alpha\in\mathrm{Inv}(w_0 w)} \mathbb{C}e_{-\alpha} = N_{w_0 w}^-,$$

we have the following.

Corollary 2.4.11. *For any ordering* $\mathrm{Inv}\left(w_0 w\right) = \left\{\beta_1, \ldots, \beta_{l(w_0 w)}\right\}$ *and any choice of root vectors* $e_{-\beta_i} \in \mathfrak{n}_{\beta_i}^-$*, there is a diffeomorphism*

$$\mathbb{C}^{l(w_0 w)} \longrightarrow \bar{N}_w^- : \boldsymbol{\zeta} \longmapsto \overleftarrow{\prod_{i=1}^{l(w_0 w)}} \exp\left(\zeta_i e_{-\beta_i}\right).$$

We might now use the same sort of heuristic reasoning we used for Schubert cells to get from coordinates on $\bar{N}_{w^{-1}}^-$ to coordinates on $N^- wAN^+ \cap K$, using the diffeomorphism $l \longmapsto \mathbf{k}\left(l\mathbf{w}\right)$:

$$\mathbf{k}\left(\left(\prod_{\alpha\in\mathrm{Inv}(w_0 w^{-1})} \exp\left(\zeta_\alpha e_{-\alpha}\right)\right)\mathbf{w}\right) \rightsquigarrow \prod_{\alpha\in\mathrm{Inv}(w_0 w^{-1})} \mathbf{k}\left(\exp\left(\zeta_\alpha e_{-\alpha}\right)\right)\mathbf{w}.$$

Note that we have

$$\mathbf{k}\left(\exp\left(\zeta_\alpha e_{-\alpha}\right)\right) = \iota_\alpha\left(\mathbf{k}\left(\begin{pmatrix}1 & 0\\ \zeta_\alpha & 1\end{pmatrix}\right)\right) = \iota_\alpha\left(N\left(\zeta_\alpha\right)\begin{pmatrix}1 & -\bar{\zeta}_\alpha\\ \zeta_\alpha & 1\end{pmatrix}\right).$$

This suggests that, if we are only concerned with obtaining a parametrization of Σ_w, the order of the factors may not matter. However, the geometric, topological, and measure-theoretic applications of the coordinate system (2.4.2) seem to depend crucially on ordering the factors using a total reflection ordering of the roots.

Example 2.4.12. For $SU(3)$, with the ordering $\beta_1 = \alpha_1 + \alpha_2$, $\beta_2 = \alpha_1$, $\beta_3 = \alpha_2$, we have the map

$$\begin{aligned}
&\boldsymbol{\zeta} \longmapsto k\left(\boldsymbol{\zeta}\right)\\
&= \begin{pmatrix}1 & 0 & 0\\ 0 & N\left(\zeta_3\right) & -N\left(\zeta_3\right)\bar{\zeta}_3\\ 0 & N\left(\zeta_3\right)\zeta_3 & N\left(\zeta_3\right)\end{pmatrix}\begin{pmatrix}N\left(\zeta_2\right) & -N\left(\zeta_2\right)\bar{\zeta}_2 & 0\\ N\left(\zeta_2\right)\zeta_2 & N\left(\zeta_2\right) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}N\left(\zeta_1\right) & 0 & -N\left(\zeta_1\right)\bar{\zeta}_1\\ 0 & 1 & 0\\ N\left(\zeta_1\right)\zeta_1 & 0 & N\left(\zeta_1\right)\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0\\ N\left(\zeta_3\right)\left(\zeta_2 - \frac{1}{N(\zeta_2)}\zeta_1\zeta_3\right) & 1 & 0\\ N\left(\zeta_3\right)\left(\zeta_2\zeta_3 + \frac{1}{N(\zeta_2)}\zeta_1\right) & \frac{\zeta_3 + N(\zeta_2)\zeta_1\bar{\zeta}_2}{1 - N(\zeta_2)\zeta_1\bar{\zeta}_2\bar{\zeta}_3} & 1\end{pmatrix}\mathbf{a}\left(k\right)\mathbf{u}\left(k\right).
\end{aligned}$$

Say we fix

$$l = \begin{pmatrix}1 & 0 & 0\\ l_1 & 1 & 0\\ l_3 & l_2 & 1\end{pmatrix}.$$

Then, we have the following relationships between ζ_i and l_i.

$$\begin{pmatrix}l_1\\ l_3\end{pmatrix} = N\left(\zeta_3\right)\begin{pmatrix}1 & -\bar{\zeta}_3\\ \zeta_3 & 1\end{pmatrix}\begin{pmatrix}\zeta_2\\ \frac{\zeta_1}{N(\zeta_2)}\end{pmatrix}, \qquad l_2 = N\left(\zeta_3\right)\begin{pmatrix}1 & -\bar{\zeta}_3\\ \zeta_3 & 1\end{pmatrix} * N\left(\zeta_2\right)\zeta_1\bar{\zeta}_2,$$

where the action $*$ is by a linear-fractional transformation. It is not clear whether or not this map is a diffeomorphism.

2.4.3 Injections into Σ_1

The element $\mathbf{w}$ can be dropped from Lemma 2.2.5 to obtain embeddings of the strata Σ_w into Σ_1,

$$\bar{N}^-_{w^{-1}}\mathbf{w}AN^+ \cap K \longrightarrow \bar{N}^-_{w^{-1}}AN^+ \cap K \subset \Sigma_1.$$

It is easy to put parameters on the latter set. The following proof is adapted from [6, 34].

Theorem 2.4.13. *For each $w \in W$ and each reduced word* $\mathrm{ww}_0 = s_{i_1} \dots s_{l(ww_0)}$*, there is a diffeomorphism*

$$\begin{aligned} \mathbb{C}^{l(ww_0)} &\longrightarrow \bar{N}^-_{w^{-1}} A N^+ \cap K \subset \Sigma_1 \\ \boldsymbol{\zeta} &\longmapsto k(\boldsymbol{\zeta}) := \overleftarrow{\prod_{i=1}^{l(ww_0)}} \iota_{\tau_j((ww_0)^{-1})} \left(N(\zeta_j) \begin{pmatrix} 1 & -\bar{\zeta}_j \\ \zeta_j & 1 \end{pmatrix} \right) \end{aligned}$$

and

$$\mathbf{a}(k(\boldsymbol{\zeta})) = \overleftarrow{\prod_{j=1}^{l(ww_0)}} \iota_{\tau_j((ww_0)^{-1})} \left(\begin{pmatrix} N(\zeta_i) & 0 \\ 0 & N(\zeta_i)^{-1} \end{pmatrix} \right).$$

Proof. We will use induction on $m := l(ww_0)$. We will write $\tilde{w} = (ww_0)^{-1}$, τ_j for $\tau_j\left((ww_0)^{-1}\right)$, and $\tilde{\mathrm{w}}_k = s_{i_1} \dots s_{i_k}$ for $k \in [m]$. We will also use the more compact notation $\mathbf{k}\left(\zeta_j e_{-\tau_j}\right)$ for the factors in the product defining $k(\boldsymbol{\zeta})$, and we will use the triangular factorization

$$\mathbf{k}\left(\exp\left(\zeta_j e_{-\tau_j}\right)\right) = \exp\left(\zeta_j e_{-\tau_j}\right) N(\zeta_j)^{h_{\tau_j}} \exp\left(-\bar{\zeta}_j e_{\tau_j}\right).$$

When $m = 1$, the theorem is trivial. For $m \geq 2$, let

$$k^{(m-1)} = \prod_{j=1}^{m-1} \mathbf{k}\left(\exp\left(\zeta_j e_{-\tau_j}\right)\right) = l_{m-1} a_{m-1} u_{m-1}.$$

where $l_{m-1} \in \exp\left(\bigoplus_{j=1}^{m-1} \mathbb{C} e_{-\tau_j}\right)$. Then,

$$k^{(m)} = \exp\left(\zeta_m f_{\tau_m}\right) N(\zeta_m)^{h_{\tau_m}} \exp\left(-\bar{z}_m e_{\tau_m}\right) l_{m-1} a_{m-1} u_{m-1}.$$

Then, we have

$$\begin{aligned} \exp\left(-\bar{\zeta}_m e_{\tau_m}\right) l_{m-1} &= \exp\left(-\bar{\zeta}_m e_{\tilde{\mathrm{w}}_{m-1}^{-1} \alpha_{i_m}}\right) l_{m-1} \\ &= \tilde{\mathbf{w}}_{m-1}^{-1} \exp\left(-\bar{\zeta}_m e_{\alpha_{i_m}}\right) \tilde{\mathbf{w}}_{m-1} l_{m-1} \\ &= \tilde{\mathbf{w}}_{m-1}^{-1} \exp\left(-\bar{\zeta}_m e_{\alpha_{i_m}}\right) \left(\tilde{\mathbf{w}}_{m-1} l_{m-1} \tilde{\mathbf{w}}_{m-1}^{-1}\right) \tilde{\mathbf{w}}_{m-1}. \end{aligned}$$

Now,

$$\tilde{\mathbf{w}}_{m-1} l_{m-1} \tilde{\mathbf{w}}_{m-1}^{-1} \in \exp\left(\bigoplus_{j=1}^{m-1} \mathbb{C} e_{-\tilde{\mathrm{w}}_{m-1}\tau_j}\right) = N^-_{\tilde{\mathrm{w}}_{m-1}}.$$

We now have

$$\exp\left(-\bar{\zeta}_m e_{\tau_m}\right) l_{m-1} = \tilde{\mathbf{w}}_{m-1}^{-1} n \tilde{\mathbf{w}}_{m-1},$$

where $n \in N^+$. Since we can decompose N^+ as

$$N^+ = N^+_{\tilde{\mathrm{w}}_{m-1}^{-1}} \bar{N}^+_{\tilde{\mathrm{w}}_{m-1}^{-1}},$$

we can decompose n as $n = n_1 n_2$, where $n_1 \in N^{+}_{\tilde{\mathrm{w}}^{-1}_{m-1}}$ and $n_2 \in \bar{N}^{+}_{\tilde{\mathrm{w}}^{-1}_{m-1}}$. Then, we have

$$k^{(m)} = l_m a_m u_m,$$

where

$$\begin{aligned} l_m =& \exp\left(\zeta_m e_{-\tau_m}\right) N\left(\zeta_m\right)^{h_{\tau_m}} \left(\tilde{\mathbf{w}}_{m-1}^{-1} n_1 \tilde{\mathbf{w}}_{m-1}\right) N\left(\zeta_m\right)^{-h_{\tau_m}} \\ a_m =& N\left(\zeta_m\right)^{h_{\tau_m}} a_{m-1}. \end{aligned}$$

Now,

$$\begin{aligned} N\left(\zeta_m\right)^{h_{\tau_m}} \left(\tilde{\mathbf{w}}_{m-1}^{-1} n_1 \tilde{\mathbf{w}}_{m-1}\right) N\left(\zeta_m\right)^{-h_{\tau_m}} \in& \tilde{\mathbf{w}}_{m-1}^{-1} \bar{N}^{+}_{\tilde{\mathrm{w}}^{-1}_{m-1}} \tilde{\mathbf{w}}_{m-1} \\ =& N^{-} \cap \tilde{\mathbf{w}}_{m-1}^{-1} N^{+} \tilde{\mathbf{w}}_{m-1} = N^{-}_{\tilde{\mathrm{w}}_{m-1}}. \end{aligned}$$

We complete the proof by noting that l_m is a unique but arbitrary element of $N^{-}_{\tilde{w}} = \bar{N}^{-}_{w^{-1}}$, since the multiplication map gives a diffeomorphism

$$\exp\left(\mathbb{C} e_{-\tau_m}\right) \times N^{-}_{\tilde{\mathrm{w}}_{m-1}} \longrightarrow N^{-}_{\tilde{w}}.$$

□

Note that the parametrization of Σ_1 given by this result agrees with that given by Proposition 2.4.1.

2.4.4 The "Lexicographically Minimal" Factorization for $SU(n)$

By Proposition 2.4.9, and since $e_{-\alpha_{ij}} = e_{ji}$, applying Corollary 2.4.11 to the lexicographically minimal reduced word $\mathrm{w}_0 = s_1 s_2 s_1 s_3 s_2 s_1 \ldots s_{n-1} s_{n-2} \ldots s_1$ yields a diffeomorphism $\mathbb{C}^{\binom{n}{2}} \longrightarrow \bar{N}_1^{-} = N^{-} \subset SL(n, \mathbb{C})$, given by

$$\boldsymbol{\zeta} \longmapsto I_n + \sum_{1 \le i < j \le n} \zeta_{m(i,j)} e_{ji} = \begin{pmatrix} 1 & & & & & \\ \zeta_1 & 1 & & & & \\ \zeta_2 & \zeta_{n+1} & 1 & & & \\ \vdots & \vdots & & \ddots & & \\ \zeta_{n-1} & \zeta_{2n-4} & \cdots & \zeta_{\binom{n}{2}-2} & 1 & \\ \zeta_n & \zeta_{2n-3} & \cdots & \zeta_{\binom{n}{2}-1} & \zeta_{\binom{n}{2}} & 1 \end{pmatrix},$$

where $m(i,j) = \binom{n}{2} - \binom{j}{2} + j - i$.

The corresponding factorization for $\Sigma_1^{SU(n)} = N^{-} H N^{+} \cap SU(n)$ is the following.

Proposition 2.4.14. *An element $k \in SU(n)$ has a triangular factorization if and only if it has a factorization*

$$k = k_1\left(\boldsymbol{\zeta}_1\right) k_2\left(\boldsymbol{\zeta}_2\right) \ldots k_{n-1}\left(\boldsymbol{\zeta}_{n-1}\right) t,$$

where $t \in T$, $\boldsymbol{\zeta}_j = (\zeta_{j1}, \ldots, \zeta_{j,n-j}) \in \mathbb{C}^{n-j}$, *the element*

$$k_j = \overleftarrow{\prod_{i=1}^{n-j}} \iota_{j+i,j}\left(N(\zeta_{ji})\begin{pmatrix} 1 & -\bar{\zeta}_{ji} \\ \zeta_{ji} & 1 \end{pmatrix}\right) \in N^-_{C,j} = \exp\left(\oplus_{i\in[j+1,n]}\mathbb{C}e_{i,j}\right)$$

has a triangular factorization with

$$\mathbf{l}(k_j) = \begin{pmatrix} I_j & & & & & \\ & 1 & & & & \\ & \frac{\zeta_{j1}}{N(\zeta_{j1})N(\zeta_{j2})\ldots N(\zeta_{j,n-j})} & 1 & & & \\ & \frac{\zeta_{2j}}{N(\zeta_{j2})\ldots N(\zeta_{j,n-j})} & & 1 & & \\ & \vdots & & & \ddots & \\ & \frac{\zeta_{j,n-j}}{N(\zeta_{j,n-j})} & & & & 1 \end{pmatrix},$$

$$\mathbf{a}(k_j) = \begin{pmatrix} N(\zeta_{j1})\ldots N(\zeta_{j,n-j}) & & & \\ & \frac{1+|\zeta_{j2}|^2+|\zeta_{j2}|^4}{N(\zeta_{j2})} & & \\ & & \ddots & \\ & & & \frac{1+|\zeta_{j,n-j}|^2+|\zeta_{j,n-j}|^4}{N(\zeta_{j,n-j})} \end{pmatrix},$$

and k *has a triangular factorization with*

$$\mathbf{l}(k) = \overrightarrow{\prod_{j=1}^{n-1}} \mathbf{au}(k_1)\ldots\mathbf{au}(k_{j-1})\,\mathbf{l}(k_j)\left(\mathbf{au}(k_1)\ldots\mathbf{au}(k_{j-1})\right)^{-1},$$

$$\mathbf{a}(k) = \prod_{j\in[n-1]}\prod_{i\in[n-j]} \iota_{j+i,j}\begin{pmatrix} N(\zeta_{ji}) & \\ & N(\zeta_{ji})^{-1} \end{pmatrix}.$$

Proof. This follows from Theorem 2.4.13, taking w_0^{-1} to be the lexicographically minimal word for w_0, writing $k = k(\boldsymbol{\eta})$ for $\boldsymbol{\eta} \in \mathbb{C}^{\binom{n}{2}}$, and letting

$$k_j = \prod_{m=\binom{n}{2}-\binom{j}{2}}^{\binom{n}{2}-\binom{j+1}{2}} \iota_{I(m),J(m)}\left(N(\eta_m)\begin{pmatrix} 1 & -\bar{\eta}_m \\ \eta_m & 1 \end{pmatrix}\right)$$

and $\zeta_{ji} = \eta_{\binom{n}{2}-\binom{j}{2}+j-i}$. The formulas for $\mathbf{l}(k_j)$ and $\mathbf{a}(k_j)$ are obtained easily by induction. □

Chapter 3
Reduced Words in Affine Weyl Groups

In our notation for the affine Weyl group, we again try to follow Carter [7] when possible. Otherwise, we follow the literature on the subject, including Shi [36], Cellini & Papi [8], Ito [16], and Lam & Pylyavskyy [23, 24]. For combinatorial aspects of affine Weyl groups, we point the reader to Björner & Brenti [3]; a less up-to-date but still excellent introduction is given in [15].

3.1 The Affine Weyl Group W_{aff}

3.1.1 The Extended Cartan Subalgebra

We will see in the next section that the extensions of $\mathfrak{h}_{\mathbb{R}}$ defined below play the role of Cartan subalgebras for (central extensions) of $L\mathfrak{g}$, the Lie algebra of LG. Thus, they are the natural domains for the action of the affine Weyl group, which plays the role of Weyl group for LG.

Let $\mathfrak{h}_{\mathbb{R}}$ be the real part of the Cartan subalgebra of $\mathfrak{g}$, equipped with the inner product $\langle \cdot, \cdot \rangle$ as defined in the previous chapter. The extended Cartan subalgebras for $\mathfrak{g}$ are the spaces $\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}} \oplus \mathbb{R}c$ and $\hat{\mathfrak{h}}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}} \oplus \mathbb{R}c \oplus \mathbb{R}d$. We extend $\langle \cdot, \cdot \rangle$ to $\hat{\mathfrak{h}}_{\mathbb{R}}$ by setting

$$\langle d, h \rangle = \langle c, h \rangle = \langle d, d \rangle = \langle c, c \rangle = 0, \qquad \langle d, c \rangle = 1.$$

Note that $\langle \cdot, \cdot \rangle$ is degenerate on $\tilde{\mathfrak{h}}_{\mathbb{R}}$, and indeed, there is no nondegenerate invariant bilinear form on $\tilde{\mathfrak{h}}_{\mathbb{R}}$; this is the reason for extending $\tilde{\mathfrak{h}}_{\mathbb{R}}$ to obtain $\hat{\mathfrak{h}}_{\mathbb{R}}$.

The dual of the extended Cartan subalgebra is $\hat{\mathfrak{h}}^*_{\mathbb{R}} = \mathfrak{h}^*_{\mathbb{R}} \oplus \mathbb{R}\gamma \oplus \mathbb{R}\delta$, where we have

$$\langle \delta, \alpha \rangle = \langle \gamma, \alpha \rangle = \langle \delta, \delta \rangle = \langle \gamma, \gamma \rangle = 0, \qquad \langle \delta, \gamma \rangle = 1.$$

The action of $\hat{\mathfrak{h}}^*_{\mathbb{R}}$ on $\hat{\mathfrak{h}}_{\mathbb{R}}$ is given by

$$\alpha(c) = \alpha(d) = \delta(h) = \delta(c) = \gamma(h) = \gamma(d) = 0$$

for $\alpha \in \mathfrak{h}^*_{\mathbb{R}}$ and $h \in \mathfrak{h}_{\mathbb{R}}$. Also, $\gamma(c) = 1$, and, for all the classical affine Lie algebras, $\delta(d) = 1$ (the full story can be found in [7]).

3.1.2 Extended Root and Coroot Systems

Just as the Weyl group W is defined in terms of the root system of $\mathfrak{g}$, W_{aff} is defined in terms of the root system of $L\mathfrak{g}$, also known as the extended root system of $\mathfrak{g}$, and denoted

$$\hat{\Phi} = \mathbb{Z}\delta \oplus (\Phi \cup \{0\}) .$$

Define $\alpha_0 = \delta - \theta$. The set $\hat{\Delta} = \{\alpha_0, \ldots, \alpha_r\}$ is a set of simple roots for $\hat{\Phi}$: Each $\alpha \in \hat{\Phi}$ can be written as a sum of simple roots $\alpha = \sum_{i=0}^{r} n_i \alpha_i$, where the integer coefficients n_i all have the same sign. The sign of α is the sign of the n_i, and we have the sets of positive and negative roots

$$\hat{\Phi}^+ = (\mathbb{Z}_{>0}\delta \oplus (\Phi \cup \{0\})) \cup \Phi^+, \qquad \hat{\Phi}^- = (\mathbb{Z}_{<0}\delta \oplus (\Phi \cup \{0\})) \cup \Phi^-.$$

Using $\langle \cdot, \cdot \rangle$ to identify $\hat{\mathfrak{h}}_{\mathbb{R}}$ and $\hat{\mathfrak{h}}_{\mathbb{R}}^*$, we can define coroots in the same way as for finite-dimensional Lie algebras. The set of simple coroots for $\hat{L}\mathfrak{g}$ is

$$\{h_0 = c - h_\theta, h_1, \ldots, h_r\}.$$

3.1.3 The Affine Weyl Group

For each simple root $\alpha_i \in \hat{\Delta}$, define the simple reflection s_i, which acts on $\alpha \in \hat{\mathfrak{h}}_{\mathbb{R}}^*$ and $h \in \hat{\mathfrak{h}}_{\mathbb{R}}$ by

$$s_i \alpha = \alpha - \alpha(h_i)\, \alpha_i, \qquad s_i h = h - \alpha_i(h)\, h_i.$$

Then the affine Weyl group W_{aff} is the infinite Coxeter group generated by the simple reflections.

A major difference between W_{aff} and W is that W_{aff} does not act transitively (or freely) on the extended root system $\hat{\Phi}$ of $\mathfrak{g}$. This fact is captured in the division of $\hat{\Phi}$ into real and imaginary sets, as follows.

Definition 3.1.1. A root $\alpha \in \hat{\Phi}$ is called real if there exist $\alpha_i \in \hat{\Delta}$ and $w \in W_{\text{aff}}$ so that $w(\alpha_i) = \alpha$. Otherwise, α is called imaginary.

For all classical Lie algebras, the sets $\hat{\Phi}_{\mathcal{R}}$ and $\hat{\Phi}_{\mathcal{I}}$ of real and imaginary roots are given by

$$\hat{\Phi}_{\mathcal{R}} = W_{\text{aff}} \cdot \hat{\Delta} = \mathbb{Z}\delta \oplus \Phi, \qquad \hat{\Phi}_{\mathcal{I}} = \hat{\Phi} \backslash \hat{\Phi}_{\mathcal{R}} = \mathbb{Z}\delta.$$

(For the full story, see Theorem 17.17 of [7].) The choice of $\hat{\Delta}$ divides $\hat{\Phi}_{\mathcal{R}}$ into positive and negative subsets as well:

$$\hat{\Phi}_{\mathcal{R}}^+ = (\mathbb{N}\delta \oplus \Phi) \cup \Phi^+, \qquad \hat{\Phi}_{\mathcal{R}}^- = (-\mathbb{N}\delta \oplus \Phi) \cup \Phi^-.$$

The distinction between real and imaginary roots is useful in understanding W_{aff}. For each root $\alpha \in \hat{\Phi}_{\mathcal{R}}^+$ and each $h \in \hat{\mathfrak{h}}_{\mathbb{R}}$, define

$$s_\alpha \beta = \beta - \beta(h_\alpha)\, \alpha, \qquad s_\alpha h = h - \alpha(h)\, \alpha,$$

so that s_α is the reflection in the hyperplane

$$H_\alpha = \ker(s_\alpha) = \left\{ h \in \hat{\mathfrak{h}}_{\mathbb{R}} \middle| \alpha(h) = 0 \right\}.$$

Then

$$W_{\text{aff}} \supset \left\{ s_\alpha \,\middle|\, \alpha \in \hat{\Phi}_{\mathbb{R}}^+ \right\}.$$

Indeed W_{aff} is generated by this set of reflections, and we can identify W_{aff} with the group of rigid automorphisms of $\hat{\mathcal{H}} = \cup_{\alpha \in \hat{\Phi}_{\mathcal{R}}^+} H_\alpha$.

3.1.4 Action on $\mathfrak{h}_{\mathbb{R}}$

The affine Weyl group W_{aff} derives its name from the fact that it acts as a group of affine reflections on $\mathfrak{h}_{\mathbb{R}}$. This action is obtained by identifying $\mathfrak{h}_{\mathbb{R}}$ with an affine subspace of $\hat{\mathfrak{h}}_{\mathbb{R}}$. Since $\alpha(c) = 0$ for all $\alpha \in \mathfrak{h}_{\mathbb{R}}^*$, W_{aff} acts trivially on $\mathbb{R}c \subset \hat{\mathfrak{h}}_{\mathbb{R}}$. Furthermore, since $\delta(wh) = (w^{-1}\delta)(h) = \delta(h)$, the action of W_{aff} on $\mathfrak{h}_{\mathbb{R}} \oplus \mathbb{R}d$ fixes the affine subspace

$$\hat{\mathfrak{h}}_{\mathbb{R},1} = \left\{ h \in \mathfrak{h}_{\mathbb{R}} \oplus \mathbb{R}d \;\middle|\; \delta(h) = 1 \right\}.$$

We identify $\hat{\mathfrak{h}}_{\mathbb{R},1}$ with $\mathfrak{h}_{\mathbb{R}}$, and we introduce the projection

$$\pi : \hat{\mathfrak{h}}_{\mathbb{R}} \longrightarrow \mathfrak{h}_{\mathbb{R}} : h + c_1 c + \frac{1}{\delta(d)} d \longmapsto h.$$

We can use π to derive the action of W_{aff} on $\mathfrak{h}_{\mathbb{R}}$. First, define the hyperplane

$$H_{k,\alpha} = \left\{ h \in \mathfrak{h}_{\mathbb{R}} \;\middle|\; \alpha(h) = k \right\},$$

and let $s_{k,\alpha}$ be the reflection in $H_{k,\alpha}$, which acts on $h \in \mathfrak{h}_{\mathbb{R}}$ by

$$s_{k,\alpha} h = h + (k - \alpha(h))\, h_\alpha.$$

Then, for $k > 0$, and $\alpha \in \Phi^+$, we have [9]

$$\pi\left(H_{k\delta-\alpha}\right) = H_{k,\alpha}, \qquad \pi \circ s_{k\delta-\alpha} = s_{k,\alpha} \circ \pi,$$

while for $k \geq 0$ and $\alpha \in \Phi^+$, we have

$$\pi\left(H_{k\delta+\alpha}\right) = H_{-k,\alpha}, \qquad \pi \circ s_{k\delta+\alpha} = s_{-k,\alpha} \circ \pi.$$

Thus, the affine Weyl group is generated by the reflections $\left\{ s_{k,\alpha} \;\middle|\; k \in \mathbb{Z}, \alpha \in \Phi^+ \right\}$, and is the rigid automorphism group of the hyperplane arrangement

$$\mathcal{H} = \bigcup_{k \in \mathbb{Z}, \alpha \in \Phi^+} H_{k,\alpha}.$$

Like all affine reflections, the affine reflection $s_{k,\alpha}$ can be written as the composition of a (non-affine) reflection and a translation:

$$s_{k,\alpha} h = (h - \alpha(h) h_\alpha) + k h_\alpha = s_\alpha h + k h_\alpha.$$

Thus, the affine action of W_{aff} on $\mathfrak{h}_{\mathbb{R}}$ further identifies W_{aff} with the semidirect product $W_{\text{aff}} = W \propto \check{T}$, where the elements $\tau \in \check{T}$ act by translation,

$$t_\tau h = h + \tau,$$

and we have the identities

$$w t_\tau w^{-1} = t_{w\tau}, \qquad s_{k,\alpha} = t_{k h_\alpha} s_\alpha, \qquad s_0 = t_{h_\theta} s_\theta. \tag{3.1.2}$$

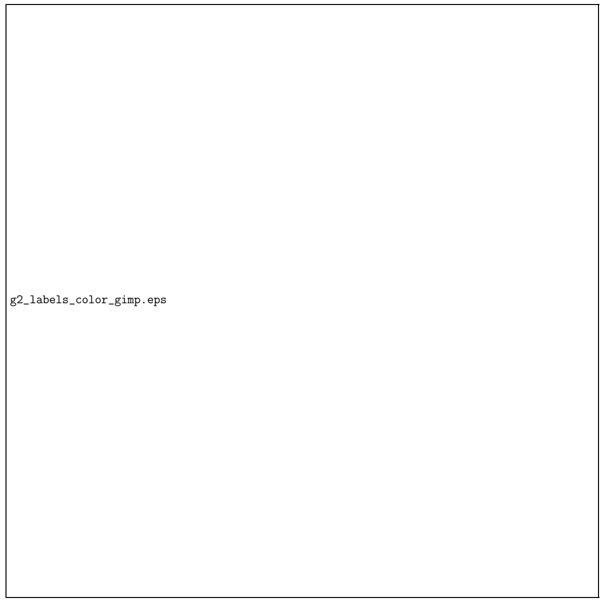


Figure 3.1: The alcove diagram for G_2.

3.1.5 Alcoves

The connected components of $\mathfrak{h}_{\mathbb{R}} \backslash \mathcal{H}$ are called alcoves. The alcoves are the fundamental domains for the action of W_{aff} on $\mathfrak{h}_{\mathbb{R}}$, and the group W_{aff} acts freely and transitively on the set $\mathcal{A}$ of alcoves. The hyperplane arrangement $\mathcal{H}$ is sometimes called the alcove diagram of $\mathfrak{g}$. Figure 3.1 shows the alcove diagram for G_2. We single out the fundamental alcove

$$A_0 = \left\{ h \in \mathfrak{h}_{\mathbb{R}} \middle| \alpha_0(h) < 1, \alpha_i(h) > 0, i = 1, \ldots, r \right\},$$

which is the bounded set carved out by $H_{\theta,1}$ and $H_{\alpha_1,0}, \ldots, H_{\alpha_n,0}$. Analogous to the fundamental Weyl chamber C_0, the fundamental alcove is the unique subset of $\mathfrak{h}_{\mathbb{R}}$ on which each simple root is positive. The set of alcoves is in correspondence with the elements of W_{aff}, and the choice of A_0 allows us to make this correspondence precise: Given an alcove A, there is exactly one $w_A \in W_{\text{aff}}$ satisfying $w_A^{-1} A_0 = A$. The geometric perspective this affords us is at the heart of our approach.

3.1.6 Reduced Words and Alcove Walks

A word of length n for $w \in W_{\text{aff}}$ is a sequence of simple reflections $s_{i_1}, s_{i_2}, \ldots, s_{i_n}$, $i_j \in [0, r]$, so that $w = s_{i_n} \ldots s_{i_2} s_{i_1}$. The length $l(w)$ of w is the minimum length of a word for w, and any word for w of length $l(w)$ is called a reduced word for w. As in the finite-dimensional case, we will use Roman typeface to distinguish reduced words from the Weyl group elements they represent, and we will sometimes say that w spells w.

Definition 3.1.3. A sequence of alcoves $P = \{A_i\}_{i\in[n]}$ is called a walk of length n from A_1 to A_n if $A_i \cap A_{i+1} = \emptyset$ and $\partial A_i \cap \partial A_{i+1} \neq \emptyset$. If there is no shorter walk from A_1 to A_n, we will call P a minimal walk.

In terms of the correspondence between W_{aff} and the set of alcoves, each word for w corresponds to a walk from A_0 to $w^{-1}A_0$, and each reduced word for w corresponds to a minimal walk. We will call such a minimal walk a minimal walk for w. Given a reduced word $\mathrm{w} = s_{i_{l(w)}} \dots s_{i_1}$, for $m \in [l(w)]$, define $\mathrm{w}_m = s_{i_m} s_{i_2} \dots s_{i_1}$. Then $\mathrm{w} = s_{i_{l(w)}} \dots s_{i_1}$ corresponds to the minimal walk $\left\{\mathrm{w}_{m-1}^{-1} A_0\right\}_{m\in[l(w)+1]}$.

3.1.7 Inversion Sets

Similarly to the finite-dimensional case, the inversion set of $w \in W_{\text{aff}}$ is the set of positive real roots mapped to negative real roots by w.

Definition 3.1.4. The inversion set of $w \in W_{\text{aff}}$ is the set

$$\mathrm{Inv}(w) = \left\{\alpha \in \hat{\Phi}^+_{\mathcal{R}} \;\middle|\; w\alpha \in \hat{\Phi}^-_{\mathcal{R}}\right\}.$$

Given a reduced word $\mathrm{w} = s_{i_{l(w)}} \dots s_{i_1}$ for w, $\mathrm{Inv}(w)$ can be calculated explicitly in the following way. For $m \in [l(w)]$, define

$$\tau_m(\mathrm{w}) = \mathrm{w}_{m-1}^{-1} \alpha_{i_m} = s_{i_1} \dots s_{i_{m-1}} \alpha_{i_m}.$$

Lemma 3.1.5.

$$\mathrm{Inv}(w) = \{\tau_m(\mathrm{w})\}_{m=1}^{l(w)}.$$

It is known that w flips $k\delta - \alpha$ if and only if $H_{k,\alpha}$ lies between A_0 and $w^{-1}A_0$, and w flips $k\delta + \alpha$ if and only if $H_{-k,\alpha}$ lies between A_0 and $w^{-1}A_0$ [9]. Thus we have the following.

Lemma 3.1.6. *If $\left\{A_0, A_1, \dots, A_{l(w)} = w^{-1}A_0\right\}$ is a minimal walk for w, and H_{k_i,β_i} is the common boundary of A_{i-1} and A_i for $i \in [l(w)]$, then*

$$\mathrm{Inv}(w) = \left\{k_i\delta - \beta_i \;\middle|\; k_i \geq 1\right\} \cup \left\{-k_i\delta + \beta_i \;\middle|\; k_i \leq 0\right\}.$$

Furthermore, a minimal alcove walk from A_0 to $w^{-1}A_0$ can only cross $H_{k,\alpha}$ if it also crosses $H_{l,\alpha}$ for all $l \in [k+1, 0]$, if $k \leq 0$, and for all $l \in [k-1]$, if $k \geq 2$. Thus, to find $\mathrm{Inv}(w)$, and indeed to identify the alcove $w^{-1}A_0$, it is sufficient to know, for each $\alpha \in \Phi^+$, the k of largest magnitude for which $H_{k,\alpha}$ lies between A_0 and $w^{-1}A_0$, and the sign of that k. This motivates the following definition.

Definition 3.1.7. Given a minimal walk $\{A_0, A_1, \ldots, A_{l(w)} = wA_0\}$ for w, and the set of hyperplanes $\{H_{k_i,\beta_i}\}_{i\in[l]}$ such that H_{k_i,β_i} is the common boundary of A_{i-1} and A_i, define

$$M_\alpha(w) = \max_{\{i|\beta_i=\alpha\}} (|k_i|), \qquad \mathrm{sgn}_\alpha(w) = \begin{cases} -1 & k_i \geq 1, \quad \beta_i = \alpha \text{ for some } i \in [n], \\ +1 & k_i \leq 0, \quad \beta_i = \alpha \text{ for some } i \in [n]. \end{cases}$$

By design, we then have the following.

Lemma 3.1.8.

$$\mathrm{Inv}(w) = \bigcup_{\substack{\alpha\in\Phi^+ \\ \mathrm{sgn}_\alpha(w)=-1}} \{\delta - \alpha, \ldots, M_\alpha(w)\delta - \alpha\} \cup \bigcup_{\substack{\alpha\in\Phi^+ \\ \mathrm{sgn}_\alpha(w)=+1}} \{\alpha, \ldots, M_\alpha(w)\delta + \alpha\},$$

$$l(w) = \sum_{\substack{\alpha\in\Phi^+ \\ \mathrm{sgn}_\alpha(w)=-1}} M_\alpha(w) + \sum_{\substack{\alpha\in\Phi^+ \\ \mathrm{sgn}_\alpha(w)=+1}} (M_\alpha(w) + 1).$$

Since $w^{-1}A_0$ lies between $H_{\mathrm{sgn}_\alpha(w)M_\alpha(w),\alpha}$ and $H_{\mathrm{sgn}_\alpha(w)(M_\alpha(w)+1),\alpha}$, an easy way to determine the numbers $\mathrm{sgn}_\alpha(w)$ and $M_\alpha(w)$ is given in the following lemma, which is key in this chapter.

Lemma 3.1.9.

$$M_\alpha(w) = \left\lfloor \left|\alpha\left(\mathrm{center}\left(w^{-1}A_0\right)\right)\right| \right\rfloor, \qquad \mathrm{sgn}_\alpha(w) = -\mathrm{sgn}\left(\alpha\left(\mathrm{center}\left(w^{-1}A_0\right)\right)\right).$$

Proof. If $H_{k,\alpha}$ is between A_0 and $w^{-1}A_0$, then for each $h \in w^{-1}A_0$, the integer k lies between the real numbers $\alpha(h)$ and 0. Furthermore, we have $\mathrm{center}\,(w^{-1}A_0) = w^{-1}\mathrm{center}\,(A_0) \in w^{-1}A_0$. □

The numbers $\mathrm{sgn}_\alpha(w)$ and $M_\alpha(w)$, which determine w (see Theorem 3.1.14 below, originally from [36]), appear at multiple points in the study of inversion sets of affine Weyl groups. For example, they arise naturally when considering biconvex sets of positive roots in the affine context.

Definition 3.1.10. A subset $I \subset \hat{\Phi}^+_{\mathcal{R}}$ is called biconvex if

1. whenever $\alpha, \beta \in I$, and $\alpha + \beta \in \hat{\Phi}$, then $\alpha + \beta \in I$.

2. whenever $\alpha, \beta \in \hat{\Phi}^+$ and $\alpha + \beta \in I$, either $\alpha \in I$ or $\beta \in I$.

The following was proven by Dyer [10].

Proposition 3.1.11. *The finite biconvex subsets of $\hat{\Phi}^+_{\mathcal{R}}$ are in bijection with the elements of W_{aff}, with the bijection given by $w \longmapsto \mathrm{Inv}(w)$.*

For $\alpha \in \Phi^+$, let $\underline{\alpha}$ be the set containing all linear combinations of δ and α which lie in $\hat{\Phi}^+_{\mathcal{R}}$:

$$\underline{\alpha} = \{\dots, 2\delta - \alpha, \delta - \alpha, \alpha, \delta + \alpha, 2\delta + \alpha, \dots\}.$$

The definition of biconvexity guarantees that if I is biconvex and $\#(I) < \infty$, then we have one of the following:

$$\begin{aligned} I \cap \underline{\alpha} &= \emptyset, \\ I \cap \underline{\alpha} &= \{\alpha, \delta + \alpha, \dots, m_\alpha \delta + \alpha\}, \\ I \cap \underline{\alpha} &= \{\delta - \alpha, 2\delta - \alpha, \dots, m_{-\alpha}\delta - \alpha\}, \end{aligned}$$

for nonnegative integers m_α. Note that only one of $m_\alpha(w)$ and $m_{-\alpha}(w)$ is defined, and we have

$$m_{\mathrm{sgn}_\alpha(w)\alpha}(w) = M_\alpha(w).$$

The Φ-tuple of integers $(m_\alpha)_{\alpha\in\Phi}$ determines the biconvex set I, and thus it determines a unique element $w \in W_{\mathrm{aff}}$ with $I = \mathrm{Inv}(w)$. In studies of the permutation realizations of W_{aff} for the classical groups, the Φ-tuple $(m_\alpha)_{\alpha\in\Phi}$ was dubbed the inversion table of w [2, 3, 11]. Lemma 3.1.8 can be restated in terms of inversion tables as follows.

Lemma 3.1.12. *Given the inversion table* $(m_\alpha(w))_{\alpha\in\Phi}$*,*

$$\mathrm{Inv}(w) = \bigcup_{\alpha\in\Phi^+} \{\alpha, \delta + \alpha, \dots, m_\alpha(w)\delta + \alpha\} \cup \bigcup_{\alpha\in\Phi^-} \{\delta + \alpha, 2\delta + \alpha, \dots, m_\alpha(w)\delta + \alpha\},$$

and $l(w) = \sum_{\alpha\in\Phi^+} (m_\alpha(w) + 1) + \sum_{\alpha\in\Phi^-} m_\alpha$.

We would be remiss if we did not mention a related Φ^+-tuple of numbers discovered by Shi [36], which he called the alcove form. Define the infinite strip

$$H^j_{k,\alpha} = \left\{ h \in \mathfrak{h}_{\mathbb{R}} \,\middle|\, k < \alpha(h) < k + j \right\},$$

so that $H^j_{k,-\alpha} = H^j_{-k-j,\alpha}$. Then, each alcove is of the form $\cap_{\alpha\in\Phi^+} H^1_{\tilde{m}_\alpha,\alpha}$, for some integers $\tilde{m}_\alpha$.

Definition 3.1.13. The alcove form of $w \in W_{\mathrm{aff}}$ is the Φ^+-tuple $(\tilde{m}_\alpha(w))_{\alpha\in\Phi^+}$ such that

$$w^{-1}A_0 = \bigcap_{\alpha\in\Phi^+} H^1_{\tilde{m}_\alpha(w),\alpha}.$$

For example, the alcove form of e is $(\tilde{m}_\alpha = 0)_{\alpha\in\Phi^+}$, since

$$A_0 = \bigcap_{\alpha\in\Phi^+} H^1_{0,\alpha},$$

and the alcove form of s_i is $\tilde{m}_{\alpha_i} = -1$ and $\tilde{m}_\alpha = 0$ for $\alpha_i \neq \alpha \in \Phi^+$, since $s_i A_0 = H^0_{-1,\alpha_i} \cap \bigcap_{\alpha_i\neq\alpha\in\Phi^+} H^1_{0,\alpha}$. Shi also characterized the Φ^+-tuples of integers which are alcove forms of elements of W_{aff} [36].

Theorem 3.1.14. *A Φ^+-tuple $(\tilde{m}_\alpha)_{\alpha\in\Phi^+}$ is the alcove form of an element $w \in W_{\mathrm{aff}}$ if and only if for any $\alpha, \beta, \gamma \in \Phi^+$ with $\gamma = \alpha + \beta$,*

$$\tilde{m}_\alpha + \tilde{m}_\beta \leq \tilde{m}_{\alpha+\beta} \leq \tilde{m}_\alpha + \tilde{m}_\beta + 1. \tag{3.1.15}$$

The integers m_α and $\tilde{m}_\alpha$ are related as follows

$$\tilde{m}_\alpha = \begin{cases} m_{-\alpha}, & \tilde{m}_\alpha \geq 1, \\ -m_\alpha - 1, & \tilde{m}_\alpha \leq 0. \end{cases} \tag{3.1.16}$$

The alcove form prompts us to identify elements of W_{aff} with what we will denote biconvex domains in $\mathfrak{h}_{\mathbb{R}}$.

Definition 3.1.17. The biconvex domain corresponding to $w \in W_{\mathrm{aff}}$ is the open set

$$B(w) = \bigcap_{\tilde{m}_\alpha(w)\leq -1} H^1_{\tilde{m}_\alpha(w),\alpha} \cap \bigcap_{\tilde{m}_\alpha(w)\geq 0} H^{\tilde{m}_\alpha(w)+1}_{0,\alpha} \subset \mathfrak{h}_{\mathbb{R}}.$$

These have the property that every minimal alcove walk for w lies in $B(w)$.

Example 3.1.18. In the affine Weyl group of type A_2, the simple roots are α_1, α_2, and $\alpha_0 = \delta - \theta = \delta - \alpha_1 - \alpha_2$. For the reduced word $\mathrm{w} = s_2 s_0 s_1 s_2 s_0 s_2 s_1$, we have

$$\begin{aligned}
&\tau_1(\mathrm{w}) = \alpha_1, \qquad \tau_2(\mathrm{w}) = s_1\alpha_2 = \theta, \qquad \tau_3(\mathrm{w}) = s_1 s_2 \alpha_0 = \delta + \alpha_1, \\
&\tau_4(\mathrm{w}) = s_1 s_2 s_0 \alpha_2 = \delta - \alpha_2, \qquad \tau_5(\mathrm{w}) = s_1 s_2 s_0 s_2 \alpha_1 = 2\delta + \alpha_1, \\
&\tau_6(\mathrm{w}) = s_1 s_2 s_0 s_2 s_1 \alpha_0 = 2\delta - \alpha_2, \\
&\tau_7(\mathrm{w}) = s_1 s_2 s_0 s_2 s_1 s_0 \alpha_2 = 3\delta + \alpha_1,
\end{aligned}$$

so that

$$\mathrm{Inv}(w) = \{\alpha_1,\ \theta,\ \delta + \alpha_1,\ \delta - \alpha_2,\ 2\delta + \alpha_1,\ 2\delta - \alpha_2,\ 3\delta + \alpha_1\},$$

and the set $\{H_{k_i,\beta_i}\}_{i\in[l(\mathrm{w})]}$ is

$$\{H_{0,\alpha_1},\ H_{0,\theta},\ H_{-1,\alpha_1},\ H_{1,\alpha_2},\ H_{-2,\alpha_1},\ H_{2,\alpha_2},\ H_{-3,\alpha_1}\}.$$

We have

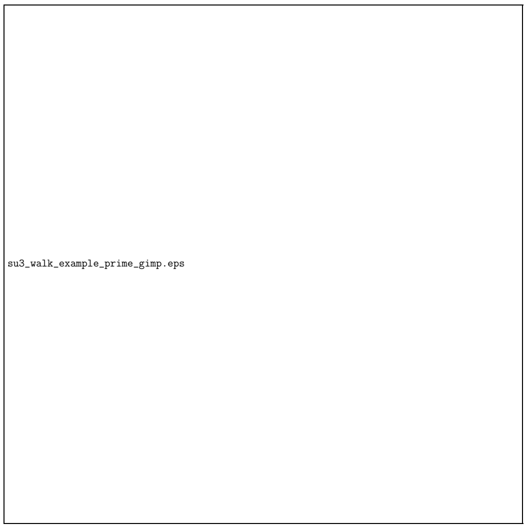
$$M_{\alpha_1}(\mathrm{w}) = 3, \qquad M_{\alpha_2}(\mathrm{w}) = 2, \qquad M_\theta(\mathrm{w}) = 0,$$

and

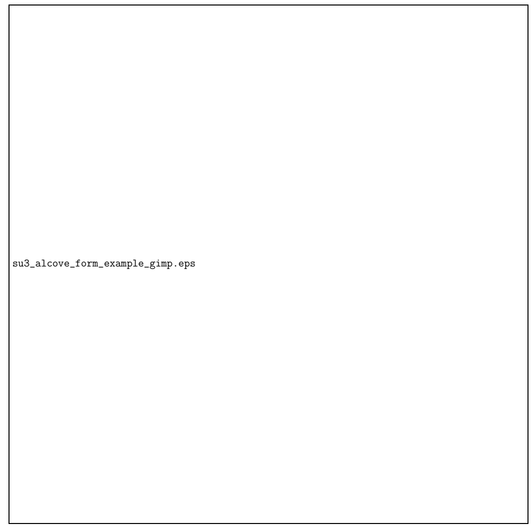
$$\mathrm{sgn}_{\alpha_1}(\mathrm{w}) = -1, \qquad \mathrm{sgn}_{\alpha_2}(\mathrm{w}) = 1, \qquad \mathrm{sgn}_\theta(\mathrm{w}) = -1.$$

We can check Lemma 3.1.9 as follows. We have $\Theta_1 = \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\Theta_2 = \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$, $\Theta_1 + \Theta_2 = (1, 0, -1) = h_\theta$, and $\mathrm{center}\,(A_0) = \frac{1}{3}(1, 0, -1)$. Also, we can expand w^{-1} using 3.1.2 and the reduced word $\mathrm{s}_\theta = \mathrm{w}_0 = s_2 s_1 s_2$:

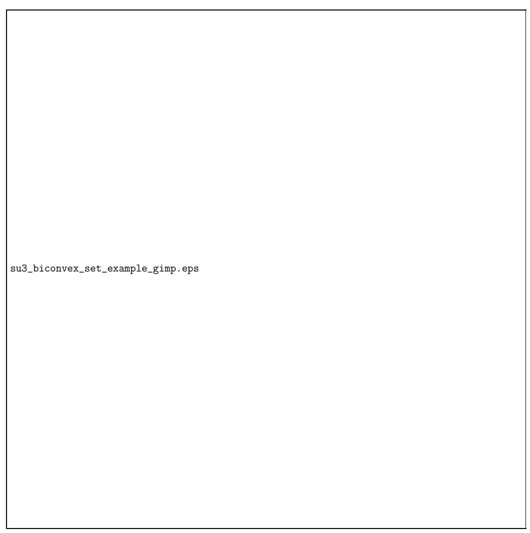
$$\begin{aligned}
w^{-1} &= s_1 s_2 s_0 s_2 s_1 s_0 s_2 = s_1 s_2 \left(t_{h_\theta} s_\theta\right) s_2 s_1 \left(t_{h_\theta} s_\theta\right) s_2 \\
&= s_1 s_2 t_{h_\theta} \left(s_2 s_1 s_2\right) s_2 s_1 t_{h_\theta} \left(s_2 s_1 s_2\right) s_2 = s_1 t_{s_2 h_\theta} t_{h_\theta} s_2 s_1 \\
&= s_1 t_{(2,-1,-1)} s_2 s_1.
\end{aligned}$$



((a)) The minimal alcove walk for w.



((b)) The alcove form for w.



((c)) The biconvex domain for w.

Figure 3.2: The minimal alcove walk, alcove form, and biconvex domain for $w = s_2s_0s_1s_2s_0s_2s_1$.

Thus, $w^{-1}\text{center}\,(A_0) = \left(-\frac{4}{3}, 2, -\frac{2}{3}\right)$, and we have

$$\alpha_1\left(w^{-1}\text{center}\,(A_0)\right) = -3 - \frac{1}{3}, \qquad \alpha_2\left(w^{-1}\text{center}\,(A_0)\right) = 2 + \frac{2}{3},$$

$$\theta\left(w^{-1}\text{center}\,(A_0)\right) = -\frac{2}{3}.$$

We also have

$$\begin{aligned}\underline{\alpha_1} \cap \mathrm{Inv}(w) &= \{\alpha_1,\ \delta+\alpha_1,\ 2\delta+\alpha_1,\ 3\delta+\alpha_1\}, \\ \underline{\alpha_2} \cap \mathrm{Inv}(w) &= \{\delta-\alpha_2,\ 2\delta-\alpha_2\}, \\ \underline{\theta} \cap \mathrm{Inv}(w) &= \{\theta\},\end{aligned}$$

so that

$$m_{\alpha_1} = 3, \qquad m_{-\alpha_2} = 2, \qquad m_\theta = 0.$$

The alcove form of w is given by

$$\tilde{m}_{\alpha_1} = -4, \qquad \tilde{m}_{\alpha_2} = 2, \qquad \tilde{m}_\theta = -1,$$

and we have

$$w^{-1}A_0 = H^1_{-4,\alpha_1} \cap H^1_{2,\alpha_2} \cap H^1_{-1,\theta}, \qquad B(w) = H^5_{-4,\alpha_1} \cap H^3_{0,\alpha_2} \cap H^2_{-1,\theta}.$$

This example is illustrated in Figure 3.2.

3.2 Infinite Reduced Words and Infinite Elements of W_{aff}

Definition 3.2.1. An infinite reduced word (or infinite reduced sequence) is a sequence of simple reflections $\mathrm{w} = \left(s_{i_j}\right)_{j\in\mathbb{N}}$ such that the length of $\mathrm{w}_p = s_{i_p}\dots s_{i_1}$ is p for each $p \in \mathbb{N}$.

We denote the set of infinite reduced words $\mathcal{RW}$. The correspondence between reduced words and minimal alcove paths tells us that infinite reduced words correspond to infinite alcove paths $\{A_i\}_{i\in\mathbb{Z}_{\geq 0}}$ all of whose initial subpaths $\{A_0, A_1, \dots, A_p\}$ are minimal. Such alcove paths will be called infinite minimal alcove paths.

Definition 3.2.2. An infinite reduced word $\mathrm{w} \in \mathcal{RW}$ is said to be periodic if there exists some $p \in \mathbb{Z}$ such that $s_{i_{j+p}} = s_{i_j}$ for all $j \in \mathbb{Z}$.

The infinite minimal alcove path corresponding to a periodic infinite reduced word has the following property, which we denote affine periodicity [34].

Definition 3.2.3. An affine periodic infinite minimal alcove path is a path with the property that there exist p and $\tau \in \check{T}$ such that $\{A_{i+p}\}_{i\in\mathbb{Z}_{\geq 0}} = \{t_\tau A_i\}_{i\in\mathbb{Z}_{\geq 0}}$.

The fact that periodic reduced words correspond to affine periodic alcove walks is pointed out in [24]. It can be proven simply as follows. Write $\mathrm{w}_p = vt_\tau$ for $\tau \in \check{T}$ and $v \in W$. Then, for all m,

$$\mathrm{w}_p^m = v^m v^{-(m-1)} t_\tau v^{m-1} v^{-(m-2)} t_\tau v^{m-2} \dots v t_\tau = v^m t_\omega,$$

where $\omega = \tau + v^{-1}\tau + \ldots + v^{-m+1}\tau$. Since W is finite, there is some m such that $v^m = 1$, and w_p^m is a translation.

We extend the concept of inversion sets to infinite reduced words as follows.

Definition 3.2.4. If $\mathrm{w} = \left(s_{i_j}\right)_{j\in\mathbb{N}}$ is an infinite reduced word, then

$$\mathrm{Inv}\,(\mathrm{w}) = \cup_{j\in\mathbb{N}} \mathrm{Inv}\,(w_j) = \{\tau_j(w)\}_{j\in\mathbb{N}}\,.$$

This leads us to the following definition of an infinite element of W_{aff}.

Definition 3.2.5. An infinite set of roots $I \subset \hat{\Phi}^+_{\mathcal{R}}$ is an infinite element of W_{aff} if it is the inversion set of an infinite reduced word.

The set of infinite elements of W_{aff} will be denoted $\mathcal{W}$. It is equal to the set of equivalence classes of $\mathcal{RW}$ under the relation $\mathrm{w} \sim \mathrm{w}'$ if $\mathrm{Inv}(\mathrm{w}) = \mathrm{Inv}(\mathrm{w}')$. As in the finite-dimensional case, we will denote the equivalence class of w by w, and we may write $w \in \mathcal{W}$; by this we intend that $\mathrm{Inv}(\mathrm{w}) \in \mathcal{W}$.

If $I \in \mathcal{W}$, then by Proposition 3.1.11 it is an infinite biconvex set, and we have one of the following [24]:

$$I \cap \underline{\alpha} = \emptyset, \tag{3.2.6}$$
$$I \cap \underline{\alpha} = \{\alpha, \delta+\alpha, \ldots, m_\alpha\delta+\alpha\}\,, \tag{3.2.7}$$
$$I \cap \underline{\alpha} = \{\delta-\alpha, 2\delta-\alpha, \ldots, m_{-\alpha}\delta-\alpha\}\,, \tag{3.2.8}$$
$$I \cap \underline{\alpha} = \{\alpha, \delta+\alpha, \ldots\}\,, \tag{3.2.9}$$
$$I \cap \underline{\alpha} = \{\delta-\alpha, 2\delta-\alpha, \ldots\}\,. \tag{3.2.10}$$

In the last two cases, we say $m_\alpha(I) = \infty$ and $m_{-\alpha}(I) = \infty$, respectively.

Definition 3.2.11. The inversion table of $I \in \mathcal{W}$ is the Φ-tuple $(m_\alpha(I))_{\alpha\in\Phi}$, with $m_\alpha(I) \in \mathbb{N} \cup \{\infty\}$, defined by (3.2.6).

Clearly, for $\mathrm{w} \in \mathcal{RW}$ we have $m_\alpha(\mathrm{w}) = \lim_{p\to\infty} m_\alpha(\mathrm{w}_p)$. The infinite values of m_α for w can thus be determined as follows.

Proposition 3.2.12. *Define the asymptotic direction of* w *to be*

$$\overline{\mathrm{dir}}(\mathrm{w}) = \lim_{p\to\infty} \frac{1}{p}\left(\mathrm{center}\left(\mathrm{w}_p^{-1}A_0\right)\right).$$

Then,

$$m_\alpha(\mathrm{w}) = \begin{cases} \infty & \alpha\left(\overline{\mathrm{dir}}(\mathrm{w})\right) < 0, \\ < \infty & \alpha\left(\overline{\mathrm{dir}}(\mathrm{w})\right) = 0, \\ 0 & \alpha\left(\overline{\mathrm{dir}}(\mathrm{w})\right) > 0. \end{cases}$$

In the last case, we have $m_{-\alpha}(\mathrm{w}) = \infty$*.*

With these concepts in hand, we are in a position to prove the major fact about infinite reduced words for affine Weyl groups needed in Chapter 4. Those only interested in the factorization results of that Chapter can skip ahead after the following.

Proposition 3.2.13. $\mathbb{N}\delta - \Phi^+ \in \mathcal{W}$*, and there exists a periodic reduced word for* $\mathbb{N}\delta - \Phi^+$*.*

Proof. There are many ways to show this result. Perhaps the most straightforward is to show that $\mathbb{N}\delta - \Phi^+$ is biconvex. We follow instead the proof appearing in [34], which predates our discovery of the work on biconvex sets. Finding any w with $\overline{\mathrm{dir}}(\mathrm{w}) \in C_0$ will prove that $\mathbb{N}\delta - \Phi^+ \in \mathcal{W}$, since if $\overline{\mathrm{dir}}(\mathrm{w}) = h \in C_0$, then $\alpha(h) > 0$ for all α implies that $m_{-\alpha} = \infty$ for all α. We construct a periodic reduced word for $\mathbb{N}\delta - \Phi^+$ as follows. Choose $h^* \in C_0 \cap \check{T}$, for example $h^* = 2\check{\rho}$, and choose a minimal alcove walk from A_0 to $A_0 + h^*$. Let $\mathrm{t}_{-\mathrm{h}^*} = s_{I_{l(t_{h^*})}} \dots s_{I_1}$ be the corresponding reduced word for $t_{-h^*} \in W_{\mathrm{aff}}$. Then the infinite reduced word $\mathrm{w} = \{s_{J_i}\}_{i \in \mathbb{Z}_{\geq 0}}$ defined by $s_{J_{i+l(t_{h^*})}} = s_{I_i}$ has

$$\overline{\mathrm{dir}}(\mathrm{w}) = \lim_{p \longrightarrow \infty} \frac{1}{p}\left(\mathrm{center}\left(t_{h^*}^p A_0\right)\right) = \lim_{p \longrightarrow \infty}\left(\frac{1}{p}\mathrm{center}\left(A_0\right) + h^*\right) = h^*.$$

This proves the result. ☐

We can also define an alcove form for an element $I \in \mathcal{W}$, using the relation (3.1.16).

Definition 3.2.14. The alcove form of $I \in \mathcal{W}$ is the Φ^+-tuple $\{\tilde{m}_\alpha(I)\}_{\alpha \in \Phi^+}$ with $\tilde{m}_\alpha(I) \in \mathbb{Z} \cup \{\pm\infty\}$ given in terms of the inversion table of I by relation (3.1.16), with the convention that $\infty - 1 = \infty$.

From Theorem 3.1.14, we obtain the following after taking limits.

Proposition 3.2.15. *A* Φ^+*-tuple* $(m_\alpha)_{\alpha \in \Phi^+}$ *is the alcove form of an element* $I \in \mathcal{W}$ *if and only if*

$$m_\alpha + m_\beta \leq m_{\alpha+\beta} \leq m_\alpha + m_\beta + 1, \tag{3.2.16}$$

where we observe the conventions that $\pm\infty \pm m = \pm\infty$ *for all* $m \in \mathbb{Z}$*,* $\pm\infty \pm \infty = \pm\infty$*, and* $\pm\infty \mp \infty$ *can be anything.*

For type A_{n-1}, the previous proposition also appears in [24].

The alcove form of an element $I \in \mathcal{W}$ allows us to relate it to the alcove diagram of $\mathfrak{h}_{\mathbb{R}}$. For $\alpha \in \Phi^+$, let

$$H_{0,\alpha}^{\infty} = \left\{ h \in \mathfrak{h}_{\mathbb{R}} \;\middle|\; \alpha(h) > 0 \right\}, \qquad H_{0,-\alpha}^{\infty} = \left\{ h \in \mathfrak{h}_{\mathbb{R}} \;\middle|\; \alpha(h) < 0 \right\}.$$

Definition 3.2.17. The biconvex subset corresponding to $I = \mathrm{Inv}(w) \in \mathcal{W}$ is the open set

$$B(I) = B(w) = \bigcap_{\alpha \in \Phi} H_{0,\alpha}^{m_\alpha(I)+1},$$

with the convention that $\infty + 1 = \infty$.

If $\mathrm{w} \in \mathcal{RW}$ is a reduced word for $I \in \mathcal{W}$, the infinite alcove path corresponding to w is contained in $B(I)$.

3.3 Coweight Elements, and their Reduced Words in Type A_{n-1}

Lemma 3.1.9, along with the fact that the fundamental coweights Θ_l satisfy $\alpha_i(\Theta_l) = \delta_{il}$, motivates us to choose the following "building blocks" for elements of W_{aff}.

Definition 3.3.1. The l^{th} coweight element is the unique element $W_l \in W_{\mathrm{aff}}$ satisfying

$$\mathrm{center}\left(W_l^{-1} A_0\right) - \mathrm{center}\left(A_0\right) = \Theta_l.$$

The inversion set $\mathrm{Inv}(W_l)$ is easy to find. By Lemma 3.1.9, for all $\alpha = \sum_{i \in [r]} n_i \alpha_i \in \Phi^+$ we have

$$m_{-\alpha}(W_l) = \lfloor \alpha(\Theta_l + \mathrm{center}(A_0)) \rfloor = \alpha(\Theta_l) = n_l, \tag{3.3.2}$$

where we use the fact that $\alpha(\mathrm{center}(A_0)) < 1$ for all $\alpha \in \Phi^+$. Defining $\Phi_l = \left\{ \alpha \in \Phi^+ \;\middle|\; \alpha(\Theta_l) > 0 \right\}$, we have

$$\mathrm{Inv}(W_l) = \left\{ k_\alpha \delta - \alpha \;\middle|\; \alpha \in \Phi_l,\ k_\alpha \in [m_{-\alpha}(W_l)] \right\}.$$

We will use these facts to identify reduced words for W_l.

We will also be interested in the smallest possible translations in the direction of Θ_l, and their reduced words.

Definition 3.3.3. The l^{th} coweight translation is the unique translation element T_l satisfying

$$\mathrm{center}\left(T_l^{-1} A_0\right) - \mathrm{center}\left(A_0\right) = k\Theta_l$$

for the smallest possible integer k.

Again applying Lemma 3.1.9, we have that $m_{-\alpha}(T_l) = k\alpha(\Theta_l)$ for some integer k which is independent of α. We will use this criterion to find reduced words for T_l.

For the Lie algebra of type A_{n-1}, corresponding to the compact, simple Lie group $SU(n)$, the affine Weyl group is also called the affine symmetric group, and denoted $\tilde{S}_n$. For $\tilde{S}_n$, we can find reduced words $\mathrm{W_l}$ for the elements W_l, as well as combinations of the reduced words $\mathrm{W_l}$ which are also reduced, directly. We now do so.

In this chapter, we will prove that a word $\mathrm{w} = s_{i_p} \ldots s_{i_1}$ is reduced and spells an element $w \in W_{\text{aff}}$ by calculating its inversion set. If $l(\mathrm{w}) = \#(\mathrm{Inv}(\mathrm{w})) = p$, then w is reduced, and if $\mathrm{Inv}(\mathrm{w}) = \mathrm{Inv}(w)$, then w spells w. By Theorem 4.15.9 in [39], for a (not necessarily reduced) word $\mathrm{w} = s_{i_p} \ldots s_{i_1}$, we have

$$\mathrm{Inv}(\mathrm{w}) = \{\tau_m(\mathrm{w})\}_{m\in[p]} \cap \Phi^+.$$

Thus, if $\tau_m(\mathrm{w})$ is positive for all $m \in [p]$, then w is reduced.

In an orthonormal basis $\{\lambda_i\}_{i\in[n]}$ with respect to the Killing form, the set of positive roots for the root system of type A_{n-1} is

$$\Phi^+ = \left\{\alpha_{ij} := \lambda_i - \lambda_{j+1} \;\middle|\; 1 \le i \le j \le n-1\right\},$$

where we will employ the convention for simple roots that $\alpha_i := \alpha_{ii} = \lambda_i - \lambda_{i+1}$. We have $\alpha_0 = \delta - \alpha_{1,n-1} = \delta - \alpha_1 - \ldots - \alpha_{n-1}$. The fundamental coweights are $\Theta_l = \lambda_1 + \ldots + \lambda_l$ for $l \in [n-1]$. Thus,

$$\Phi_l = \left\{\alpha_{ij} \;\middle|\; i \in [l],\ j \in [l, n-1]\right\},$$

$$\mathrm{Inv}(W_l) = \delta - \Phi_l = \left\{\delta - \alpha_{ij} \;\middle|\; i \in [l],\ j \in [l, n-1]\right\}. \tag{3.3.4}$$

For type A_{n-1}, we define

$$h^* = \begin{cases} \check{\rho} & n \text{ odd}, \\ 2\check{\rho} & n \text{ even}. \end{cases}$$

Given $\mathrm{w} = s_{i_n} \ldots s_{i_1} \in \tilde{S}_n$, we recall from (1.3.4) the notation

$$\mathrm{w}^{(k)} = s_{i_n+k} \ldots s_{i_1+k},$$

where in this section indices should be read mod n.

Proposition 3.3.5. *In $\tilde{S}_n$, for $l \in [n-1]$, let*

$$\mathrm{c_l} = s_{l+1}s_{l+2} \ldots s_{n-1}s_0, \qquad \mathrm{d_l} = s_{l-1} \ldots s_1 s_0.$$

(Note that s_0 appears only once in the words $\mathrm{c_l}$ and $\mathrm{d_l}$, so that $l(\mathrm{c_i}) = n-i$, $l(\mathrm{d_i}) = i$, and $\mathrm{c_{n-1}} = \mathrm{d_1} = s_0$.) Then the l^{th} coweight element W_l has reduced words

$$\mathrm{W_l} = \overleftarrow{\prod_{i=1}^{l-1}} \mathrm{c_l}^{(i)}, \qquad \mathrm{W_l'} = \overleftarrow{\prod_{i=1}^{n-1-l}} \mathrm{d_l}^{(-i)}.$$

We remark that $\mathrm{W}_1 = \mathrm{W}'_1 = s_2 s_3 \dots s_{n-1} s_0$, and $\mathrm{W}_{\mathrm{n}-1} = \mathrm{W}'_{\mathrm{n}-1} = s_{n-2} s_{n-1} \dots s_1 s_0$. $\blacksquare$

Example 3.3.6. In the simplest case, $SU(2)$, we have $\mathrm{c}_1 = s_0$ and $\mathrm{d}_1 = s_0$, so that $\mathrm{W}_1 = s_0$. In the simplest interesting case, $SU(3)$, we have

$$\mathrm{c}_1 = s_2 s_0, \qquad \mathrm{c}_2 = s_0, \qquad \mathrm{d}_1 = s_0, \qquad \mathrm{d}_2 = s_1 s_0.$$

Thus,

$$\mathrm{W}_1 = \mathrm{c}_1 = s_2 s_0, \qquad \mathrm{W}_2 = \mathrm{c}_2^{(1)} \mathrm{c}_2 = s_{0+1} s_0 = s_1 s_0,$$

while

$$\mathrm{W}'_1 = \mathrm{d}_1^{(-1)} \mathrm{d}_1 = s_{0-1} s_0 = s_2 s_0, \qquad \mathrm{W}'_2 = \mathrm{d}_2 = s_1 s_0.$$

In the case of $SU(4)$, we have

$$\mathrm{c}_1 = s_2 s_3 s_0, \qquad \mathrm{c}_2 = s_3 s_0, \qquad \mathrm{c}_3 = s_0, \qquad \mathrm{d}_1 = s_0, \qquad \mathrm{d}_2 = s_1 s_0, \qquad \mathrm{d}_3 = s_2 s_1 s_0,$$

so that

$$\mathrm{W}_1 = \mathrm{c}_1 = s_2 s_3 s_0, \qquad \mathrm{W}_2 = \mathrm{c}_2^{(1)} \mathrm{c}_2 = s_{3+1} s_{0+1} s_3 s_0 = s_0 s_1 s_3 s_0,$$

$$\mathrm{W}_3 = \mathrm{c}_3^{(2)} \mathrm{c}_3^{(1)} \mathrm{c}_3 = s_{0+2} s_{0+1} s_0 = s_2 s_1 s_0,$$

and

$$\mathrm{W}'_1 = \mathrm{d}_1^{(-2)} \mathrm{d}_1^{(-1)} \mathrm{d}_1 = s_{0-2} s_{0-1} s_0 = s_2 s_3 s_0, \quad \mathrm{W}'_2 = \mathrm{d}_2^{(-1)} \mathrm{d}_2 = s_{1-1} s_{0-1} s_1 s_0 = s_0 s_3 s_1 s_0,$$

$$\mathrm{W}_3 = \mathrm{d}_3 = s_2 s_1 s_0.$$

Proof. According to our convention for reduced words, the symbol W_l denotes a (known) reduced word for W_l. Thus, in this proof, we will denote $\overleftarrow{\prod}_{i=1}^{l-1} \mathrm{c}_\mathrm{l}^{(i)}$ by W. We will simultaneously show that W is reduced, and that $\mathrm{Inv}\,(\mathrm{W}) = \mathrm{Inv}\,(W_l)$, by calculating $\tau_m\,(\mathrm{W})$ for $m \in [l(n-l)]$. Showing that each $\tau_m\,(\mathrm{W})$ is positive will imply that W is reduced; we will then show that $\tau_m\,(\mathrm{W})$ runs over $\mathrm{Inv}\,(W_l)$ as m runs over $[l(n-l)]$.

Let $m \in [l(n-l)]$, and let $m = p(n-l) + q$, where $q \in [n-l]$, and $p \in [0, l-1]$. Thus, p tells us in which "multiple" of c_l index m lies, and q tells us how far into $\mathrm{c}_\mathrm{l}^{(p)}$ index m lies. We will see that $\tau_m\,(\mathrm{W})$ is a sum of simple roots with adjacent indices, in which each simple root has multiplicity one. The lower index of summation depends on q, and the upper index of summation depends on p:

$$\tau_m\,(\mathrm{W}_\mathrm{l}) = \sum_{j=1-q}^{p} \alpha_j,$$

where indices are to be read mod n. This sum has $p+q$ terms, since $1-q \geq l+1-n = l+1 \bmod n$ implies that $1-q \geq p$.

To prove this, first recall that

$$s_i\alpha_j = \begin{cases} -\alpha_i & j = i \\ \alpha_j + \alpha_i & j \in \{i-1, i+1\} \\ \alpha_j & \text{otherwise} \end{cases}$$

where, again, indices should be read mod n. We calculate

$$\begin{aligned}
\tau_m(\mathrm{W}) &= \left(\prod_{i=0}^{\xrightarrow{p-1}} s_i s_{i-1} \dots s_{i+l+1} \right) s_p s_{p-1} \dots s_{p-q+2} \alpha_{p-q+1} \\
&= \left(\prod_{i=0}^{\xrightarrow{p-2}} s_i s_{i-1} \dots s_{i+l+1} \right) s_{p-1} s_{p-2} \dots s_{p+l} \sum_{j=p-q+1}^{p} \alpha_j \\
&= \left(\prod_{i=0}^{\xrightarrow{p-2}} s_i s_{i-1} \dots s_{i+l+1} \right) s_{p-1} s_{p-2} \dots s_{p-q} \sum_{j=p-q+1}^{p} \alpha_j \\
&= \left(\prod_{i=0}^{\xrightarrow{p-2}} s_i s_{i-1} \dots s_{i+l+1} \right) \sum_{j=p-q}^{p} \alpha_j \\
&= \dots = \sum_{j=1-q}^{p} \alpha_j,
\end{aligned}$$

where in the third equality we used the fact that $p-q+1 \geq p-(n-l)+1 = p+l+1$ mod n. This shows that $\tau_m(\mathrm{W})$ is positive for all m.

To see that $\mathrm{Inv}(\mathrm{W}) = \mathrm{Inv}(W_l)$, note that

$$\sum_{j=1-q}^{p} \alpha_j = \alpha_0 + \left(\sum_{j=1}^{p} + \sum_{j=n-q+1}^{n-1} \right) \alpha_j = \alpha_0 + \alpha_{1,p} + \alpha_{n-q+1,n-1} = \delta - \alpha_{p+1,n-q}.$$

Since $p+1 \in [l]$ and $n-q \in [l, n-1]$, $\alpha_{p+1,n-q} \in \Phi_l$. Furthermore, as m runs over $[l(n-l)]$, $p+1$ runs over $[l]$, and for each value of $p+1$, $n-q$ runs over $[l, n-1]$. Thus, recalling (3.3.4), we see that $\mathrm{Inv}(\mathrm{W}) = \mathrm{Inv}(W_l)$. The proof for W_1' is similar. $\square$

We also have the following.

Corollary 3.3.7. *In $\tilde{S}_n$, the word W_1 flips the roots in $\mathrm{Inv}(W_l)$ in lexicographical order with respect to the alphabet $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. In other words, $\delta - (\lambda_i - \lambda_j)$ is flipped before $\delta - (\lambda_k - \lambda_l)$ if $i < k$, or if $i = k$ and $j > l$.*

Example 3.3.8. For $n = 5$, we have

$$\Phi_3 = \{\delta - \lambda_1 + \lambda_4,\ \delta - \lambda_1 + \lambda_5,\ \delta - \lambda_2 + \lambda_4,\ \delta - \lambda_2 + \lambda_5,\ \delta - \lambda_3 + \lambda_4,\ \delta - \lambda_3 + \lambda_5\}. \blacksquare$$

To find the order in which these roots are flipped by W_l, we weight λ_j less than λ_i if $j > i$. Thus, the roots containing the summand $-\lambda_1$ come first; out of the two of these, the root containing the summand $+\lambda_5$ is "smaller" than the root containing the summand $+\lambda_4$, and so on. The (total reflection) ordering $\prec$ given to these roots by W_l is thus

$$\delta-\lambda_1+\lambda_5 \prec \delta-\lambda_1+\lambda_4 \prec \delta-\lambda_2+\lambda_5 \prec \delta-\lambda_2+\lambda_4 \prec \delta-\lambda_3+\lambda_5 \prec \delta-\lambda_3+\lambda_4.$$

Recall that we associate the positive root α_{ij} for $SU(n)$ with the $(i,j)^{\text{th}}$ superdiagonal entry of an $n \times n$ matrix. From this perspective, the roots in Φ_l correspond to the block of entries to the northeast of $(l, l+1)$, the entry corresponding to α_l. The total inversion ordering $\prec$ that W_l gives to Φ_l corresponds to reading this block from right to left, and from top to bottom.

Example 3.3.9. Again considering $n = 5$, we have the correspondences

$$\Phi^+ \longleftrightarrow \begin{pmatrix} & \alpha_1 & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ & & \alpha_2 & \alpha_{24} & \alpha_{25} \\ & & & \alpha_3 & \alpha_{35} \\ & & & & \alpha_4 \\ & & & & \end{pmatrix}, \qquad \Phi_3 \longleftrightarrow \begin{pmatrix} & & & \alpha_{14} & \alpha_{15} \\ & & & \alpha_{24} & \alpha_{25} \\ & & & \alpha_3 & \alpha_{35} \\ & & & & \\ & & & & \end{pmatrix},$$

$$\prec \longleftrightarrow \begin{array}{ccc} \alpha_{14} & \leftarrow & \alpha_{15} \\ & \searrow & \\ \alpha_{24} & \leftarrow & \alpha_{25}. \\ & \searrow & \\ \alpha_3 & \leftarrow & \alpha_{35} \end{array}$$

We can put the words W_l together to obtain other reduced words. Just how we do this is the content of the next theorem.

Theorem 3.3.10. *Let $w \in \tilde{S}_n$ be such that for positive integers a_i,*

$$\text{center}\left(w^{-1}A_0\right) - \text{center}\left(A_0\right) = \sum_{i\in[n-1]} a_i\Theta_i$$

let $P = \sum_i a_i$, and let $J \in \mathbb{Z}^p$ be a vector of a_1 ones, followed by a_2 twos, a_3 threes, and so on. For any $\sigma \in S_P$ and $m \in [P]$, define $K_{\sigma,m} = \sum_{i\in[m]} J_{\sigma(i)}$. Then, for any $\sigma \in S_P$, w has the reduced word

$$\mathrm{W}(\boldsymbol{a}, \sigma) = \mathrm{W}_{\mathrm{J}_{\sigma(\mathrm{P})}}^{\left(K_{\sigma,P-1}\right)} \dots \mathrm{W}_{\mathrm{J}_{\sigma(2)}}^{(K_{\sigma,1})} \mathrm{W}_{\mathrm{J}_{\sigma(1)}}.$$

We will need the following lemma.

Lemma 3.3.11.

$$\left(\mathrm{W}_{l}^{(k)}\right)^{-1}\alpha_j = \begin{cases} \delta + \alpha_k & j = k + l, \\ \alpha_{k+l} - \delta & j = k + 2l, \\ \alpha_{j-l} & \textit{otherwise.} \end{cases}$$

Proof. These are straightforward calculations, where all indices are to be taken mod n. If $j \in [k + l + 1, k + 2l - 1]$, then

$$\begin{aligned} \left(\mathrm{W}_{l}^{(k)}\right)^{-1}\alpha_j &= \prod_{p=0}^{l-1} \left(s_{k+p} \ldots s_{k+p+l+1}\right)\alpha_j \\ &= \prod_{p=0}^{j-k-l} \left(s_{k+p} \ldots s_{k+p+l+1}\right) s_{n-l+j} \ldots s_{j+1}\alpha_j \\ &= \prod_{p=0}^{j-k-l-1} \left(s_{k+p} \ldots s_{k+p+l+1}\right) s_{n-l+j-1} \ldots s_j \sum_{i=j}^{n-l+j} \alpha_i \\ &= \prod_{p=0}^{j-k-l-2} \left(s_{k+p} \ldots s_{k+p+l+1}\right) s_{n-l+j-2} \ldots s_{j-1}\alpha_{n-l+j} \\ &= \alpha_{j+n-l} = \alpha_{j-l}. \end{aligned}$$

Next, if $j \in [k + 2l + 1, k + l - 1]$, then

$$\begin{aligned} \left(\mathrm{W}_{l}^{(k)}\right)^{-1}\alpha_j &= \prod_{p=0}^{l-2} \left(s_{k+p} \ldots s_{k+p-n+l+1}\right) s_{k+l-1} \ldots s_{j-1}\alpha_j \\ &= \prod_{p=0}^{l-2} \left(s_{k+p} \ldots s_{k+p-n+l+1}\right) s_{k+l-1} \ldots s_j \left(\alpha_j + \alpha_{j-1}\right) \\ &= \prod_{p=0}^{l-2} \left(s_{k+p} \ldots s_{k+p-n+l+1}\right) s_{k+l-1} \ldots s_{j+1} \left(\alpha\right)_{j-1} \\ &= \prod_{p=0}^{l-3} \left(s_{k+p} \ldots s_{k+p-n+l+1}\right) \alpha_{j-2} \\ &= \ldots = \alpha_{j-l}. \end{aligned}$$

Now, if $j = k + l$, then

$$\begin{aligned}
\left(\mathrm{W}_{\mathrm{l}}^{(k)}\right)^{-1}\alpha_j &= \prod_{p=0}^{l-2}\left(s_{k+p}\dots s_{k+p-n+l+1}\right)s_{k+l-1}\dots s_{k+2l-n}\alpha_{k+l}\\
&= \prod_{p=0}^{l-3}\left(s_{k+p}\dots s_{k+p-n+l+1}\right)s_{k+l-2}\dots s_{k+2l-n-1}\left(\alpha_{k+l-1}+\alpha_{k+l}\right)\\
&= s_k\dots s_{k-n+l+1}\sum_{i=k+1}^{k+l}\alpha_i\\
&= s_k\sum_{i=k+1}^{k-1}\alpha_i = 2\alpha_k + \sum_{i=k+1}^{k-1}\alpha_i.
\end{aligned}$$

Finally, if $j = k + 2l$, then

$$\begin{aligned}
\left(\mathrm{W}_{\mathrm{l}}^{(k)}\right)^{-1}\alpha_j &= \prod_{p=0}^{l-2}\left(s_{k+p}\dots s_{k+p-n+l+1}\right)s_{k+l-1}\dots s_{k+2l-n}\alpha_{k+2l}\\
&= \prod_{p=0}^{l-2}\left(s_{k+p}\dots s_{k+p-n+l+1}\right)s_{k+l-1}\dots s_{k+2l+1}\left(-\alpha_{k+2l}\right)\\
&= \prod_{p=0}^{l-3}\left(s_{k+p}\dots s_{k+p-n+l+1}\right)s_{k+l-2}\dots s_{k+2l-1}\left(-\sum_{k+2l}^{k+l-1}\alpha i\right)\\
&= \prod_{p=0}^{l-3}\left(s_{k+p}\dots s_{k+p-n+l+1}\right)\left(-\sum_{k+2l-1}^{k+l-1}\alpha i\right)\\
&= -\sum_{i=k+l+1}^{k+l-1}\alpha_i.
\end{aligned}$$

□

Proof (Theorem 3.3.10). Our strategy will be to determine $\mathrm{Inv}\left(\mathrm{W}\left(\boldsymbol{a},\sigma\right)\right)$ by determining $m_{-\alpha}\left(\mathrm{W}\left(\boldsymbol{a},\sigma\right)\right)$ for all $\alpha \in \Phi^+$. We will show that $\#\left(\mathrm{Inv}\left(\mathrm{W}\left(\boldsymbol{a},\sigma\right)\right)\right) = \sum_{l\in[n-1]} a_l l(n-l)$, which is the number of simple reflections in the expression $\mathrm{W}\left(\boldsymbol{a},\sigma\right)$. This will allow us to conclude that $\mathrm{W}\left(\boldsymbol{a},\sigma\right)$ is reduced. We will also show that $m_{-\alpha}\left(\mathrm{W}\left(\boldsymbol{a},\sigma\right)\right) = m_{-\alpha}(w)$ for each $\alpha \in \Phi^+$, allowing us to conclude that $\mathrm{W}\left(\boldsymbol{a},\sigma\right)$ spells w.

First, we determine $\mathrm{Inv}(w)$ and $l(w)$ by the same method. For $\alpha_{ij} \in \Phi^+$, we have

$$\alpha_{ij}\left(\sum_{l\in[n-1]} a_l\Theta_l\right) = \sum_{l\in[n-1]} a_l\alpha_{ij}\left(\Theta_l\right) = \sum_{i\le l<j} a_l,$$

and thus by (3.3.2),

$$m_{-\alpha}(w) = \sum_{i \le l < j} a_l.$$

By Lemma 3.1.12, we have

$$l(w) = \sum_{1 \le i < j \le n} m_{-\alpha_{ij}}(w) = \sum_{1 \le i < j \le n} \sum_{l \in [i,j-1]} a_l = \sum_{l \in [n-1]} a_l l(n-l).$$

Now, we calculate $m_{-\alpha}(\mathrm{W}(\boldsymbol{a}, \sigma))$ for $\alpha \in \Phi^+$. For any $i, j \in \mathbb{Z}$, write $\alpha_{i,j}^{(k)} = \alpha_{i+k} + \ldots + \alpha_{j+k}$, with indices taken mod n, and note that $\delta^{(k)} = \delta$. Also, for a set $S \subset X$, let χ_S be the characteristic function of S, so that for $x \in X$, $\chi_S(x) = 1$ if and only if $x \in S$. We will show the following.

1. $\mathrm{Inv}\left(\mathrm{W}_{\mathrm{l}}^{(k)}\right) = \mathrm{Inv}\left(\mathrm{W}_{\mathrm{l}}\right)^{(k)}$.
2. $\left(\mathrm{W}_{\mathrm{l}}^{(k)}\right)^{-1} (\delta - \alpha_{ij})^{(k+l)} = (\delta - \alpha_{ij})^{(k)} + \chi_{[i,j-1]}(l)\delta.$

Putting these together, along with the fact that $w\delta = \delta$ for all $w \in W_{\mathrm{aff}}$, we have

$$\begin{aligned}\left(\mathrm{W}_{\mathrm{l}}^{(k)}\right)^{-1} \mathrm{Inv}\left(\mathrm{W}_{\mathrm{l'}}^{(k+l)}\right) &= \left(\delta + \left(\mathrm{Inv}\left(W_l\right) \cap \mathrm{Inv}\left(W_{l'}\right)\right)\right)^{(k)} \cup \left(\mathrm{Inv}\left(W_{l'}\right) \backslash \mathrm{Inv}\left(W_l\right)\right)^{(k)} \\ &= \left(2\delta - \left(\Phi_l \cap \Phi_{l'}\right)\right)^{(k)} \cup \left(\delta - \left(\Phi_{l'} \backslash \Phi_l\right)\right)^{(k)}.\end{aligned}$$

Thus,

$$m_{-\alpha_{ij}}\left(\mathrm{W}_{\mathrm{l}}^{(k)} \mathrm{W}_{\mathrm{l'}}^{(k+l)}\right) = \chi_{[i,j-1]}(l) + \chi_{[i,j-1]}\left(l'\right),$$

and, continuing in this way, we have

$$m_{-\alpha_{ij}}\left(\mathrm{W}\left(\sigma, \boldsymbol{a}\right)\right) = \sum_{k \in [P]} \chi_{[i,j-1]}\left(J_{\sigma(k)}\right) = \sum_{l \in [i,j-1]} a_l = m_{-\alpha_{ij}}(w). \tag{3.3.12}$$

To show (1), let $m \in [l(n-l)]$, and let $m = pl + q(n-l)$, where $p \in [0, l-1]$ and $q \in [n-l]$. We have

$$\tau_m\left(\mathrm{W}_{\mathrm{l}}^{(k)}\right) = \left(\mathrm{W}_{\mathrm{l}}^{(k)}\right)_{m-1}^{-1} \alpha_{k+p-q+1} = \sum_{k+1-q}^{k+p} \alpha_i = \left(\delta - \alpha_{p+1,n-q}\right)^{(k)},$$

so that

$$\mathrm{Inv}\left(\mathrm{W}_{\mathrm{l}}^{(k)}\right) = \mathrm{Inv}\left(\mathrm{W}_{\mathrm{l}}\right)^{(k)}.$$

To show (2), let $m \in [l'(n-l')]$, and let $m = pl' + q(n-l')$, where $p \in [0, l'-1]$ and $q \in [n-l']$. We want to determine $\left(\mathrm{W}_{\mathrm{l}}^{(k)}\right)^{-1} \left(\delta - \alpha_{p+1,n-q}\right)^{(k+l)}$. We have

$$\left(\delta - \alpha_{p+1,n-q}\right)^{(k+l)} = \sum_{k+l+1-q}^{k+l+p} \alpha_i,$$

where indices are to be taken mod n, and this sum contains $p+q$ terms.

First, assume that $p \le l-1$ and $q \le n-l$, so that $(\delta - \alpha_{p+1,n-q})^{(k+l)} \in \delta - (\Phi_l)^{(k+l)}$ (we already know it is in $\delta - (\Phi_{l'})^{(k+l)}$), and $k+2l \notin [k+l+1-q, k+p]$. Then,

$$\begin{aligned}
\left(\mathrm{W}_{\mathrm{l}}^{(k)}\right)^{-1} \sum_{k+l+1-q}^{k+l+p} \alpha_i &= \left(\mathrm{W}_{\mathrm{l}}^{(k)}\right)^{-1} \left(\sum_{k+l+1-q}^{k+l-1} \alpha_i + \alpha_{k+l} + \sum_{k+l+1}^{k+l+p} \alpha_i \right) \\
&= \sum_{k+1-q}^{k-1} \alpha_i + 2\alpha_k + \sum_{k+1}^{k-1} \alpha_i + \sum_{k+1}^{k+p} \alpha_i \\
&= \sum_{k+1-q}^{k+p} \alpha_i + \sum_{0}^{n-1} \alpha_i \\
&= \delta + \sum_{k+1-q}^{k+p} \alpha_i = (2\delta - \alpha_{p+1,n-q})^{(k)} .
\end{aligned}$$

Next, if either $p \ge l$ or $q \ge n-l+1$ (both is not possible), then

$$\begin{aligned}
\left(\mathrm{W}_{\mathrm{l}}^{(k)}\right)^{-1} \sum_{k+l+1-q}^{k+l+p} \alpha_i &= \left(\mathrm{W}_{\mathrm{l}}^{(k)}\right)^{-1} \left(\sum_{\substack{i \in [k+l+1-q, k+l+p] \\ i \notin \{k+l, k+2l\}}} \alpha_i + \alpha_{k+l} + \alpha_{k+2l} \right) \\
&= \sum_{\substack{i \in [k+1-q, k+p] \\ i \notin \{k, k+l\}}} \alpha_i + \alpha_k + \sum_{1}^{n-1} \alpha_i - \sum_{k+l+1}^{k+l-1} \alpha_i \\
&= \sum_{\substack{i \in [k+1-q, k+p] \\ i \notin \{k, k+l\}}} \alpha_i + \alpha_k + \alpha_{k+l} \\
&= \sum_{k+1-q}^{k+p} \alpha_i = (\delta - \alpha_{p+1,n-q})^{(k)} .
\end{aligned}$$

Thus, we have shown (2). Applying it iteratively to $\mathrm{W}(\sigma, \boldsymbol{a})$ yields (3.3.12), showing that $\mathrm{W}(\sigma, \boldsymbol{a})$ spells w. □

Defining

$$\mathrm{W}_0 := \mathrm{W}_{\mathrm{n-1}}^{(\binom{n-1}{2})} \ldots \mathrm{W}_3{}^{(3)} \mathrm{W}_2{}^{(1)} \mathrm{W}_1,$$

we can apply this theorem to obtain reduced words

$$\mathrm{t}_{\mathrm{h}^*} = \begin{cases} \mathrm{W}_0 & n \text{ odd}, \\ \mathrm{W}_0{}^{(\binom{n}{2})} \mathrm{W}_0 & n \text{ even}. \end{cases}$$

We also have the following.

Corollary 3.3.13. *The roots in* $\mathbb{N}\delta - \Phi_l$ *are flipped in lexicographic order with respect to the alphabet* $\{\delta, \lambda_1, \ldots, \lambda_n\}$ *by the reduced word* $\overleftarrow{\prod}_{i=0}^{\infty} \mathrm{c}_{\mathrm{l}}^{(i)}$.

3.4 Permutation Realizations of the Classical Affine Weyl Groups

The permutation realizations of the classical affine Weyl groups are discussed at length in [3]. Our presentation differs from that one in two ways. First, we consider the fundamental interval for the action of the affine Weyl groups of types B_n, $\tilde{C}_n$, and $\tilde{D}_n$ to be $[2n]$ rather than $[-n, n]$. Second, we "absorb" the mirrors at 0 and $n+1$ by moving them to $1/2$ and $n + 1/2$. This makes our formulas for the inversion set of a permutation different from those appearing in [3]. We apologize to the reader for this change of convention, which results from our late discovery of the existing literature on the permutation representations for these types.

Let $P(\mathbb{Z})$ be the group of permutations of $\mathbb{Z}$, with composition giving the group operation. We will write $p \in P(\mathbb{Z})$ either as an explicit map on integers $i \longmapsto p(i)$, or as an infinite vector with the value $p(i)$ appearing in position i, and a semicolon appearing between position 0 and position 1. Thus, $r_{3+1/2}$, the reflection in $3 + \frac{1}{2}$, is defined by $r_{3+1/2}(i) = 7 - i$, and can be written

$$r_{3+1/2} = (\ldots, 9, 8; 7, 6, 5, 4, 3, 2, 1, 0, \ldots).$$

For $p \in P(\mathbb{Z})$ and $m \in \frac{1}{2}\mathbb{Z}$, we define $L_m(p) = \left\{n > m \;\middle|\; p(n) < m\right\}$, the set of values moved left of m by p, and $R_m(p) = \left\{n < m \;\middle|\; p(n) > m\right\}$, the set of values moved right of m by p.

Definition 3.4.1. A permutation $p \in P(\mathbb{Z})$ is locally finite if $L_{1/2}(p) < \infty$, and $R_{1/2}(p) = L_{1/2}(p)$. We denote the subgroup of locally finite permutations as $P_{\text{fin}}(\mathbb{Z})$. A permutation $p \in P(\mathbb{Z})$ is locally even at m if $L_m(p) = R_m(p) = 2k$ for $k \in \mathbb{Z}_{\geq 0}$. We denote the subgroup of permutations locally even at m by E_m.

3.4.1 Type A_{n-1}

Lusztig was the first to notice that the affine Weyl group of type A_{n-1}, also called the affine symmetric group $\tilde{S}_n$, can be realized as a subgroup of $P(\mathbb{Z})$.

Proposition 3.4.2. *The group $\tilde{S}_n$ is the group of elements $w \in P_{\text{fin}}(\mathbb{Z})$ satisfying*

$$\sum_{i=1}^{n} w(i) = \sum_{i=1}^{n} i = \binom{n}{2}, \qquad w(i+n) = w(i) + n.$$

We will think of elements of $\tilde{S}_n$ acting on positions, so that $(wv)(i) = v(w(i))$. We will denote the simple reflections for $\tilde{S}_n$ by $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$; these are the transpositions $(i, i+1)$, where indices are taken mod n, so that multiplying by $(i, i+1)$ on the right swaps $w(i + kn)$ with $w(i + 1 + kn)$, for all k.

Since $w(i + n) = w(i) + n$, the values $w(1), \ldots, w(n)$ uniquely determine w. Put another way, the set $[n]$ is a fundamental domain for the action of $\tilde{S}_n$. Thus, we will frequently refer to $w \in \tilde{S}_n$ by its window notation $w = [w(1), \ldots, w(n)]$. For example, the window notation for $\sigma_0 \in \tilde{S}_4$ is $\sigma_0 = [0, 2, 3, 5]$.

The coroot lattice $\check{T}$ is identified with the subset of $\mathbb{Z}^n$ perpendicular to $\mathbf{1} = (1, 1, \ldots, 1)$, and for $\tau = (\tau_1, \ldots, \tau_n) \in \check{T}$,

$$t_\tau[w(1), \ldots, w(n)] = [w(1) + n\tau_1, \ldots, w(n) + n\tau_n]. \tag{3.4.3}$$

The inversion table of $w \in \tilde{S}_n$ can be determined from its permutation representation. Recall that the set of positive roots for A_{n-1} is

$$\Phi^+ = \left\{\alpha_{ij} := \lambda_i - \lambda_{j+1} \,\middle|\, 1 \le i \le j \le n-1\right\}.$$

Proposition 3.4.4. *For $w \in \tilde{S}_n$, define*

$$\hat{m}_{ij}(w) = \left\lceil \frac{w^{-1}(i) - w^{-1}(j)}{n} \right\rceil.$$

Then the inversion table of w is

$$m_{\operatorname{sgn}(\hat{m}_{ij}(w))\alpha_{ij}}(w) = \begin{cases} -\hat{m}_{ij}(w), & \hat{m}_{ij}(w) < 0, \\ \hat{m}_{ij}(w) - 1, & \hat{m}_{ij}(w) \ge 0. \end{cases}$$

Example 3.4.5. To illustrate these ideas, we calculate the window notation and the inversion table for our running example w $= s_2s_0s_1s_2s_0s_2s_1$, a reduced word in the affine Weyl group of type A_2. Since we are not concerned with representatives of Weyl group elements, Iwasawa decompositions, or triangular decompositions in this section, we will use boldface type to make window notation calculations easier to follow. Thus, we will use bold type either to follow particular values (or their representatives mod n) through a series of manipulations, as in

$$\sigma_0\sigma_2\sigma_1[1, \mathbf{2}, 3] = \sigma_0\sigma_2[\mathbf{2}, 1, 3] = \sigma_0[\mathbf{2}, 3, 1] = [-2, 3, \mathbf{5}],$$

or to indicate the changes taking place in a series of manipulations, as in

$$\sigma_0\sigma_2\sigma_1[1, 2, 3] = \sigma_0\sigma_2[\mathbf{2}, \mathbf{1}, 3] = \sigma_0[2, \mathbf{3}, \mathbf{1}] = [-\mathbf{2}, 3, \mathbf{5}].$$

To calculate the window notation for w, we act on the window notation for the identity permutation $e = [1, 2, 3]$. Thus,

$$\begin{aligned} s_2s_0s_1s_2s_0s_2s_1 =& s_2s_0s_1s_2s_0s_2s_1[1, 2, 3] = s_2s_0s_1s_2s_0s_2[\mathbf{2}, \mathbf{1}, 3] = s_2s_0s_1s_2s_0[2, \mathbf{3}, \mathbf{1}] \\ =& s_2s_0s_1s_2[-\mathbf{2}, 3, \mathbf{5}] = s_2s_0s_1[-2, \mathbf{5}, \mathbf{3}] = s_2s_0[\mathbf{5}, -\mathbf{2}, 3] = s_2[\mathbf{0}, -2, \mathbf{8}] \\ =& [0, \mathbf{8}, -\mathbf{2}]. \end{aligned}$$

In applying s_0, we use the fact that $w(i+n) = w(i) + n$. For example, s_0 swaps $s_2s_1(3)$ with $s_2s_1(4)$, and we have $s_2s_1(4) = s_2s_1(1) + 3 = -1 + 3 = 2$. Alternatively, we can write the permutation s_2s_1 as

$$s_2s_1 = (\ldots; -4, -3, -5; -1, 0, -2; 2, 3, 1; 5, 6, 4; \ldots),$$

where we've placed a semicolon between 0 and 1, mod n, and we have

$$\begin{aligned} &s_0(\ldots; -4, -3, -5; -1, 0, -2; 2, 3, 1; 5, 6, 4; \ldots) \\ &= (\ldots; -8, -3, -1; -5, 0, 2; -2, 3, 5; 1, 6, 8; \ldots). \end{aligned}$$

Next we calculate the inversion table of w. We have

$$\begin{aligned} \mathrm{w}^{-1}(1) =& \mathrm{w}^{-1}(-2+3) = \mathrm{w}^{-1}(-2) + 3 = 6, \\ \mathrm{w}^{-1}(2) =& \mathrm{w}^{-1}(8-6) = \mathrm{w}^{-1}(8) - 6 = 3 - 6 = -3, \\ \mathrm{w}^{-1}(3) =& \mathrm{w}^{-1}(0+3) = \mathrm{w}^{-1}(0) + 3 = 1 + 3 = 4. \end{aligned}$$

So, we have

$$\hat{m}_{12} = \left\lceil \frac{6-(-3)}{3} \right\rceil = 4, \qquad \hat{m}_{23} = \left\lceil \frac{-3-4}{3} \right\rceil = -2, \qquad \hat{m}_{12} = \left\lceil \frac{6-4}{3} \right\rceil = 1.$$

Thus, applying Proposition 3.4.4, we recover the inversion table

$$m_{\alpha_1}(\mathrm{w}) = 3, \qquad m_{-\alpha_2}(\mathrm{w}) = 2, \qquad m_{\alpha_3}(\mathrm{w}) = 0.$$

3.4.2 Types B_n & C_n

The root systems of types B_n and C_n (for $n \geq 2$) are dual to each other, so that the roots for one are the coroots for the other. Thus, they share the same Weyl group W.

In an orthonormal basis $\{\lambda_i\}_{i \in [n]}$ with respect to the Killing form, the set of positive roots for B_n (and the set of positive coroots for C_n) is

$$\Phi^+ = \left\{ \lambda_i - \lambda_j, \;\; \lambda_i + \lambda_j, \;\; \lambda_k \;\middle|\; 1 \leq i < j \leq n, \;\; k \in [n] \right\}$$

The set of positive roots for C_n (and the set of positive coroots for B_n) is

$$\Phi^+ = \left\{ \lambda_i - \lambda_j, \quad \lambda_i + \lambda_j, \quad 2\lambda_k \quad\middle|\quad 1 \leq i \leq j \leq n, \quad k \in [n] \right\}.$$

The long roots $\pm 2\lambda_i$ for C_n can be thought of as the centers of the faces of a cube in $\mathbb{R}^n$, centered at the origin and with edges of length 4. The Weyl group W (for both B_n and C_n) is the automorphism group of this cube. It is isomorphic to the semidirect product $S_n \propto (S_2)^n$. The subgroup S_n acts by simultaneously permuting the face centers $\{2\lambda_i\}_{i\in[n]}$ and the face centers $\{-2\lambda_i\}_{i\in[n]}$, while the abelian subgroup $(S_2)^n$ changes the signs of the face centers, swapping $2\lambda_i$ and $-2\lambda_i$. The group W is also isomorphic to the subgroup of the symmetric group S_{2n}, acting on symbols $\{1, 2, \ldots, 2n\}$, which commutes with the permutation $r_{n+\frac{1}{2}}(i) = 2n+1-i$. The group W is generated by the elements $s_i = (i, i+1)(2n-i, 2n+1-i)$ for $i \in [n-1]$ and the element $s_n = (n, n+1)$. In terms of our cube, the permutation $r_{n+\frac{1}{2}}$ swaps the face centers $2\lambda_i$ and $-2\lambda_i$ for each $i \in [n]$.

The affine Weyl groups of types B_n and C_n are not isomorphic, since the highest root for B_n is $\theta = \lambda_1 + \lambda_2$, while the highest root for C_n is $\theta = 2\lambda_1$. Indeed, the affine Weyl group of type B_n is defined only for $n \geq 4$, while the affine Weyl group of type C_n is defined for $n \geq 2$. However, the previous discussion regarding the Weyl group for these types suggests permutation representations of the corresponding affine Weyl groups. These were first written down rigorously by Eriksson & Eriksson [11], although these authors acknowledged that this was not their first appearance in the literature.

Proposition 3.4.6. *For $n \geq 2$, the affine Weyl group of type C_n is the subgroup of $\tilde{S}_{2n}$ commuting with the reflection $r_{n+\frac{1}{2}}$. For $n \geq 4$, the affine Weyl group of type B_n is the subgroup of $\tilde{S}_{2n} \cap E_{\frac{1}{2}}$ commuting with $r_{n+\frac{1}{2}}$.*

For W_{aff} of type C_n, the simple reflections are the transpositions $s_i = (i, i+1)(2n-i, 2n+1-i)$ for $i \in [n-1]$, $s_n = (n, n+1)$, and $s_0 = (0, 1)$, where the indices are taken mod n. For W_{aff} of type B_n, the simple reflections are the transpositions $s_i = (i, i+1)(2n-i, 2n+1-i)$ for $i \in [n-1]$, $s_n = (n, n+1)$, and $s_0 = (-1, 1)(0, 2)$, where the indices are again taken mod n. In terms of the simple reflections for $\tilde{S}_{2n}$, we have $s_i = \sigma_i \sigma_{2n-i}$ when $i \in [n-1]$ and $s_n = \sigma_n$ for both types, while for C_n $s_0 = \sigma_0$ and for B_n $s_0 = \sigma_0 \sigma_1 \sigma_0$.

For both types, we identify $\mathbb{Z}^n \subset \mathfrak{h}_{\mathbb{R}}$ with

$$\left\{(\boldsymbol{z}, -\boldsymbol{z}) \in \mathbb{Z}^{2n} \;\middle|\; \boldsymbol{z} \in \mathbb{Z}^n\right\} \subset \check{T} \subset \mathfrak{sl}(2n, \mathbb{C}),$$

so that $\lambda_i \in \mathfrak{h}_{\mathbb{R}}$ is identified with $\lambda_i - \lambda_{2n+1-i} \in \check{T}$, and translation by λ_i is defined (as an element of $\tilde{S}_{2n}$) for all i, even though it may not be an element of W_{aff} for B_n or C_n. The coroot lattices for types B_n and C_n can then be thought of as sublattices of the coroot lattice for type A_{2n-1}, so that translation by an element of the coroot lattice for type B_n or C_n is also given by (3.4.3).

We recover the inversion table for $w \in W_{\text{aff}}$ as follows.

Proposition 3.4.7. *For $w \in W_{\text{aff}} \subset P_{\text{fin}}(\mathbb{Z})$ and $\alpha \in \Phi^+$, calculate $\hat{m}_\alpha(w)$ as follows.*

For type B_n,

$$\hat{m}_{\lambda_i-\lambda_j}(w)=\hat{m}_{ij}(w),\quad \hat{m}_{\lambda_i}(w)=\left\lceil\frac{\hat{m}_{i(2n+1-i)}(w)}{2}\right\rceil,\quad \hat{m}_{\lambda_i+\lambda_j}(w)=\hat{m}_{i(2n+1-j)}(w).$$

For type C_n,

$$\hat{m}_{\lambda_i-\lambda_j}(w)=\hat{m}_{ij}(w),\quad \hat{m}_{2\lambda_i}(w)=\hat{m}_{i(2n+1-i)}(w),\quad \hat{m}_{\lambda_i+\lambda_j}(w)=\hat{m}_{i(2n+1-j)}(w).$$

Then, the inversion table of w is

$$m_{\mathrm{sgn}(\hat{m}_\alpha(w))\alpha}(w)=\begin{cases}-\hat{m}_\alpha(w), & \hat{m}_\alpha(w)<0,\\ \hat{m}_\alpha(w)-1, & \hat{m}_\alpha(w)>0.\end{cases}$$

Example 3.4.8. In type B_4, consider $\mathrm{w}=s_3s_4s_2s_0s_1s_2$. We calculate the window notation and the inversion table for this reduced word by acting on the window notation for the identity permutation $e=[1,2,3,4,5,6,7,8]$. Thus,

$$\begin{aligned}s_3s_4s_2s_0s_1s_2 =& s_3s_4s_2s_0s_1s_2[1,2,3,4,5,6,7,8]=s_3s_4s_2s_0s_1[1,\mathbf{3},\mathbf{2},4,5,\mathbf{7},\mathbf{6},8]\\ =& s_3s_4s_2s_0[\mathbf{3},\mathbf{1},2,4,5,7,\mathbf{8},\mathbf{6}]=s_3s_4s_2[\mathbf{0},-\mathbf{2},2,4,5,7,\mathbf{11},\mathbf{9}]\\ =& s_3s_4[0,\mathbf{2},-\mathbf{2},4,5,\mathbf{11},\mathbf{7},9]=s_3[0,2,-2,\mathbf{5},\mathbf{4},11,7,9]\\ =& [0,2,\mathbf{5},-\mathbf{2},\mathbf{11},\mathbf{4},7,9].\end{aligned}$$

Now,

$$\begin{aligned}&\mathrm{w}^{-1}(1)=\mathrm{w}^{-1}(9-8)=\mathrm{w}^{-1}(9)-8=0,\qquad \mathrm{w}^{-1}(2)=2,\\ &\mathrm{w}^{-1}(3)=\mathrm{w}^{-1}(11-8)=\mathrm{w}^{-1}(11)-8=-3,\qquad \mathrm{w}^{-1}(4)=6,\\ &\mathrm{w}^{-1}(5)=3,\qquad \mathrm{w}^{-1}(6)=\mathrm{w}^{-1}(-2+8)=\mathrm{w}^{-1}(-2)+8=12,\\ &\mathrm{w}^{-1}(7)=7,\qquad \mathrm{w}^{-1}(8)=\mathrm{w}^{-1}(0+8)=\mathrm{w}^{-1}(0)+8=9.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\hat{m}_{13}=\left\lceil\frac{0-(-3)}{8}\right\rceil=1,\qquad \hat{m}_{16}=\left\lceil\frac{0-12}{8}\right\rceil=-1,\qquad \hat{m}_{18}=\left\lceil\frac{0-9}{8}\right\rceil=-1,\\ &\hat{m}_{23}=\left\lceil\frac{2-(-3)}{8}\right\rceil=1,\qquad \hat{m}_{26}=\left\lceil\frac{2-12}{8}\right\rceil=-1,\qquad \hat{m}_{34}=\left\lceil\frac{-3-6}{8}\right\rceil=-1,\\ &\hat{m}_{36}=\left\lceil\frac{-3-12}{8}\right\rceil=-1,\qquad \hat{m}_{45}=\left\lceil\frac{6-3}{8}\right\rceil=1,\end{aligned}$$

with $\hat{m}_{ij}=0$ for all other $i\in[4]$ and $j\in[i+1,9-i]$. Thus, applying Proposition 3.4.7, we recover the inversion table

$$\begin{aligned}&m_{\lambda_1-\lambda_3}(\mathrm{w})=1,\qquad m_{-(\lambda_1+\lambda_3)}(\mathrm{w})=1,\qquad m_{\lambda_2-\lambda_3}(\mathrm{w})=1,\\ &m_{-(\lambda_2+\lambda_3)}(\mathrm{w})=1,\qquad m_{-(\lambda_3-\lambda_4)}(\mathrm{w})=1,\qquad m_{\lambda_4}(\mathrm{w})=1,\end{aligned}$$

with $m_\alpha(\mathrm{w}) = 0$ for all other $\alpha \in \Phi$, and

$$\mathrm{Inv}\,(\mathrm{w}) = \{\lambda_1 - \lambda_3,\ \delta - (\lambda_1 + \lambda_3)\,,\ \lambda_2 - \lambda_3,\ \delta - (\lambda_2 + \lambda_3)\,,\ \delta - (\lambda_3 - \lambda_4)\,,\ \lambda_4\}\,.$$

In type C_2, consider the reduced word $\mathbf{w} = s_2 s_0 s_1$. We have

$$s_2 s_0 s_1[1, 2, 3, 4] = s_2 s_0[\mathbf{2}, \mathbf{1}, \mathbf{4}, \mathbf{3}] = s_2[-\mathbf{1}, 1, 4, \mathbf{6}] = [-1, \mathbf{4}, \mathbf{1}, 6],$$

so that

$$\mathrm{w}^{-1}(1) = 3, \qquad \mathrm{w}^{-1}(2) = \mathrm{w}^{-1}(6-4) = \mathrm{w}^{-1}(6) - 4 = 0,$$
$$\mathrm{w}^{-1}(3) = \mathrm{w}^{-1}(-1+4) = \mathrm{w}^{-1}(-1) + 4 = 5, \qquad \mathrm{w}^{-1}(4) = 2.$$

Thus,

$$\hat{m}_{12} = \left\lceil \frac{3-0}{4} \right\rceil = 1, \qquad \hat{m}_{14} = \left\lceil \frac{3-2}{4} \right\rceil = 1, \qquad \hat{m}_{23} = \left\lceil \frac{0-5}{4} \right\rceil = -1,$$

with $\hat{m}_{ij} = 0$ for all other $i \in [4]$ and $j \in [i+1, 5-i]$. Again applying Proposition 3.4.7, we recover

$$m_{\lambda_1 - \lambda_2}(\mathrm{w}) = 1, \qquad m_{2\lambda_1}(\mathrm{w}) = 1, \qquad m_{-2\lambda_2}(\mathrm{w}) = 1,$$

with $m_\alpha(\mathrm{w}) = 0$ for all other $\alpha \in \Phi$, and

$$\mathrm{Inv}\,(\mathrm{w}) = \{\lambda_1 - \lambda_2,\ 2\lambda_1,\ \delta - 2\lambda_2\}\,.$$

3.4.3 Type D_n

The long roots of a root system of type B_n form a root system of type D_n. In an orthonormal basis $\{\lambda_i\}_{i\in[n]}$, the positive simple roots of D_n are the vectors $\alpha_i := \lambda_i - \lambda_{i+1}$ for $i = 1, \ldots, n-1$, and $\alpha_n = \lambda_{n-1} + \lambda_n$. The set of positive roots is

$$\Phi^+ = \left\{ \lambda_p \pm \lambda_q \;\middle|\; 1 \le p < q \le n \right\}.$$

It is thus possible to recover a permutation representation of W_{aff} for D_n (defined for $n \ge 5$) from the permutation representation of W_{aff} for B_n, and we have the following.

Proposition 3.4.9. *For $n \ge 5$, the affine Weyl group of type D_n is the subgroup of $\tilde{S}_{2n} \cap E_{\frac{1}{2}} \cap E_{n+\frac{1}{2}}$ commuting with $r_{n+\frac{1}{2}}$.*

The simple reflections for this group are the transpositions $s_i = (i, i+1)(2n-i, 2n+1-i)$ for $i \in [n-2]$, $s_0 = (-1, 1)(0, 2)$, and $s_n = (n-1, n+1)(n, n+2)$, where the indices are as usual taken mod n. In terms of the simple reflections for $\tilde{S}_{2n}$, we have $s_i = \sigma_i \sigma_{2n-i}$ when $i \in [n-2]$, $s_0 = \sigma_0 \sigma_1 \sigma_0$, and $s_n = \sigma_{n-1} \sigma_n \sigma_{n-1}$.

Finally, we have the following.

Proposition 3.4.10. *For type D_n, $w \in W_{\text{aff}}$, and $\alpha \in \Phi^+$, define $\hat{m}_\alpha(w)$ to be*

$$\hat{m}_{\lambda_i-\lambda_j}(w) = \hat{m}_{ij}(w), \qquad \hat{m}_{\lambda_i+\lambda_j} = \hat{m}_{i(2n+1-j)}(w).$$

Then, the inversion table of w is

$$m_{\text{sgn}(\hat{m}_\alpha(w))\alpha}(w) = \begin{cases} -\hat{m}_\alpha(w), & \hat{m}_\alpha(w) < 0, \\ \hat{m}_\alpha(w) - 1, & \hat{m}_\alpha(w) > 0. \end{cases}$$

Example 3.4.11. In type D_6, consider $\mathrm{w} = s_0s_2s_3s_5s_6s_4$. We calculate the window notation and the inversion table for this reduced word by acting on the window notation for the identity permutation $e = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]$. Thus,

$$\begin{aligned} s_0s_2s_3s_5s_6s_4 =& s_0s_2s_3s_5s_6s_4[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] \\ =& s_0s_2s_3s_5s_6[1, 2, 3, \mathbf{5}, \mathbf{4}, 6, 7, \mathbf{9}, \mathbf{8}, 10, 11, 12] \\ =& s_0s_2s_3s_5[1, 2, 3, 5, \mathbf{7}, \mathbf{9}, \mathbf{4}, \mathbf{6}, 8, 10, 11, 12] \\ =& s_0s_2s_3[1, 2, 3, 5, \mathbf{9}, \mathbf{7}, \mathbf{6}, \mathbf{4}, 8, 10, 11, 12] \\ =& s_0s_2[1, 2, \mathbf{5}, \mathbf{3}, 9, 7, 6, 4, \mathbf{10}, \mathbf{8}, 11, 12] = s_0[1, \mathbf{5}, \mathbf{2}, 3, 9, 7, 6, 4, 10, \mathbf{11}, \mathbf{8}, 12] \\ =& [-\mathbf{4}, \mathbf{0}, 2, 3, 9, 7, 6, 4, 10, 11, \mathbf{13}, \mathbf{17}]. \end{aligned}$$

Now,

$$\begin{aligned} &\mathrm{w}^{-1}(1) = \mathrm{w}^{-1}(13 - 12) = -1, && \mathrm{w}^{-1}(7) = 6, \\ &\mathrm{w}^{-1}(2) = 3, && \mathrm{w}^{-1}(8) = \mathrm{w}^{-1}(-4 + 12) = 13, \\ &\mathrm{w}^{-1}(3) = 4, && \mathrm{w}^{-1}(9) = 5, \\ &\mathrm{w}^{-1}(4) = 8, && \mathrm{w}^{-1}(10) = 9, \\ &\mathrm{w}^{-1}(5) = \mathrm{w}^{-1}(17 - 12) = 0, && \mathrm{w}^{-1}(11) = 10, \\ &\mathrm{w}^{-1}(6) = 7, && \mathrm{w}^{-1}(12) = \mathrm{w}^{-1}(0 + 12) = 14. \end{aligned}$$

Thus, we have

$$\hat{m}_{18} = -1, \qquad \hat{m}_{25} = \hat{m}_{35} = \hat{m}_{45} = \hat{m}_{46} = \hat{m}_{47} = 1,$$

with $\hat{m}_{ij} = 0$ for all other $i \in [6]$ and $j \in [i+1, 12-i]$. Thus, applying Proposition 3.4.10, we recover the inversion table

$$m_{-(\lambda_1+\lambda_6)}(\mathrm{w}) = 1, \qquad m_{\lambda_2-\lambda_5}(\mathrm{w}) = m_{\lambda_3-\lambda_5}(\mathrm{w}) = m_{\lambda_4-\lambda_5}(\mathrm{w}) = m_{\lambda_4-\lambda_6} = m_{\lambda_4+\lambda_6} = 1,$$

with $m_\alpha(\mathrm{w}) = 0$ for all other $\alpha \in \Phi$, so that

$$\mathrm{Inv}(\mathrm{w}) = \{\delta - (\lambda_1 - \lambda_6),\ \lambda_2 - \lambda_5,\ \lambda_3 - \lambda_5,\ \lambda_4 - \lambda_5,\ \lambda_4 - \lambda_6,\ \lambda_4 + \lambda_6\}.$$

3.5 Reduced Words for the Coweight Elements of Other Classical Types

In this section, we find reduced words for the coweight elements of the affine Weyl groups of types B_n, C_n, and D_n.

Most of the proofs involve calculating the window notation for a reduced word, and then calculating the inversion table of that reduced word from its window notation. We will follow the logic of Theorem 3.3.10. Namely, to show that $\mathrm{w} = s_{i_p} \dots s_{i_1}$ is reduced and spells w, we calculate the window notation of w; we calculate $m_\alpha(\mathrm{w})$ from this window notation; we calculate $l(\mathrm{w})$ from $m_\alpha(\mathrm{w})$ and show that it equals p; and we show that $m_\alpha(\mathrm{w}) = m_\alpha(w)$ for all $\alpha \in \Phi^+$. For $\sigma \in P(\mathbb{Z})$, we define the notation

$$\mathrm{w}^\sigma = s_{\sigma(i_n)} \dots s_{\sigma(i_1)},$$

where, in this section, indices should be taken mod n.

3.5.1 Type $\tilde{B}_n$

The fundamental coweights for type B_n are the vectors $\Theta_l = \lambda_1 + \dots + \lambda_l$ for $l \in [n]$. If l is even, then $\Theta_l \in \check{T}$ since

$$\Theta_l = (\lambda_1 + \lambda_2) + \dots + (\lambda_{l-1} + \lambda_l) .$$

If l is odd, then $2\Theta_l \in \check{T}$. If l' is also odd (for definiteness let $l' > l$), then $\Theta_l + \Theta_{l'} \in \check{T}$, since $l' - l$ is even, and we have

$$\Theta_l + \Theta_l' = 2\lambda_1 + \dots + 2\lambda_l + (\lambda_{l+1} + \lambda_{l+2}) + \dots + (\lambda_{l'-1} + \lambda_{l'}) .$$

Thus the shortest element of $\check{T} \cap C_0$ is given by

$$h^* := \begin{cases} \Theta_1 + \dots + \Theta_n & n = 0, 3 \text{ mod } 4 \\ 2\Theta_1 + \dots + \Theta_n & n = 1, 2 \text{ mod } 4. \end{cases}$$

We also have

$$\Phi_l = \left\{ \lambda_i - \lambda_j, \;\; \lambda_i + \lambda_k, \;\; \lambda_i \;\middle|\; i \in [l], \;\; j \in [l+1, n], \;\; k \in [i+1, n] \right\},$$

$$\mathrm{Inv}\,(W_l) = \left\{ \delta - (\lambda_i - \lambda_j)\,,\; \delta - (\lambda_i + \lambda_k)\,,\; \delta - \lambda_i \;\middle|\; i \in [l],\, j \in [l+1, n],\, k \in [i+1, n] \right\}$$
$$\cup \left\{ 2\delta - (\lambda_i + \lambda_j) \;\middle|\; i, j \in [l] \right\} .$$

Proposition 3.5.1. *In W_{aff} of type B_n, for $l \in [n]$, define*

$$\mathrm{c}_l = s_l s_{l+1} \dots s_{n-1} s_n s_{n-1} \dots s_3 s_2 s_0 .$$

(Note that $l(c_i) = 2n - i$.) Then for $l \in [2, n]$, the l^{th} coweight element for W_{aff} of type B_n has reduced word

$$\mathrm{W}_{\mathrm{l}} = \begin{cases} (\mathrm{c}_{\mathrm{l}}^{\sigma_0}\mathrm{c}_{\mathrm{l}})^s & l = 2s, \\ \mathrm{c}_{\mathrm{l}}\,(\mathrm{c}_{\mathrm{l}}^{\sigma_0}\mathrm{c}_{\mathrm{l}})^s & l = 2s+1, \end{cases}$$

while

$$\mathrm{W}_1 = s_0\mathrm{c}_2 = s_0 s_2 s_3 \ldots s_{n-1} s_n s_{n-1} \ldots s_3 s_2 s_0.$$

Example 3.5.2. In the simplest case, B_4, we have

$$\mathrm{c}_1 = s_1s_2s_3s_4s_3s_2s_0, \qquad \mathrm{c}_2 = s_2s_3s_4s_3s_2s_0, \qquad \mathrm{c}_3 = s_3s_4s_3s_2s_0, \qquad \mathrm{c}_4 = s_4s_3s_2s_0,$$

and

$$\mathrm{W}_1 = s_0s_2s_3s_4s_3s_2s_0, \qquad \mathrm{W}_2 = s_2s_3s_4s_3s_2s_1s_2s_3s_4s_3s_2s_0,$$

$$\mathrm{W}_3 = s_3s_4s_3s_2s_0s_3s_4s_3s_2s_1s_3s_4s_3s_2s_0, \qquad \mathrm{W}_4 = (s_4s_3s_2s_1s_4s_3s_2s_0)^2.$$

In the case of B_5, we have

$$\mathrm{c}_1 = s_1s_2s_3s_4s_5s_4s_3s_2s_0, \qquad \mathrm{c}_2 = s_2s_3s_4s_5s_4s_3s_2s_0,$$

$$\mathrm{c}_3 = s_3s_4s_5s_4s_3s_2s_0, \qquad \mathrm{c}_4 = s_5s_4s_3s_2s_0,$$

and

$$\mathrm{W}_1 = s_0s_2s_3s_4s_5s_4s_3s_2s_0, \qquad \mathrm{W}_2 = s_2s_3s_4s_5s_4s_3s_2s_1s_2s_3s_4s_5s_4s_3s_2s_0,$$

$$\mathrm{W}_3 = s_3s_4s_5s_4s_3s_2s_0s_3s_4s_5s_4s_3s_2s_1s_3s_4s_5s_4s_3s_2s_0, \quad \mathrm{W}_4 = (s_4s_5s_4s_3s_2s_1s_4s_5s_4s_3s_2s_0)^2,$$

$$\mathrm{W}_5 = s_5s_4s_3s_2s_0\,(s_5s_4s_3s_2s_1s_5s_4s_3s_2s_0)^2.$$

Proof. We act on the window notation for the identity permutation $e = [1, \ldots, 2n]$ with W_{l}, and calculate the inversion table of the result. Acting on the window notation for e, the word $\mathrm{c}_{\mathrm{l}}^{\sigma_0}\mathrm{c}_{\mathrm{l}}$ pulls two values from the ends of the subsequent and previous windows, and then moves these values into positions $l-1, l$ and $2n-l+1, 2n-l+2$, respectively:

$$\begin{aligned}
&\mathrm{c}_{\mathrm{l}}^{\sigma_0}\mathrm{c}_{\mathrm{l}}[1, \ldots, 2n] \\
&= \mathrm{c}_{\mathrm{l}}^{\sigma_0} s_l s_{l+1} \ldots s_{n-1} s_n s_{n-1} \ldots s_3 s_2 [\mathbf{-1}, \mathbf{0}, 3, \ldots, 2n-2, \mathbf{2n+1}, \mathbf{2n+2}] \\
&= \mathrm{c}_{\mathrm{l}}^{\sigma_0}[\mathbf{-1}, 3, \ldots, l, \mathbf{2n+1}, l+1, \ldots, 2n-l, \mathbf{0}, 2n-l+1, \ldots, 2n-2, \mathbf{2n+2}] \\
&= [3, \ldots, l, \mathbf{2n+1}, \mathbf{2n+2}, l+1, \ldots, 2n-l, \mathbf{-1}, \mathbf{0}, 2n-l+1, \ldots, 2n-2].
\end{aligned}$$

Thus, if $l = 2s$, we have

$$\begin{aligned}
(\mathrm{c}_{\mathrm{l}}^{\sigma_0}\mathrm{c}_{\mathrm{l}})^s &= [\mathbf{2n+1}, \ldots, \mathbf{2n+l}, l+1, \ldots, 2n-l, \mathbf{-l+1}, \ldots, \mathbf{0}] \\
&= e + [2n, \ldots, 2n, 0, \ldots, 0, -2n, \ldots, -2n] = t_{(1,\ldots,1,0,\ldots,0,-1,\ldots,-1)} = t_{\Theta_l}.
\end{aligned}$$

Alternatively, if $l = 2s + 1$, we have

$$\begin{aligned}
&\mathrm{c}_{\mathrm{l}}\left(\mathrm{c}_{\mathrm{l}}^{\sigma_0}\mathrm{c}_{\mathrm{l}}\right)^s \\
&= \mathrm{c}_{\mathrm{l}}[l, \mathbf{2n+1}, \ldots, \mathbf{2n+l-1}, l+1, \ldots, 2n-l, \mathbf{-l+2}, \ldots, \mathbf{0}, 2n-l+1] \\
&= s_l \ldots s_n \ldots s_2[\mathbf{-2n}, \mathbf{-l+1}, 2n+2, \ldots, 2n+l-1, l+1, \ldots \\
&\qquad \ldots, 2n-l, -l+2, \ldots, -1, \mathbf{2n+l}, \mathbf{4n+1}] \\
&= [-2n, 2n+2, \ldots, \mathbf{2n+l}, l+1, \ldots, 2n-l, \mathbf{-l+1}, \ldots, -1, 4n+1] \\
&= \sigma_0[2n+1, \ldots, 2n+l, l+1, \ldots, 2n-l, -l+1, \ldots, -1, 0] \\
&= \sigma_0 t_{(1,\ldots,1,0,\ldots,0,-1,\ldots,-1)} = \sigma_0 t_{\Theta_l},
\end{aligned}$$

where in the last line we are thinking of Θ_l as an element of $\check{T} \subset \mathfrak{sl}(2n, \mathbb{C})$. Thus, we have

$$\begin{aligned}
\mathrm{W}_{\mathrm{l}}^{-1}(1) &= \mathrm{W}_{\mathrm{l}}^{-1}(4n+1) - 4n = -2n, \\
\mathrm{W}_{\mathrm{l}}^{-1}(2n) &= \mathrm{W}_{\mathrm{l}}^{-1}(-2n) + 4n = 4n+1,
\end{aligned}$$

and

$$\mathrm{W}_{\mathrm{l}}^{-1}(i) = \begin{cases} \mathrm{W}_{\mathrm{l}}^{-1}(2n+i) - 2n = i - 2n & i \in [2, l] \\ i & i \in [l+1, 2n-l] \\ \mathrm{W}_{\mathrm{l}}^{-1}(-(2n+1-i)+1) + 2n = i + 2n & i \in [2n-l+1, 2n-1]. \end{cases}$$

This gives us

$$\hat{m}_{\lambda_i - \lambda_j} = \begin{cases} 0 & i, j \in [1, l] \\ -1 & i \in [1, l], \quad j \in [l+1, n], \end{cases} \qquad \hat{m}_{\lambda_k} = \begin{cases} -3 & k = 1 \\ -2 & k \in [1, l] \\ 0 & k \in [l+1, n], \end{cases}$$

$$\hat{m}_{\lambda_i + \lambda_j} = \begin{cases} -2 & i, j \in [1, l] \\ -1 & i \in [1, l], \quad j \in [l+1, n] \\ 0 & i, j \in [l+1, n], \end{cases}$$

so that

$$m_{-(\lambda_i - \lambda_j)} = \begin{cases} 0 & i, j \in [1, l] \\ 1 & i \in [1, l], \quad j \in [l+1, n], \end{cases} \qquad m_{-\lambda_i} = \begin{cases} 1 & i \in [1, l] \\ 0 & i \in [l+1, n], \end{cases}$$

$$m_{-(\lambda_i + \lambda_j)} = \begin{cases} 2 & i, j \in [1, l] \\ 1 & i \in [1, l], \quad j \in [l+1, n] \\ 0 & i, j \in [l+1, n]. \end{cases}$$

Thus, the inversion set of $\mathrm{W_l}$ is $\mathrm{Inv}\,(W_l)$. Furthermore,

$$\begin{aligned} l\,(\mathrm{W_l}) = &\sum_{1\le i<j\le n} \left(m_{-(\lambda_i-\lambda_j)}\,(\mathrm{W_l}) + m_{-(\lambda_i+\lambda_j)}\,(\mathrm{W_l})\right) + \sum_{k\in[n]} m_{-\lambda_i}\,(\mathrm{W_l}) \\ =& l(n-l) + l + 2\tbinom{l}{2} + l(n-l) = 2nl - l^2, \end{aligned}$$

which is the number of simple reflections in $\mathrm{W_l}$, and we conclude that $\mathrm{W_l}$ is reduced. Next,

$$\begin{aligned} s_0\mathrm{c}_2 &= s_0s_2s_3\dots s_{n-1}s_ns_{n-1}\dots s_3s_2s_0[1,2,3,\dots,2n-2,2n-1,2n] \\ &= s_0s_2s_3\dots s_{n-1}s_ns_{n-1}\dots s_3s_2[\mathbf{-1},\mathbf{0},3,\dots,2n-2,\mathbf{2n+1},\mathbf{2n+2}] \\ &= s_0[-1,\mathbf{2n+1},3,\dots,2n-2,\mathbf{0},2n+2] \\ &= [\mathbf{-2n},\mathbf{2},3,\dots,2n-2,\mathbf{2n-1},\mathbf{4n+1}] \\ &= \sigma_0[2n+1,2,\dots,2n-1,0] = \sigma_0 t_{(1,0,\dots,0,-1)} = \sigma_0 t_{\Theta_1}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{W}_1^{-1}(1) =& \mathrm{W}_1^{-1}(4n+1) - 4n = -2n, \\ \mathrm{W}_1^{-1}(2n) =& \mathrm{W}_1^{-1}(-2n) + 4n = 4n+1, \end{aligned}$$

and $\mathrm{W}_1^{-1}(i) = i$ for $i \in [2, 2n-1]$. This gives

$$m_{-(\lambda_1-\lambda_k)} = -1, \qquad m_{-\lambda_1} = -1, \qquad m_{-(\lambda_1+\lambda_k)} = -1,$$

for $k \in [2, n]$, as desired. □

We then have the following.

Corollary 3.5.3. *In type B_n, the reduced word $\mathrm{W_l}$ flips the roots in $\mathrm{Inv}\,(W_l)$ according to the (total inversion) order $\prec$, where $\prec$ is determined by the following conditions.*

1. *For fixed k, the roots $k\delta - \alpha$ are flipped in lexicographic order with respect to the alphabet $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.*
2. *For $j \in [2, r]$, and $i \in [j-1]$,*

$$\delta - (\lambda_j + \lambda_l) \prec 2\delta - (\lambda_i + \lambda_j) \prec \delta - (\lambda_j + \lambda_{l+1})\,.$$

We note that the first condition on $\prec$ can be restated as follows:

(1a) If there exists i such that $\alpha \in \Phi_i \cap \Phi_l$ and $i < j$ for all j such that $\beta \in \Phi_j \cap \Phi_l$, then $\delta - \alpha \prec \delta - \beta$.

(1b) If $\alpha, \beta \in \Phi_i \cap \Phi_l$, and $\mathrm{ht}\,(\alpha) > \mathrm{ht}\,(\beta)$, then $k\delta - \alpha \prec k\delta - \beta$ $(k = 1, 2)$.

Example 3.5.4. For $n = 6$ and $l = 2$, we have

$$\begin{aligned}
&\delta - (\lambda_1 + \lambda_2) \prec \delta - (\lambda_1 + \lambda_3) \prec \delta - (\lambda_1 + \lambda_4) \prec \delta - (\lambda_1 + \lambda_5) \prec \delta - (\lambda_1 + \lambda_6)\\
&\prec \delta - \lambda_1 \prec \delta - (\lambda_1 - \lambda_6) \prec \delta - (\lambda_1 - \lambda_5) \prec \delta - (\lambda_1 - \lambda_4) \prec \delta - (\lambda_1 - \lambda_3)\\
&\prec 2\delta - (\lambda_1 + \lambda_2) \prec \delta - (\lambda_2 + \lambda_3) \prec \delta - (\lambda_2 + \lambda_4) \prec \delta - (\lambda_2 + \lambda_5)\\
&\prec \delta - \lambda_2 \prec \delta - (\lambda_2 - \lambda_6) \prec \delta - (\lambda_2 - \lambda_5) \prec \delta - (\lambda_2 - \lambda_4) \prec \delta - (\lambda_2 - \lambda_3)\,,
\end{aligned}$$

while for $n = 6$ and $l = 5$, we have

$$\begin{aligned}
&\delta - (\lambda_1 + \lambda_2) \prec \delta - (\lambda_1 + \lambda_3) \prec \delta - (\lambda_1 + \lambda_4) \prec \delta - (\lambda_1 + \lambda_5) \prec \delta - (\lambda_1 + \lambda_6)\\
&\prec \delta - \lambda_1 \prec \delta - (\lambda_1 - \lambda_6)\\
&\prec \delta - (\lambda_2 + \lambda_3) \prec \delta - (\lambda_2 + \lambda_4) \prec \delta - (\lambda_2 + \lambda_5)\\
&\prec 2\delta - (\lambda_1 + \lambda_2) \prec 2\delta - (\lambda_1 + \lambda_3) \prec 2\delta - (\lambda_1 + \lambda_4) \prec 2\delta - (\lambda_1 + \lambda_5)\\
&\prec \delta - (\lambda_2 + \lambda_6) \prec \delta - \lambda_2 \prec \delta - (\lambda_2 - \lambda_6)\\
&\prec \delta - (\lambda_3 + \lambda_4) \prec \delta - (\lambda_3 + \lambda_5)\\
&\prec 2\delta - (\lambda_2 + \lambda_3) \prec 2\delta - (\lambda_2 + \lambda_4) \prec 2\delta - (\lambda_2 + \lambda_5)\\
&\prec \delta - (\lambda_3 + \lambda_6) \prec \delta - \lambda_3 \prec \delta - (\lambda_3 - \lambda_6)\\
&\prec \delta - (\lambda_4 + \lambda_5)\\
&\prec 2\delta - (\lambda_3 + \lambda_4) \prec 2\delta - (\lambda_3 + \lambda_5)\\
&\prec \delta - (\lambda_4 + \lambda_6) \prec \delta - \lambda_4 \prec \delta - (\lambda_4 - \lambda_6)\\
&\prec 2\delta - (\lambda_4 + \lambda_5)\\
&\prec \delta - (\lambda_5 + \lambda_6) \prec \delta - \lambda_5 \prec \delta - (\lambda_5 - \lambda_6)\,.
\end{aligned}$$

In combining the reduced words $\mathrm{W_l}$, the following lemma on commutation relations will be useful. It also provides us with reduced words for the coweight translations T_l.

Lemma 3.5.5. *In type B_n, for odd numbers l, l' and even numbers k, k', we have*

$$\mathrm{W_k W_{k'}} = \mathrm{W_{k'} W_k} = t_{\Theta_k + \Theta_{k'}}, \qquad \mathrm{W_{l'}^{\sigma_0} W_l} = \mathrm{W_l^{\sigma_0} W_{l'}} = t_{\Theta_l + \Theta_{l'}},$$
$$\mathrm{W_k^{\sigma_0} W_l} = \mathrm{W_l W_k} = \sigma_0 t_{\Theta_k + \Theta_l}.$$

Proof. The commutation relation for even coweight elements is due to the fact that these elements are translations by elements of $\check{T}$. For the remaining relations, we first calculate the window notation for $\mathrm{W_l^{\sigma_0}}$. We have

$$\begin{aligned}
&\mathrm{c_l c_l^{\sigma_0}}[1, \ldots, 2n]\\
&= \mathrm{c_l}[2, \ldots, l, \mathbf{2n}, l+1, \ldots, 2n-l, \mathbf{1}, 2n-l+1, \ldots, 2n-1]\\
&= s_l s_{l+1} \ldots s_{n-1} s_n s_{n-1} \ldots s_3 s_2 [\mathbf{-2}, \mathbf{-1}, 4, \ldots, l, \mathbf{2n}, l+1, \ldots, 2n-l, \mathbf{1},\\
&\qquad 2n-l+1, \ldots, 2n-3, \mathbf{2n+2}, \mathbf{2n+3}]\\
&= [\mathbf{-2}, 4, \ldots, l, \mathbf{2n}, \mathbf{2n+1}, l+1, \ldots, 2n-l, \mathbf{-1}, \mathbf{1}, 2n-l+1, \ldots, 2n-3, \mathbf{2n+3}].
\end{aligned}$$

Thus, for $k = 2s$, we have

$$
\begin{aligned}
&(\mathrm{c_k c_k^{\sigma_0}})^s\,[1,\ldots,2n]\\
&= \mathrm{c_k c_k^{\sigma_0}}[-k+2,k,2n,2n+2,\ldots,2n+k-2,k+1,\ldots\\
&\qquad \ldots,2n-k,-k-1,\ldots,-1,1,2n+1-k,2n+k-1]\\
&= \mathrm{c_k}[k,2n,2n+2,\ldots,\mathbf{2n+k-1},k+1,\ldots,2n-k,\mathbf{-k+2},\ldots,-1,1,2n+1-k]\\
&= s_k\ldots s_n\ldots s_2[\mathbf{-2n+1},\mathbf{-k+1},2n+2,\ldots,2n+k-1,k+1,\ldots\\
&\qquad \ldots,2n-k,-k+2,\ldots,-1,\mathbf{2n+k},\mathbf{4n}]\\
&= [-2n+1,2n+2,\ldots,\mathbf{2n+k},k+1,\ldots,2n-k,-k+1,\ldots,-1,\mathbf{4n}]\\
&= e+[-2n,2n,\ldots,2n,0,\ldots,0,-2n,\ldots,-2n,2n]\\
&= t_{(-1,1,\ldots,1,0,\ldots,0,-1,\ldots,-1,1)} = t_{\sigma_0\Theta_k}.
\end{aligned}
$$

Now, let k be even and l be odd. Then we have

$$\mathrm{W_k^{\sigma_0}W_l} = t_{\sigma_0\Theta_k}\sigma_0 t_{\Theta_l} = \sigma_0\sigma_0 t_{\sigma_0\Theta_k}\sigma_0 t_{\Theta_l} = \sigma_0 t_{\sigma_0\sigma_0\Theta_k} t_{\Theta_l} = \sigma_0 t_{\Theta_k+\Theta_l},$$

while

$$\mathrm{W_l W_k} = \sigma_0 t_{\Theta_l} t_{\Theta_k} = \sigma_0 t_{\Theta_k+\Theta_l}.$$

To check the commutation relation for odd elements, let $l = 2s+1$ and $l' = 2s'+1$, and let $l' < l$. Then,

$$
\begin{aligned}
&\mathrm{W_{l'}^{\sigma_0}W_l}\\
&= \mathrm{W_{l'}^{\sigma_0}}[-2n,2n+2,\ldots,2n+l,l+1,\ldots,2n-l,-l+1,\ldots,-1,4n+1]\\
&= (\mathrm{c_{l'}^{\sigma_0}c_{l'}})^{s'}\,[2n+2,\ldots,2n+l',\mathbf{4n+1},2n+l'+1,\ldots,2n+l,l+1,\ldots\\
&\qquad \ldots,2n-l,-l+1,\ldots,-l',\mathbf{-2n},-l'+1,\ldots,-1]\\
&= (\mathrm{c_{l'}^{\sigma_0}c_{l'}})^{s'-1}\,[2n+4,\ldots,2n+l',4n+1,\mathbf{4n+2},\mathbf{4n+3},2n+l'+1,\ldots,2n+l,l+1,\ldots\\
&\qquad \ldots,2n-l,-l+1,\ldots,-l',\mathbf{-2n-2},\mathbf{-2n-1},-2n,-l'+1,\ldots,-3]\\
&= \mathrm{c_{l'}^{\sigma_0}c_{l'}}[2n+l'-1,2n+l',4n+1,\ldots,4n+l'-2,2n+l'+1,\ldots,2n+l,l+1,\ldots\\
&\qquad \ldots,2n-l,-l+1,\ldots,-l',-2n-l'+2,\ldots,-2n,-l',-l'+1]\\
&= [4n+1,\ldots,4n+l',2n+l'+1,\ldots,2n+l,l+1,\ldots\\
&\qquad \ldots,2n-l,-l+1,\ldots,-l',-2n-l',\ldots,-2n]\\
&= t_{(2,\ldots,2,1,\ldots,1,0,\ldots,0,-1,\ldots,-1,-2,\ldots,-2)} = t_{\Theta_l+\Theta_{l'}},
\end{aligned}
$$

∎

while

$$\begin{aligned}
&\mathrm{W}_{\mathrm{l}}^{\sigma_0}\mathrm{W}_{\mathrm{l}'}\\
&= \mathrm{W}_{\mathrm{l}}^{\sigma_0}[-2n, 2n+2, \ldots, 2n+l', l'+1, \ldots, 2n-l', -l'+1, \ldots, -1, 4n+1]\\
&= (\mathrm{c}_{\mathrm{l}}^{\sigma_0}\mathrm{c}_{\mathrm{l}})^s\,[2n+2, \ldots, 2n+l', l'+1, \ldots, l, \mathbf{4n+1}, l+1, \ldots\\
&\qquad \ldots, 2n-l, \mathbf{-2n}, 2n-l+1, \ldots, 2n-l', -l'+1, \ldots, -1]\\
&= (\mathrm{c}_{\mathrm{l}}^{\sigma_0}\mathrm{c}_{\mathrm{l}})^{s-1}\,[2n+4, \ldots, 2n+l', l'+1, \ldots, l, 4n+1, \mathbf{4n+2}, \mathbf{4n+3}, l+1, \ldots\\
&\qquad \ldots, 2n-l, \mathbf{-2n-2}, \mathbf{-2n-1}, -2n, 2n-l+1, \ldots, 2n-l', -l'+1, \ldots, -3]\\
&= \mathrm{c}_{\mathrm{l}}^{\sigma_0}\mathrm{c}_{\mathrm{l}}[l-1, l, 4n+1, \ldots, 4n+l', 2n+l'+1, \ldots, 2n+l-2, l+1, \ldots, 2n-l,\\
&\qquad -l+3, \ldots, -l', -2n-l'+1, \ldots, -2n, 2n-l+1, 2n-l+2]\\
&= [4n+1, \ldots, 4n+l', 2n+l'+1, \ldots, 2n+l, l+1, \ldots\\
&\qquad \ldots, 2n-l, -l+1, \ldots, -l', -2n-l'+1, \ldots, -2n]\\
&= t_{(2,\ldots,2,1,\ldots,1,0,\ldots,0,-1,\ldots,-1,-2,\ldots,-2)} = t_{\Theta_l+\Theta_{l'}}.
\end{aligned}$$

□

We can now specify reduced words for the coweight translations T_l.

Proposition 3.5.6. *In type B_n, for $l \in [2, n]$, the l^{th} coweight translation has reduced word*

$$\mathrm{T}_{\mathrm{l}} = \begin{cases} (\mathrm{c}_{\mathrm{l}}^{\sigma_0}\mathrm{c}_{\mathrm{l}})^s & l = 2s,\\ (\mathrm{c}_{\mathrm{l}}^{\sigma_0}\mathrm{c}_{\mathrm{l}})^{2s+1} & l = 2s+1,\end{cases}$$

while the first coweight translation has reduced words

$$\mathrm{T}_1 = \mathrm{c}_1^2, \qquad \mathrm{T}'_1 = s_1\mathrm{c}_2^{\sigma_0}s_0\mathrm{c}_2.$$

Proof. The coweight words for $l \in [2, n]$ follow from the calculation in the proof of Proposition 3.5.1, and from Proposition 3.5.5. For $l = 1$, we have

$$\begin{aligned}
\mathrm{T}'_1 &= s_1s_2\ldots s_ns_{n-1}\ldots s_2s_1[-2n, 2, 3, \ldots, 2n-2, 2n-1, 4n+1]\\
&= [\mathbf{4n+1}, 2, 3, \ldots, 2n-2, 2n-1, \mathbf{-2n}]\\
&= e + [4n, 0, \ldots, 0, -4n] = t_{(2,0,\ldots,0,-2)} = t_{2\Theta_1}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\mathrm{T}_1 &= \mathrm{c}_1s_1s_2\ldots s_{n-1}s_ns_{n-1}\ldots s_3s_2[\mathbf{-1}, \mathbf{0}, 3, \ldots, 2n-2, \mathbf{2n+1}, \mathbf{2n+2}]\\
&= \mathrm{c}_1[2n+1, \mathbf{-1}, 3, \ldots, 2n-2, \mathbf{2n+2}, 0]\\
&= s_1s_2\ldots s_{n-1}s_ns_{n-1}\ldots s_3s_2[\mathbf{2}, \mathbf{-2n}, 3, \ldots, 2n-2, \mathbf{4n+1}, \mathbf{2n-1}]\\
&= [\mathbf{4n+1}, 2, 3, \ldots, 2n-2, 2n-1, \mathbf{-2n}]\\
&= e + [4n, 0, \ldots, 0, -4n] = t_{(2,0,\ldots,0,-2)} = t_{2\Theta_1}.
\end{aligned}$$

□

We can put the words $\mathrm{W_l}$ together to get reduced words for other elements as follows.

Theorem 3.5.7. *For type B_n, let $w \in W_{\text{aff}}$ be such that for positive integers a_i,*

$$\text{center}\left(w^{-1}A_0\right) - \text{center}\left(A_0\right) = \sum_{l\in[n]} a_l\Theta_l,$$

let $P = \sum_l a_l$, and let $J \in \mathbb{Z}^P$ be a vector of a_1 ones, followed by a_2 twos, a_3 threes, and so on. For any $\sigma \in S_P$ and $m \in [P]$, define $K_{\sigma,m} = \sum_{i\in[m]} J_{\sigma(i)}$. Then, for any $\sigma \in S_P$, w has the reduced word

$$\mathrm{W}\left(\boldsymbol{a},\sigma\right) = \mathrm{W}_{\mathrm{J}_{\sigma(\mathrm{P})}}^{\sigma_0^{K_{\sigma,P-1}}} \dots \mathrm{W}_{\mathrm{J}_{\sigma(3)}}^{\sigma_0^{K_{\sigma,2}}} \mathrm{W}_{\mathrm{J}_{\sigma(2)}}^{\sigma_0^{J_{\sigma(1)}}} \mathrm{W}_{\mathrm{J}_{\sigma(1)}}.$$

Proof. By 3.3.2, we have

$$m_{-\alpha}(w) = \alpha\left(\sum_{l\in[n-1]} a_l\Theta_l\right) = \sum_{l\in[n-1]} a_l\alpha\left(\Theta_l\right).$$

More precisely, we have

$$m_{-\lambda_i+\lambda_j} = \sum_{l\in[i,j-1]} a_l, \qquad m_{-\lambda_i} = \sum_{l\in[i,n]} a_l, \qquad m_{-\lambda_i-\lambda_j} = \sum_{l\in[i,j-1]} a_l + 2\sum_{l\in[j,n]} a_l.$$

Thus,

$$\begin{aligned} l(w) &= \sum_{1\le i<j\le n}\sum_{l\in[i,j-1]} a_l + \sum_{i\in[n]}\sum_{l\in[i,n]} a_l + \sum_{1\le i<j\le n}\left(\sum_{l\in[i,j-1]} a_l + 2\sum_{l\in[j,n]} a_l\right) \\ &= \sum_{l\in[n]} \left(2l(n-l) + l + 2\tbinom{l}{2}\right) a_l = \sum_{l\in[n]} (2n-l)la_l, \end{aligned}$$

which is the number of reflections in $\mathrm{W}\left(\boldsymbol{a},\sigma\right)$.

We will show that particular choices of σ yield reduced words for w; all other reduced words will follow from applying the commutation relations of Lemma 3.5.5. Let $E = \sum_{l \text{ even}} a_l$, and choose σ so that $J_{\sigma(i)}$ is even for $i \in [E]$ and $J_{\sigma(i)}$ is odd for $i \in [E+1, P]$. There are two cases we wish to consider, the case when $O = \sum_{l \text{ odd}} a_l$ is odd, and the case when it is even. In the latter case, $\sum_{l\in[n]} a_l\Theta_l \in \check{T}$, and we can write down this translation by pairing off the odd coroots and writing down the resulting translations using Lemma 3.5.5. More precisely, letting

$$\tau_o = \sum_{l \text{ odd}} a_l\Theta_l, \qquad \tau_e = \sum_{l \text{ even}} a_l\Theta_l$$

$$\mathrm{W}\left(\sigma,\boldsymbol{a}\right)=\overleftarrow{\prod_{i=E+1}^{P}}\mathrm{W}_{\mathrm{J}_{\sigma(\mathrm{i})}}^{\sigma_0^{K_{\sigma,i}}}\prod_{i\in[E]}\mathrm{W}_{\mathrm{J}_{\sigma(\mathrm{i})}}=t_{\tau_o}t_{\tau_e}.$$

In the case that O is odd, we need only apply $\mathrm{W}_{\mathrm{J}_{\sigma(\mathrm{P})}}$ to a translation of the aforementioned type. Now, let

$$\sum_{i\in[P-1]}\Theta_{J_{\sigma(i)}}=\sum_{l\in[n]}b_l\Theta_l,$$

and let $B_i=\sum_{l\in[i,n]}b_l$. Then, if $l=J_{\sigma(P)}$,

$$\begin{aligned}
&\mathrm{W}\left(\sigma,\boldsymbol{a}\right)=\mathrm{W}_{\mathrm{l}}[1+2nB_1,2+2nB_2,\ldots,n+2nB_n,n+1-2nB_n,\ldots,2n-2nB_1]\\
&=\sigma_0t_{(1,\ldots,1,0,\ldots,0,-1,\ldots,-1)}[1+2nB_1,\ldots,n+2nB_n,n+1-2nB_n,\ldots,2n-2nB_1]\\
&=\sigma_0[1+2n\left(B_1+1\right),\ldots,l+2n\left(B_l+1\right),l+1+2nB_{l+1},\ldots,n+2nB_n,\\
&\qquad n+1-2nB_n,\ldots,2n-l-2nB_{l+1},2n+1-l-2n\left(B_l+1\right),\ldots,2n-2n\left(B_1+1\right)]\\
&=[\mathbf{2n-2n\left(B_1+2\right)},2+2n\left(B_2+1\right),\ldots,l+2n\left(B_l+1\right),\\
&\qquad l+1+2nB_{l+1},\ldots,n+2nB_n,n+1-2nB_n,\ldots,2n-l-2nB_{l+1},\\
&\qquad 2n+1-l-2n\left(B_l+1\right),\ldots,2n-1-2nB_2,\mathbf{1+2n\left(B_1+2\right)}].
\end{aligned}$$

∎

Thus,

$$\mathrm{W}\left(\sigma,\boldsymbol{a}\right)^{-1}(1)=-2n\left(B_1+1\right),\qquad \mathrm{W}\left(\sigma,\boldsymbol{a}\right)^{-1}(2n)=1+2n\left(B_1+2\right),$$

and otherwise

$$\mathrm{W}\left(\sigma,\boldsymbol{a}\right)^{-1}(i)=\begin{cases}i-2n\left(B_i+1\right) & i\in[2,l]\\ i-2nB_i & i\in[l+1,n]\\ i+2nB_{2n+1-i} & i\in[n+1,2n-l]\\ i+2n\left(B_{2n+1-i}+1\right) & i\in[2n+1-l,2n-1].\end{cases}$$

Thus,

$$\hat{m}_{\lambda_1}=\left\lceil\frac{-1-2n\left(2B_1+3\right)}{4n}\right\rceil=\left\lceil\frac{-2B_1-3}{2}\right\rceil=-\sum_{l\in[1,n]}b_l-1,$$

and for $j \in [2,l]$ and $k \in [l+1,n]$,

$$\begin{aligned}
\hat{m}_{\lambda_1-\lambda_j} &= \left\lceil \frac{-j-2n\left(B_1-B_j\right)}{2n} \right\rceil = -\sum_{l\in[1,j-1]} b_l, \\
\hat{m}_{\lambda_1+\lambda_j} &= \left\lceil \frac{-2n-1+j-2n\left(B_1+B_j+2\right)}{2n} \right\rceil = -\sum_{l\in[1,j-1]} b_l - 2\sum_{l\in[j,n]} b_l - 2, \\
\hat{m}_{\lambda_1-\lambda_k} &= \left\lceil \frac{-k-2n\left(B_1-B_j+1\right)}{2n} \right\rceil = -\sum_{l\in[1,j-1]} b_l - 1, \\
\hat{m}_{\lambda_1+\lambda_k} &= \left\lceil \frac{-k-2n\left(B_1+B_j+1\right)}{2n} \right\rceil = -\sum_{l\in[1,j-1]} b_l - 2\sum_{l\in[j,n]} b_l - 1.
\end{aligned}$$

Also, for $2 \leq i < j \leq l$ and $k \in [2,l]$,

$$\begin{aligned}
\hat{m}_{\lambda_i-\lambda_j} &= \left\lceil \frac{i-j-2n\left(B_i-B_j\right)}{2n} \right\rceil = -\sum_{l\in[i,j-1]} b_l, \\
\hat{m}_{\lambda_k} &= \left\lceil \frac{-2n\left(2B_i+2\right)}{4n} \right\rceil = -\sum_{l\in[i,n]} b_l - 1, \\
\hat{m}_{\lambda_i+\lambda_j} &= \left\lceil \frac{i+j-3-2n\left(B_i+B_j+3\right)}{2n} \right\rceil = -\sum_{l\in[i,j-1]} b_l - 2\sum_{l\in[j,n]} b_l - 2,
\end{aligned}$$

while for $2 \leq i \leq l < j \leq n$ and $k \in [l,n]$,

$$\begin{aligned}
\hat{m}_{\lambda_i-\lambda_j} &= \left\lceil \frac{i-j-1-2n\left(B_i-B_j+1\right)}{2n} \right\rceil = -\sum_{l\in[i,j-1]} b_l - 1, \\
\hat{m}_{\lambda_j} &= \left\lceil \frac{-2n\left(2B_i\right)}{4n} \right\rceil = -\sum_{l\in[i,n]} b_l, \\
\hat{m}_{\lambda_i+\lambda_j} &= \left\lceil \frac{i+j-2-2n\left(B_i+B_j+2\right)}{2n} \right\rceil = -\sum_{l\in[i,j-1]} b_l - 2\sum_{l\in[j,n]} b_l - 1, \\
\hat{m}_{\lambda_k} &= \left\lceil \frac{i+j-2-2n\left(B_i+B_j+1\right)}{2n} \right\rceil = -\sum_{l\in[i,j-1]} b_l - 2\sum_{l\in[j,n]} b_l.
\end{aligned}$$

But these give exactly the values of $m_{-\alpha}$ that w has, so $W\left(\sigma, \boldsymbol{a}\right)$ is a reduced word for w. □

As an application of this theorem, we have reduced words

$$\mathrm{t_{h*}} = \begin{cases} \mathrm{W_n}\left(\mathrm{W_{n-1}W_{n-2}}\right)^{\sigma_0}\mathrm{W_{n-3}}\ldots\mathrm{W_5W_4}\left(\mathrm{W_3W_2}\right)^{\sigma_0}\mathrm{W_1} & n = 0 \bmod 4 \\ \left(\mathrm{W_nW_{n-1}}\right)^{\sigma_0}\mathrm{W_{n-2}W_{n-3}}\ldots\left(\mathrm{W_5W_4}\right)^{\sigma_0}\mathrm{W_3W_2W_1^{\sigma_0}W_1} & n = 1 \bmod 4 \\ \mathrm{W_n^{\sigma_0}W_{n-1}W_{n-2}W_{n-3}^{\sigma_0}}\ldots\mathrm{W_5W_4}\left(\mathrm{W_3W_2}\right)^{\sigma_0}\mathrm{W_1} & n = 2 \bmod 4 \\ \mathrm{W_nW_{n-1}}\left(\mathrm{W_{n-2}W_{n-3}}\right)^{\sigma_0}\ldots\left(\mathrm{W_5W_4}\right)^{\sigma_0}\mathrm{W_3W_2W_1^{\sigma_0}W_1} & n = 3 \bmod 4. \end{cases}$$

Example 3.5.8. For $n = 4$, we have

$$\begin{aligned} \mathrm{t_{h*}} &= \mathrm{W_4W_3^{\sigma_0}W_2^{\sigma_0}W_1} \\ &= \left(s_4s_3s_2s_1s_4s_3s_2s_0\right)^2 s_3s_4s_3s_2s_1s_3s_4s_3s_2s_0s_3s_4s_3s_2s_1 \\ &\times s_2s_3s_4s_3s_2s_0s_2s_3s_4s_3s_2s_1s_0s_2s_3s_4s_3s_2s_0, \end{aligned}$$

which gives $\mathrm{Inv}\,(t_{h^*})$ the total reflection ordering

$$\begin{aligned}
&\delta-(\lambda_1+\lambda_2) \prec \delta-(\lambda_1+\lambda_3) \prec \delta-(\lambda_1+\lambda_4) \prec \delta-\lambda_1 \prec \delta-(\lambda_1-\lambda_4) \\
&\quad \prec \delta-(\lambda_1-\lambda_3) \prec \delta-(\lambda_1-\lambda_2) \prec \\
&2\delta-(\lambda_1+\lambda_2) \prec 2\delta-(\lambda_1+\lambda_3) \prec 2\delta-(\lambda_1+\lambda_4) \prec 2\delta-\lambda_1 \prec 2\delta-(\lambda_1-\lambda_4) \\
&\quad \prec 2\delta-(\lambda_1-\lambda_3) \prec 3\delta-(\lambda_1+\lambda_2) \prec \delta-(\lambda_2+\lambda_3) \prec \delta-(\lambda_2+\lambda_4) \prec \delta-\lambda_2 \\
&\quad \prec \delta-(\lambda_2-\lambda_4) \prec \delta-(\lambda_2-\lambda_3) \prec \\
&4\delta-(\lambda_1+\lambda_2) \prec 3\delta-(\lambda_1+\lambda_3) \prec 3\delta-(\lambda_1+\lambda_4) \prec 3\delta-\lambda_1 \prec 3\delta-(\lambda_1-\lambda_4) \\
&\quad \prec 2\delta-(\lambda_2+\lambda_3) \prec 5\delta-(\lambda_1+\lambda_2) \prec 2\delta-(\lambda_2+\lambda_4) \prec 2\delta-\lambda_2 \prec 2\delta-(\lambda_2-\lambda_4) \\
&\quad \prec 4\delta-(\lambda_1+\lambda_3) \prec 3\delta-(\lambda_2+\lambda_3) \prec \delta-(\lambda_3+\lambda_4) \prec \delta-\lambda_3 \prec \delta-(\lambda_3-\lambda_4) \\
&6\delta-(\lambda_1+\lambda_2) \prec 5\delta-(\lambda_1+\lambda_3) \prec 4\delta-(\lambda_1+\lambda_4) \prec 4\delta-\lambda_1 \\
&\quad \prec 4\delta-(\lambda_2+\lambda_3) \prec 3\delta-(\lambda_2+\lambda_4) \prec 7\delta-(\lambda_1+\lambda_2) \prec 3\delta-\lambda_2 \\
&\quad \prec 2\delta-(\lambda_3+\lambda_4) \prec 6\delta-(\lambda_1+\lambda_3) \prec 5\delta-(\lambda_2+\lambda_3) \prec 2\delta-\lambda_3 \\
&\quad \prec 5\delta-(\lambda_1+\lambda_4) \prec 4\delta-(\lambda_2+\lambda_4) \prec 3\delta-(\lambda_3+\lambda_4) \prec \delta-\lambda_4.
\end{aligned}$$

∎

3.5.2 Type C_n

The fundamental coweights for C_n are the vectors $\Theta_l = \lambda_1 + \ldots + \lambda_l$ and the vector $\Theta_n = \frac{1}{2}(\lambda_1 + \ldots + \lambda_n)$. The vector $h^* = \sum_{l\in[n-1]}\Theta_l + 2\Theta_n$ is in the coroot lattice $\check{T}$, since

$$h^* = n\lambda_1 + (n-1)\lambda_2 + \ldots + 2\lambda_{n-1} + \lambda_n$$

and λ_i is a coroot for each i. We also have

$$\Phi_l = \left\{ \lambda_i - \lambda_j,\ \ 2\lambda_i,\ \ \lambda_i + \lambda_k \ \middle|\ i \in [l],\ \ j \in [l+1,n],\ \ k \in [i+1,n] \right\},$$

$$\begin{aligned} \operatorname{Inv}(W_l) = &\left\{ \delta - (\lambda_i - \lambda_j),\ \delta - 2\lambda_i,\ \delta - (\lambda_i + \lambda_k) \ \middle|\ i \in [l],\ j \in [l+1,n],\ k \in [i+1,n] \right\} \\ &\cup \left\{ 2\delta - 2\lambda_i,\ \ 2\delta - (\lambda_i + \lambda_j) \ \middle|\ i \in [l],\ \ j \in [i+1,l] \right\} \end{aligned}$$

for $l \in [n-1]$, and

$$\operatorname{Inv}(\mathrm{W_n}) = \left\{ \delta - 2\lambda_i,\ \ \delta - (\lambda_i + \lambda_j) \ \middle|\ i \in [n],\ \ j \in [i+1,n] \right\}.$$

Thus, letting $i \in [n-1]$, we have

$$l\left(W_i\right) = 2i(n-i) + 2i + 2\tbinom{i}{2} = (2n+1-i)\,i, \qquad l\left(W_n\right) = \tbinom{n}{2} + n = \tbinom{n+1}{2}.$$

Proposition 3.5.9. *In the affine Weyl group of type C_n, for $l \in [n]$, define*

$$\mathrm{c_l} = s_l s_{l+1} \ldots s_{n-1} s_n s_{n-1} \ldots s_2 s_1 s_0, \qquad \mathrm{d_l} = s_0 s_1 \ldots s_{l-1},$$

so that $l\left(\mathrm{c_i}\right) = 2n + 1 - i$ and $l\left(\mathrm{d_i}\right) = i$. Then, letting $l \in [n-1]$, the coweight elements have reduced words

$$\mathrm{W_l} = \mathrm{c_l}^l, \qquad \mathrm{W_n} = \overleftarrow{\prod_{l=1}^{n}} \mathrm{d_l}.$$

Example 3.5.10. For $n = 2$, we have

$$\mathrm{c_0} = s_0 s_1 s_2 s_1 s_0, \qquad \mathrm{c_1} = s_1 s_2 s_1 s_0, \qquad \mathrm{d_0} = s_0, \qquad \mathrm{d_1} = s_0 s_1,$$

and

$$\mathrm{W_1} = s_1 s_2 s_1 s_0, \qquad \mathrm{W_2} = s_0 s_1 s_0.$$

For $n = 3$, we have

$$\begin{array}{lll} \mathrm{c_0} = s_0 s_1 s_2 s_3 s_2 s_1 s_0, & \mathrm{c_1} = s_1 s_2 s_3 s_2 s_1 s_0, & \mathrm{c_2} = s_2 s_3 s_2 s_1 s_0, \\ \mathrm{d_0} = s_0, & \mathrm{d_1} = s_0 s_1, & \mathrm{d_2} = s_0 s_1 s_2, \end{array}$$

and

$$\mathrm{W_1} = s_1 s_2 s_3 s_2 s_1 s_0, \qquad \mathrm{W_2} = s_2 s_3 s_2 s_1 s_0 s_2 s_3 s_2 s_1 s_0, \qquad \mathrm{W_3} = s_0 s_1 s_2 s_0 s_1 s_0.$$

For $n = 4$, we have

$$\mathrm{c_0} = s_0 s_1 s_2 s_3 s_4 s_3 s_2 s_1 s_0, \qquad \mathrm{c_1} = s_1 s_2 s_3 s_4 s_3 s_2 s_1 s_0,$$

$$\mathrm{c_2} = s_2 s_3 s_4 s_3 s_2 s_1 s_0, \qquad \mathrm{c_3} = s_3 s_4 s_3 s_2 s_1 s_0,$$

$$\mathrm{d_0} = s_0, \qquad \mathrm{d_1} = s_0 s_1, \qquad \mathrm{d_2} = s_0 s_1 s_2, \qquad \mathrm{d_2} = s_0 s_1 s_2 s_3,$$

and

$$\mathrm{W}_1 = s_1s_2s_3s_4s_3s_2s_1s_0, \qquad \mathrm{W}_2 = (s_2s_3s_4s_3s_2s_1s_0)^2$$
$$\mathrm{W}_3 = (s_3s_4s_3s_2s_1s_0)^3, \qquad \mathrm{W}_4 = s_0s_1s_2s_3s_0s_1s_2s_0s_1s_0.$$

Proof. Note first that for all $i \in [n]$, we have $l(\mathrm{W_i}) = l(W_i)$. We act on the window notation for the identity permutation $e = [1, \ldots, 2n]$ with $\mathrm{W_l}$, and calculate the inversion table of the result. The effect of $\mathrm{c_l}$ on w is to swap $w(1)$ and $w(0)$, and $w(2n)$ and $w(2n+1)$, and to rotate the latter values into positions $2n+1-l$ and l, respectively:

$$\begin{aligned}\mathrm{c_l}[1,\ldots,2n] =& s_ls_{l+1}\ldots s_{n-1}s_ns_{n-1}\ldots s_2s_1[\mathbf{0}, 2, \ldots, 2n-1, \mathbf{2n+1}] \\ =&[2,\ldots,l,\mathbf{2n+1}, l+1, \ldots, 2n-l, \mathbf{0}, 2n+1-l, \ldots, 2n-1].\end{aligned}$$

Thus, for $l \in [1, n-1]$, we have

$$\begin{aligned}\mathrm{c_l}^l =& \mathrm{c_l}^{l-1}[2,\ldots,l,\mathbf{2n+1},l+1,\ldots,2n-l,\mathbf{0},2n+1-l,\ldots,2n-1] \\ =&[\mathbf{2n+1},\ldots,\mathbf{2n+l},l+1,\ldots,2n-l,\mathbf{-l+1},\ldots,\mathbf{0}] \\ =&e+[2n,\ldots,2n,0,\ldots,0,-2n,\ldots,-2n] = t_{\Theta_l}.\end{aligned}$$

Next,

$$\begin{aligned}\mathrm{W_n} = \overleftarrow{\prod_{l=1}^{n}} \mathrm{d_l}[1,\ldots,2n] =& \overleftarrow{\prod_{l=2}^{n}} \mathrm{d_l}[\mathbf{0},2,\ldots,2n-1,\mathbf{2n+1}] \\ =& \overleftarrow{\prod_{l=3}^{n}} \mathrm{d_l}[\mathbf{-1},0,3,\ldots,2n-2,2n+1,\mathbf{2n+2}] \\ =&[-n+1,\ldots,0,2n+1,\ldots,3n] \\ =&e+[-n,\ldots,-n,n,\ldots,n].\end{aligned}$$

Thus,

$$(\mathrm{W_n})^{-1}(i) = \begin{cases}(\mathrm{W_n})^{-1}(2n+i)-2n = i-n & i \in [1,n] \\ (\mathrm{W_n})^{-1}(i-2n)+2n = i+n & i \in [n+1,2n],\end{cases}$$

so that for $1 \le i < j \le n$,

$$m_{-(\lambda_i-\lambda_j)} = 0, \qquad m_{-(\lambda_i+\lambda_j)} = 1, \qquad m_{-2\lambda_i} = 1,$$

and $\mathrm{Inv}(\mathrm{W_n}) = \mathrm{Inv}(W_n)$. □

We have the following corollary.

Corollary 3.5.11. *In type C_n, for $l \in [n-1]$, the reduced word $\mathrm{W_l}$ flips the roots in $\mathrm{Inv}(W_l)$ in the (total inversion) order $\prec$ determined by the following conditions.*

1. *For fixed* p*, the roots* $p\delta - \alpha$ *are flipped in lexicographic order with respect to the alphabet* $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

2. *For* $i \in [l]$,
$$\delta - (\lambda_i + \lambda_n) \prec 2\delta - 2\lambda_i \prec \delta - (\lambda_i - \lambda_n)\,.$$

3. *For* $j \in [2, r]$*, and* $i \in [j-1]$,
$$\delta - (\lambda_j + \lambda_l) \prec 2\delta - (\lambda_i + \lambda_j) \prec \delta - (\lambda_j + \lambda_{l+1})\,.$$

The reduced word $\mathrm{W_n}$ *flips the roots in* $\hat{\Phi}_n$ *in reverse lexicographic order with respect to the alphabet* $\{\lambda_n, \lambda_{n-1}, \dots, \lambda_1\}$.

Again, the first condition on the order for $l \in [n-1]$ can be restated as follows:

(1a) If $\alpha \in \Phi_i \cap \Phi_l$ and $\beta \in \Phi_j \cap \Phi_l$, then if $i < j$, $\delta - \alpha \prec \delta - \beta$.

(1b) If $\alpha, \beta \in \Phi_i \cap \Phi_l$, and $\mathrm{ht}\,(\alpha) > \mathrm{ht}\,(\beta)$, then $k\delta - \alpha \prec k\delta - \beta$ $(k = 1, 2)$.

Example 3.5.12. For $n = 5$ and $l = 3$, we have

$$\begin{aligned}
&\delta - 2\lambda_1 \prec \delta - (\lambda_1 + \lambda_2) \prec \delta - (\lambda_1 + \lambda_3) \prec \delta - (\lambda_1 + \lambda_4) \prec \delta - (\lambda_1 + \lambda_5)\\
&\prec 2\delta - 2\lambda_1 \prec \delta - (\lambda_1 - \lambda_5) \prec \delta - (\lambda_1 - \lambda_4)\\
&\prec \delta - 2\lambda_2 \prec \delta - (\lambda_2 + \lambda_3) \prec 2\delta - (\lambda_1 + \lambda_2) \prec \delta - (\lambda_2 + \lambda_4) \prec \delta - (\lambda_2 + \lambda_5)\\
&\prec 2\delta - 2\lambda_2 \prec \delta - (\lambda_2 - \lambda_5) \prec \delta - (\lambda_2 - \lambda_4)\\
&\prec \delta - 2\lambda_3 \prec 2\delta - (\lambda_1 + \lambda_3) \prec 2\delta - (\lambda_2 + \lambda_3) \prec \delta - (\lambda_3 + \lambda_4) \prec \delta - (\lambda_3 + \lambda_5)\\
&\prec 2\delta - 2\lambda_3 \prec \delta - (\lambda_2 - \lambda_5) \prec \delta - (\lambda_2 - \lambda_4)
\end{aligned}$$

while for $n = 5$ and $l = 5$, we have

$$\begin{aligned}
&\delta - 2\lambda_1 \prec \delta - (\lambda_1 + \lambda_2) \prec \delta - 2\lambda_2 \prec \delta - (\lambda_1 + \lambda_3) \prec \delta - (\lambda_2 + \lambda_3) \prec \delta - 2\lambda_3\\
&\prec \delta - (\lambda_1 + \lambda_4) \prec \delta - (\lambda_2 + \lambda_4) \prec \delta - (\lambda_3 + \lambda_4) \prec \delta - 2\lambda_4\\
&\prec \delta - (\lambda_1 + \lambda_5) \prec \delta - (\lambda_2 + \lambda_5) \prec \delta - (\lambda_3 + \lambda_5) \prec \delta - (\lambda_4 + \lambda_5) \prec \delta - 2\lambda_5.
\end{aligned}$$

We find two reduced words for T_n. Recall that $r_{n/2}$ is the reflection in $\frac{n}{2}$ given by $r_{n/2}(i) = n - i$. For a word $\mathrm{w} = s_{i_m} \dots s_{i_1}$, let $\mathrm{w}^{r_{n/2}}$ denote the reduced word obtained by applying $r_{n/2}$ to the indices of w:

$$\mathrm{w}^{r_{n/2}} = s_{n-i_m} \cdots s_{n-i_1}.$$

Proposition 3.5.13. *In type C_n, for $l \in [n-1]$, the coweight translation T_l has reduced word* $\mathrm{T_l} = \mathrm{W_l}$. *The coweight translation T_n has reduced words*

$$\mathrm{T_n} = \mathrm{W_n}^{r_{n/2}} \mathrm{W_n}, \qquad \mathrm{T'_n} = \mathrm{c_n^{\mathit{n}}}.$$

Proof. The calculation in the previous proof of the window notation of $\mathrm{W_l}$ shows that it is a translation. Furthermore, this calculation for $\mathrm{c_l^{\mathit{l}}}$ is valid when $l = n$, in which case we have

$$\mathrm{c_n^{\mathit{n}}} = e + [2n, \dots, 2n, -2n, \dots, -2n] = t_{2\Theta_n}.$$

For our other reduced word, we have

$$\begin{aligned}
\mathrm{W_n}^{r_{n/2}} \mathrm{W_n} &= \mathrm{W_n}^{r_{n/2}} [-n+1, \dots, 0, 2n+1, \dots, 3n] \\
&= \prod_{l=2}^{\overleftarrow{n}} \mathrm{d_l}^{r_{n/2}} [-n+1, \dots, -1, \mathbf{2n+1}, \mathbf{0}, 2n+2, \dots, 3n] \\
&= \prod_{l=3}^{\overleftarrow{n}} \mathrm{d_l}^{r_{n/2}} [-n+1, \dots, -2, 2n+1, \mathbf{2n+2}, \mathbf{-1}, 0, 2n+3, \dots, 3n] \\
&= [2n+1, \dots, 3n, -n+1, \dots, 0] \\
&= e + [2n, \dots, 2n, -2n, \dots, -2n] = t_{2\Theta_n}.
\end{aligned}$$

Since $l\left(t_{2\Theta_n}\right) = 2l\left(W_n\right)$, the words we have given for T_n are reduced. □

Since the lattice of translations is an abelian subgroup of W_{aff}, we have the following.

Corollary 3.5.14. *In type C_n, let $w \in W_{\text{aff}}$ be such that for positive integers a_i,*

$$\text{center}\left(w^{-1} A_0\right) - \text{center}\left(A_0\right) = \sum_{l \in [n-1]} a_l \Theta_l + 2a_n \Theta_n,$$

let $P = \sum_l a_l$, and let $J \in \mathbb{Z}^P$ be a vector of a_1 ones, followed by a_2 twos, a_3 threes, and so on. Then, for any $\sigma \in S_P$, w has the reduced word

$$\mathrm{W}\left(\boldsymbol{a}, \sigma\right) = \mathrm{T_{J_{\sigma(P)}}} \dots \mathrm{T_{J_{\sigma(2)}}} \mathrm{T_{J_{\sigma(1)}}}.$$

As an application of this corollary, we have the reduced words

$$\mathrm{t_{h*}} = \mathrm{T_n T_{n-1} \dots T_2 T_1}, \qquad \mathrm{t'_{h*}} = \mathrm{T'_n T_{n-1} \dots T_2 T_1}.$$

Example 3.5.15. For $n = 2$, we have

$$\mathrm{t_{h^*}} = s_2 s_1 s_2 s_0 s_1 s_0 s_1 s_2 s_1 s_0, \qquad \mathrm{t'_{h*}} = s_2 s_1 s_0 s_2 s_1 s_0 s_1 s_2 s_1 s_0.$$

The reduced word $\mathrm{t_{h^*}}$ gives $\mathrm{Inv}\,(t_{h*})$ the ordering

$$\begin{aligned}&\delta-2\lambda_1 \;\prec\; \delta-(\lambda_1+\lambda_2) \;\prec\; 2\delta-2\lambda_1 \;\prec\; \delta-(\lambda_1-\lambda_2) \;\prec\\ &3\delta-2\lambda_1 \;\prec\; 2\delta-(\lambda_1+\lambda_2) \;\prec\; \delta-2\lambda_2 \;\prec\\ &4\delta-2\lambda_1 \;\prec\; 3\delta-(\lambda_1+\lambda_2) \;\prec\; 2\delta-2\lambda_2.\end{aligned}$$

For $n=3$, we have

$$\mathrm{t_{h^*}} = s_3s_2s_1s_3s_2s_3s_0s_1s_2s_0s_1s_0s_2s_3s_2s_1s_0s_2s_3s_2s_1s_0s_1s_2s_3s_2s_1s_0,$$

which gives $\mathrm{Inv}\,(t_{h*})$ the ordering

$$\begin{aligned}&\delta-2\lambda_1 \;\prec\; \delta-(\lambda_1+\lambda_2) \;\prec\; \delta-(\lambda_1+\lambda_3) \;\prec\; 2\delta-2\lambda_1 \;\prec\; \delta-(\lambda_1-\lambda_3)\\ &\quad\prec\; \delta-(\lambda_1-\lambda_2)\\ &3\delta-2\lambda_1 \;\prec\; 2\delta-(\lambda_1+\lambda_2) \;\prec\; 2\delta-(\lambda_1+\lambda_3) \;\prec\; 4\delta-2\lambda_1 \;\prec\; 2\delta-(\lambda_1-\lambda_3)\\ &\quad\prec\; \delta-2\lambda_2 \;\prec\; 3\delta-(\lambda_1+\lambda_2) \;\prec\; \delta-(\lambda_2+\lambda_3) \;\prec\; 2\delta-2\lambda_2 \;\prec\; \delta-(\lambda_2-\lambda_3) \;\prec\\ &5\delta-2\lambda_1 \;\prec\; 4\delta-(\lambda_1+\lambda_2) \;\prec\; 3\delta-2\lambda_2 \;\prec\; 3\delta-(\lambda_1+\lambda_3) \;\prec\; 2\delta-(\lambda_2+\lambda_3)\\ &\quad\prec\; \delta-2\lambda_3 \;\prec\\ &6\delta-2\lambda_1 \;\prec\; 5\delta-(\lambda_1+\lambda_2) \;\prec\; 4\delta-2\lambda_2 \;\prec\; 4\delta-(\lambda_1+\lambda_3) \;\prec\; 3\delta-(\lambda_2+\lambda_3)\\ &\quad\prec\; 2\delta-2\lambda_3.\end{aligned}$$

For $n=4$, we have

$$\begin{aligned}\mathrm{t_{h^*}} =& s_4s_3s_2s_1s_4s_3s_2s_4s_3s_4s_0s_1s_2s_3s_0s_1s_2s_0s_1s_0\\ &\times \left(s_3s_2s_3s_2s_1s_0\right)^3 \left(s_2s_3s_2s_1s_0\right)^2 s_1s_2s_3s_2s_1s_0,\end{aligned}$$

which gives $\mathrm{Inv}\,(t_{h*})$ the ordering

$$\begin{aligned}
&\delta - 2\lambda_1 \prec \delta - (\lambda_1 + \lambda_2) \prec \delta - (\lambda_1 + \lambda_3) \prec \delta - (\lambda_1 + \lambda_4) \\
&\quad \prec 2\delta - 2\lambda_1 \prec \delta - (\lambda_1 - \lambda_4) \prec \delta - (\lambda_1 - \lambda_3) \prec \delta - (\lambda_1 - \lambda_2) \prec \\
&3\delta - 2\lambda_1 \prec 2\delta - (\lambda_1 + \lambda_2) \prec 2\delta - (\lambda_1 + \lambda_3) \prec 2\delta - (\lambda_1 + \lambda_4) \\
&\quad \prec 4\delta - 2\lambda_1 \prec 2\delta - (\lambda_1 - \lambda_4) \prec 2\delta - (\lambda_1 - \lambda_3) \prec \\
&\ \ \delta - 2\lambda_2 \prec 3\delta - (\lambda_1 + \lambda_2) \prec \delta - (\lambda_2 + \lambda_3) \prec \delta - (\lambda_2 + \lambda_4) \\
&\quad \prec 2\delta - 2\lambda_2 \prec \delta - (\lambda_2 - \lambda_4) \prec \delta - (\lambda_2 - \lambda_3) \prec \\
&5\delta - 2\lambda_1 \prec 4\delta - (\lambda_1 + \lambda_2) \prec 3\delta - (\lambda_1 + \lambda_3) \prec 3\delta - (\lambda_1 + \lambda_4) \prec 6\delta - 2\lambda_1 \\
&\quad \prec 3\delta - (\lambda_1 - \lambda_4) \prec 3\delta - 2\lambda_2 \prec 2\delta - (\lambda_2 + \lambda_3) \prec 5\delta - (\lambda_1 + \lambda_2) \\
&\quad \prec 2\delta - (\lambda_2 + \lambda_4) \prec 4\delta - 2\lambda_2 \prec 2\delta - (\lambda_2 - \lambda_4) \prec \delta - 2\lambda_3 \prec 4\delta - (\lambda_1 + \lambda_3) \\
&\quad \prec 3\delta - (\lambda_2 + \lambda_3) \prec \delta - (\lambda_3 + \lambda_4) \prec 2\delta - 2\lambda_3 \prec \delta - (\lambda_3 - \lambda_4) \prec \\
&7\delta - 2\lambda_1 \prec 6\delta - (\lambda_1 + \lambda_2) \prec 5\delta - \lambda_2 \prec 5\delta - (\lambda_1 + \lambda_3) \prec 4\delta - (\lambda_2 + \lambda_3) \\
&\quad \prec 3\delta - \lambda_3 \prec 4\delta - (\lambda_1 + \lambda_4) \prec 3\delta - (\lambda_2 + \lambda_4) \prec 2\delta - (\lambda_3 + \lambda_4) \\
&\quad \prec \delta - \lambda_4 \prec \\
&8\delta - 2\lambda_1 \prec 7\delta - (\lambda_1 + \lambda_2) \prec 6\delta - \lambda_2 \prec 6\delta - (\lambda_1 + \lambda_3) \prec 5\delta - (\lambda_2 + \lambda_3) \\
&\quad \prec 4\delta - \lambda_3 \prec 5\delta - (\lambda_1 + \lambda_4) \prec 4\delta - (\lambda_2 + \lambda_4) \prec 3\delta - (\lambda_3 + \lambda_4) \prec 2\delta - \lambda_4.
\end{aligned}$$

3.5.3 Type $\tilde{D}_n$

The fundamental coweights for D_n are the vectors $\Theta_l = \lambda_1 + \ldots + \lambda_l$ for $l \in [n-2]$, $\Theta_{n-1} = \frac{1}{2}(\lambda_1 + \ldots + \lambda_{n-1} - \lambda_n)$, and $\Theta_n = \frac{1}{2}(\lambda_1 + \ldots + \lambda_n)$. Since $(\lambda_i - \lambda_j) + (\lambda_i + \lambda_j) = 2\lambda_i$, the coroot lattice for D_n is the same as that for B_n. As in B_n, if l is even, then $\Theta_l \in \check{T}$, if l is odd, then $2\Theta_l \in \check{T}$, and if l, l' are odd with $l < l'$, then $\Theta_l + \Theta_{l'} \in \check{T}$. If n is odd, then

$$\Theta_{n-1} + \Theta_n = \lambda_1 + \ldots + \lambda_{n-1} \in \check{T}.$$

If n is even, then $\Theta_{n-1} + \Theta_n + \Theta_l \in \check{T}$ for any odd l. Thus, the shortest element of $\check{T} \cap C_0$ for type D_n is

$$h^* := \begin{cases} \Theta_1 + \ldots + \Theta_n & n = 0, 1 \bmod 4 \\ 2\Theta_1 + \ldots + \Theta_n & n = 2, 3 \bmod 4. \end{cases}$$

We have

$$\begin{aligned}
\mathrm{Inv}\,(W_l) = &\left\{\delta - (\lambda_i - \lambda_j)\,,\ \delta - (\lambda_i + \lambda_k) \;\middle|\; i \in [1, l],\ j \in [l+1, n],\ k \in [1, n]\right\} \\
&\cup \left\{2\delta - (\lambda_i + \lambda_j) \;\middle|\; 1 \le i < j \le l\right\}
\end{aligned}$$

for $l \in [n-2]$, and

$$\operatorname{Inv}(W_{n-1}) = \left\{\delta - (\lambda_i - \lambda_n)\,,\ \delta - (\lambda_i + \lambda_j) \;\middle|\; 1 \le i < j \le n-1\right\},$$
$$\operatorname{Inv}(W_n) = \left\{\delta - (\lambda_i + \lambda_j) \;\middle|\; 1 \le i < j \le n\right\}.$$

We also have

$$l\,(W_i) = (2n-1-i)\,i, \qquad l\,(W_{n-1}) = l\,(W_n) = \tbinom{n}{2}.$$

Proposition 3.5.16. *In W_{aff} of type D_n, for $l \in [n-2]$, define*

$$\mathrm{c_l} = s_l s_{l+1} \dots s_{n-2} s_n s_{n-1} \dots s_3 s_2 s_0, \qquad \mathrm{d_l} = s_{l-2} s_{l-1} \dots s_2 s_1 s_{l-1} s_{l-2} \dots s_2 s_0,$$

so that $l\,(\mathrm{c_i}) = 2n-1-i$ and $l\,(\mathrm{c_i}) = 2i-3$. Then for $l \in [2, n-2]$, the l^{th} coweight element for W_{aff} of type D_n has reduced word

$$\mathrm{W_l} = \begin{cases} \left(\mathrm{c_l^{\sigma_0} c_l}\right)^s & l = 2s, \\ \mathrm{c_l}\left(\mathrm{c_l^{\sigma_0} c_l}\right)^s & l = 2s+1, \end{cases}$$

and there are reduced words

$$\mathrm{W_1} = s_0 \mathrm{c_2}, \qquad \mathrm{W_n} = \overrightarrow{\prod_{l=n \bmod 2}^{\lfloor \frac{n}{2} \rfloor}} \mathrm{d_{2l}}, \qquad \mathrm{W_{n-1}} = \mathrm{W_n^{\sigma_{n-1}}}.$$

Example 3.5.17. In the simplest case, when $n = 5$, we have

$$\mathrm{W_1} = s_0 s_2 s_3 s_5 s_4 s_3 s_2 s_0, \quad \mathrm{W_2} = s_2 s_3 s_5 s_4 s_3 s_2 s_1 s_2 s_3 s_5 s_4 s_3 s_2 s_0,$$

$$\mathrm{W_3} = s_3 s_5 s_4 s_3 s_2 s_0 s_3 s_5 s_4 s_3 s_2 s_1 s_3 s_5 s_4 s_3 s_2 s_0, \quad \mathrm{W_4} = s_1 s_2 s_0 s_3 s_2 s_1 s_5 s_3 s_2 s_0,$$
$$\mathrm{W_5} = s_1 s_2 s_0 s_3 s_2 s_1 s_4 s_3 s_2 s_0.$$

When $n = 6$, we have

$$\mathrm{W_1} = s_0 s_2 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_0, \quad \mathrm{W_2} = s_2 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1 s_2 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_0,$$

$$\mathrm{W_3} = s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_0 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_0,$$
$$\mathrm{W_4} = \left(s_4 s_6 s_5 s_4 s_3 s_2 s_1 s_4 s_6 s_5 s_4 s_3 s_2 s_0\right)^2, \quad \mathrm{W_5} = s_2 s_1 s_3 s_2 s_0 s_4 s_3 s_2 s_1 s_6 s_4 s_3 s_2 s_0,$$
$$\mathrm{W_6} = s_0 s_2 s_1 s_3 s_2 s_0 s_4 s_3 s_2 s_1 s_5 s_4 s_3 s_2 s_0.$$

Proof. First, note that the number of simple reflections in W_i is $l(W_i)$. To prove that W_l is reduced for $l \in [n-1]$, we calculate its window notation and its inversion table.

Just as for type B_n, the word $\mathrm{c}_\mathrm{l}^{(1,0)}\mathrm{c}_\mathrm{l}$ acts on the window notation for $e = [1,\ldots,2n]$ by pulling two values from the ends of the subsequent and previous windows, and then moving these values into positions $l-1$, l and $2n-l+1$, $2n-l+2$, respectively:

$$
\begin{aligned}
&\mathrm{c}_\mathrm{l}^{\sigma_0}\mathrm{c}_\mathrm{l}[1,\ldots,2n]\\
&= \mathrm{c}_\mathrm{l}^{\sigma_0} s_l s_{l+1}\ldots s_{n-2}s_n s_{n-1}\ldots s_3 s_2[\mathbf{-1},\mathbf{0},3,\ldots,2n-2,\mathbf{2n+1},\mathbf{2n+2}]\\
&= \mathrm{c}_\mathrm{l}^{\sigma_0} s_l\ldots s_{n-2}s_n[\mathbf{-1},3,\ldots,n,\mathbf{0},\mathbf{2n+1},n+1,\ldots,2n-2,\mathbf{2n+2}]\\
&= \mathrm{c}_\mathrm{l}^{\sigma_0} s_l\ldots s_{n-2}[\mathbf{-1},3,\ldots,n-1,\mathbf{2n+1},n+1,n,\mathbf{0},n+2,\ldots,2n-2,\mathbf{2n+2}]\\
&= \mathrm{c}_\mathrm{l}^{\sigma_0}[\mathbf{-1},3,\ldots,l,\mathbf{2n+1},l+1,\ldots,n-1,n+1,\\
&\qquad n,n+2,\ldots,2n-l,\mathbf{0},2n-l+1,\ldots,2n-2,\mathbf{2n+2}]\\
&= s_l\ldots s_{n-2}s_n[3,\ldots,l,\mathbf{2n+1},l+1,\ldots,n-1,n+1,\mathbf{-1},\\
&\qquad \mathbf{2n+2},n,n+2,\ldots,2n-l,\mathbf{0},2n-l+1,\ldots,2n-2]\\
&= s_l\ldots s_{n-2}[3,\ldots,l,\mathbf{2n+1},l+1,\ldots,n-1,\mathbf{2n+2},n,\\
&\qquad n+1,\mathbf{-1},n+2,\ldots,2n-l,\mathbf{0},2n-l+1,\ldots,2n-2]\\
&= [3,\ldots,l,\mathbf{2n+1},\mathbf{2n+2},l+1,\ldots,2n-l,\mathbf{-1},\mathbf{0},2n-l+1,\ldots,2n-2].
\end{aligned}
$$

Thus, if $l \in [2,n-2]$ and $l = 2s$, we have

$$
\begin{aligned}
\left(\mathrm{c}_\mathrm{l}^{\sigma_0}\mathrm{c}_\mathrm{l}\right)^s &= [2n+1,\ldots,2n+l,l+1,\ldots,2n-l,-l+1,\ldots,0]\\
&= e + [2n,\ldots,2n,0,\ldots,0,-2n,\ldots,-2n]\\
&= t_{(1,\ldots,1,0,\ldots,0,-1,\ldots,-1)} = t_{\Theta_l}.
\end{aligned}
$$

Alternatively, if $l \in [2,n-2]$ and $l = 2s+1$, we have

$$
\begin{aligned}
&\mathrm{c}_\mathrm{l}\left(\mathrm{c}_\mathrm{l}^{\sigma_0}\mathrm{c}_\mathrm{l}\right)^s\\
&= \mathrm{c}_\mathrm{l}[l,2n+1,\ldots,2n+l-1,l+1,\ldots,2n-l,-l+2,\ldots,0,2n-l+1]\\
&= s_l s_{l+1}\ldots s_{n-2}s_n s_{n-1}\ldots s_3 s_2[\mathbf{-2n},\mathbf{-l+1},2n+2,\ldots,2n+l-1,l+1,\ldots\\
&\qquad \ldots,2n-l,-l+2,\ldots,-1,\mathbf{2n+l},\mathbf{4n+1}]\\
&= s_l s_{l+1}\ldots s_{n-2}s_n[-2n,2n+2,\ldots,2n+l-1,l+1,\ldots,n,\mathbf{-l+1},\\
&\qquad \mathbf{2n+l},n+1,\ldots,2n-l,-l+2,\ldots,-1,4n+1]\\
&= s_l s_{l+1}\ldots s_{n-2}[-2n,2n+2,\ldots,2n+l-1,l+1,\ldots,n-1,\mathbf{2n+l},\mathbf{n+1},\\
&\qquad \mathbf{n},\mathbf{-l+1},n+2,\ldots,2n-l,-l+2,\ldots,-1,4n+1]\\
&= [-2n,2n+2,\ldots,2n+l-1,\mathbf{2n+l},l+1,\ldots,n-1,n+1,\\
&\qquad n,n+2,\ldots,2n-l,\mathbf{-l+1},-l+2,\ldots,-1,4n+1]\\
&= \sigma_0\sigma_n[2n+1,\ldots,2n+l,l+1,\ldots,2n-l,-l+1,\ldots,0]\\
&= \sigma_0\sigma_n t_{(1,\ldots,1,0,\ldots,0,-1,\ldots,-1)} = \sigma_0\sigma_n t_{\Theta_l}.
\end{aligned}
$$

Thus, we have

$$\mathrm{W}_{\mathrm{l}}^{-1}(1) = -2n, \qquad \mathrm{W}_{\mathrm{l}}^{-1}(2n) = 4n+1,$$

and

$$\mathrm{W}_{\mathrm{l}}^{-1}(i) = \begin{cases} i-2n & i \in [2,l] \\ i & i \in [l+1, 2n-l] \\ i+2n & i \in [2n-l+1, 2n-1]. \end{cases}$$

This gives us

$$\hat{m}_{ij} = \begin{cases} 0 & i,j \in [1,l] \\ -1 & i \in [1,l], \quad j \in [l+1,n], \end{cases} \qquad \hat{m}_{i(2n+1-j)} = \begin{cases} -2 & i,j \in [1,l] \\ -1 & i \in [1,l], \quad j \in [l+1,n] \\ 0 & i,j \in [l+1,n]. \end{cases}$$

So, whether l is even or odd, we have

$$m_{-(\lambda_i-\lambda_j)} = \begin{cases} 0 & i,j \in [1,l] \\ 1 & i \in [1,l], \quad j \in [l+1,n], \end{cases} \qquad m_{-(\lambda_i+\lambda_j)} = \begin{cases} 2 & i,j \in [1,l] \\ 1 & i \in [1,l], \quad j \in [l+1,n] \\ 0 & i,j \in [l+1,n], \end{cases} \qquad \blacksquare$$

and the inversion set of W_{l} is Φ_l. We conclude that W_{l} is reduced, and spells W_l.

We calculate the window notation for W_1 separately. So,

$$\begin{aligned} s_0\mathrm{c}_2 &= s_0 s_2 s_3 \dots s_{n-2} s_n s_{n-1} \dots s_3 s_2 s_0 [1,2,3,\dots,2n-2,2n-1,2n] \\ &= s_0 s_2 s_3 \dots s_{n-2} s_n s_{n-1} \dots s_3 s_2 [\mathbf{-1},\mathbf{0},3,\dots,2n-2,\mathbf{2n+1},\mathbf{2n+2}] \\ &= s_0 s_2 s_3 \dots s_{n-2} s_n [-1,3,\dots,n,\mathbf{0},\mathbf{2n+1},n+1,\dots,2n-2,2n+2] \\ &= s_0 s_2 s_3 \dots s_{n-2} [-1,3,\dots,n-1,\mathbf{2n+1},\mathbf{n+1},\mathbf{n},\mathbf{0},n+2,\dots,2n-2,2n+2] \\ &= s_0 [-1,\mathbf{2n+1},3,\dots,n-1,n+1,n,n+2,\dots,2n-2,\mathbf{0},2n+2] \\ &= [-2n,2,3,\dots,n-1,n+1,n,n+2,\dots,2n-2,2n-1,4n+1] \\ &= \sigma_0 \sigma_n t_{\Theta_1}. \end{aligned}$$

Thus,

$$\mathrm{W}_1^{-1}(1) = -2n, \quad \mathrm{W}_1^{-1}(n) = n+1, \quad \mathrm{W}_1^{-1}(n+1) = n, \quad \mathrm{W}_1^{-1}(2n) = 4n+1,$$

and $\mathrm{W}_1^{-1}(i) = i$ for $i \in [2, n-1] \cup [n+2, 2n-1]$. This gives

$$m_{\lambda_1-\lambda_i} = -1, \qquad m_{\lambda_1+\lambda_i} = -1,$$

for $i \in [2,n]$, which shows that $\mathrm{Inv}\,(\mathrm{W}_1) = \Phi_1$, and thus that W_1 is reduced and spells W_1.

Next, we calculate the window notation for $\mathrm{W_n}$. If n is even, say $n = 2s$, we have

$$\begin{aligned}
&\overrightarrow{\prod_{l=0}^{s}} \mathrm{d_{2l}}[1, \ldots, 2n] \\
&= \overrightarrow{\prod_{l=0}^{s-1}} \mathrm{d_{2l}} s_{n-2} \ldots s_1 s_{n-1} \ldots s_2 [\mathbf{-1}, \mathbf{0}, 3, \ldots, n, n+1, \ldots, 2n-2, \mathbf{2n+1}, \mathbf{2n+2}] \\
&= \overrightarrow{\prod_{l=0}^{s-1}} \mathrm{d_{2l}} [3, \ldots, n, \mathbf{-1}, \mathbf{0}, \mathbf{2n+1}, \mathbf{2n+2}, n+1, \ldots, 2n-2] \\
&= \overrightarrow{\prod_{l=0}^{s-2}} \mathrm{d_{2l}} [5, \ldots, n, \mathbf{-3}, \mathbf{-2}, -1, 0, 2n+1, 2n+2, \mathbf{2n+3}, \mathbf{2n+4}, n+1, \ldots, 2n-4]. \\
&= s_0[n-1, n, -n+3, \ldots, 0, 2n+1, \ldots, 3n-2, n+1, n+2] \\
&= [-n+1, -n+2, -n+3, \ldots, 0, 2n+1, \ldots, 3n-2, 3n-1, 3n] \\
&= e + [-n, \ldots, -n, n, \ldots, n],
\end{aligned}$$

and

$$\mathrm{W_n^{-1}}(i) = \begin{cases} i - n & i \in [n] \\ i + n & i \in [n+1, 2n]. \end{cases}$$

If n is odd, say $n = 2s + 1$, we have

$$\begin{aligned}
&\overrightarrow{\prod_{l=1}^{s}} \mathrm{d_l}[1, \ldots, 2n] \\
&= s_1 s_2 s_0 [n-2, n-1, n, -n+4, \ldots, 0, 2n+1, \ldots, 3n-3, n+1, n+2, n+3] \\
&= s_1 s_2 [\mathbf{-n+2}, \mathbf{-n+3}, n, -n+4, \ldots, 0, 2n+1, \ldots, 3n-3, n+1, \mathbf{3n-2}, \mathbf{3n-1}] \\
&= [n, \mathbf{-n+2}, \mathbf{-n+3}, -n+4, \ldots, 0, 2n+1, \ldots, 3n-3, \mathbf{3n-2}, \mathbf{3n-1}, n+1] \\
&= \sigma_0 \left(e + [-n, \ldots, -n, n, \ldots, n] \right),
\end{aligned}$$

and

$$\mathrm{W_n^{-1}}(n) = 1, \quad \mathrm{W_n^{-1}}(n+1) = 2n, \quad \mathrm{W_n^{-1}}(i) = \begin{cases} i - n & i \in [n-1] \\ i + n & i \in [n+2, 2n]. \end{cases}$$

In either case, for $1 \leq i < j \leq n$,

$$\hat{m}_{\lambda_i - \lambda_j} = 0, \qquad \hat{m}_{\lambda_i + \lambda_j} = -1,$$

as desired. Thus $\mathrm{W_n}$ is reduced and spells W_n.

Finally, we calculate the window notation for $\mathrm{W_{n-1}}$. If n is even, say $n = 2s$, we have

$$\begin{aligned}
&\overrightarrow{\prod_{l=0}^{s-1}} \mathrm{d}_{2l}^{\sigma_{n-1}}[1,\ldots,2n] \\
&= \overrightarrow{\prod_{l=0}^{s-2}} \mathrm{d}_{2l} s_{n-2}\ldots s_1 s_n s_{n-2}\ldots s_2[-\mathbf{1},\mathbf{0},3,\ldots,n,n+1,\ldots,2n-2,\mathbf{2n+1},\mathbf{2n+2}] \\
&= \overrightarrow{\prod_{l=0}^{s-2}} \mathrm{d}_{2l} s_{n-2}\ldots s_1 s_n[-1,3,\ldots,n-1,\mathbf{0},n,n+1,\mathbf{2n+1},n+2,\ldots,2n-2,2n+2] \\
&= \overrightarrow{\prod_{l=0}^{s-2}} \mathrm{d}_{2l} s_{n-2}\ldots s_1[-1,3,\ldots,n-1,n+1,\mathbf{2n+1},\mathbf{0},n,n+2,\ldots,2n-2,2n+2] \\
&= \overrightarrow{\prod_{l=0}^{s-2}} \mathrm{d}_{2l}[3,\ldots,n-1,n+1,-\mathbf{1},2n+1,0,\mathbf{2n+2},n,n+2,\ldots,2n-2] \\
&= s_0[n-1,n+1,-n+3,\ldots,-1,2n+1,0,2n+2,\ldots,3n-2,n,n+2] \\
&= [-n,-n+2,-n+3,\ldots,-1,2n+1,0,2n+2,\ldots,3n-2,3n-1,3n+1] \\
&= \sigma_0\sigma_n\left(e+\sigma_0[-n,\ldots,-n,n,\ldots,n]\right).
\end{aligned}$$

so that

$$\mathrm{W_{n-1}^{-1}}(1) = -n, \quad \mathrm{W_n^{-1}}(n) = 2n+1, \quad \mathrm{W_n^{-1}}(n+1) = 0, \quad \mathrm{W_n^{-1}}(2n) = 3n+1,$$

and otherwise

$$\mathrm{W_n^{-1}}(i) = \begin{cases} i-n & i \in [2,n-1] \\ i+n & i \in [n+2,2n-1]. \end{cases}$$

If n is odd, say $n = 2s+1$, we have

$$\overrightarrow{\prod_{l=1}^{s}} \mathrm{d}_l^{(n-1,n)}[1,\ldots,2n] = [n+1,-n+2,\ldots,-1,2n+1,0,2n+2,\ldots,3n-1,n],$$

so that

$$\mathrm{W_{n-1}^{-1}}(1) = -n, \quad \mathrm{W_{n-1}^{-1}}(n) = 2n, \quad \mathrm{W_{n-1}^{-1}}(n+1) = 1, \quad \mathrm{W_{n-1}^{-1}}(2n) = 3n+1,$$

and otherwise

$$\mathrm{W_{n-1}^{-1}}(i) = \begin{cases} i-n & i \in [2,n-1] \\ i+n & i \in [n+2,2n-1]. \end{cases}$$

In either case, for $1 \leq i < j \leq n-1$,

$$\hat{m}_{\lambda_i - \lambda_j} = 0, \qquad \hat{m}_{\lambda_i + \lambda_j} = -1,$$

while for $i \in [n-1]$,

$$\hat{m}_{\lambda_i - \lambda_n} = -1, \qquad \hat{m}_{\lambda_i + \lambda_n} = 0.$$

This is what we seek. Thus $\mathrm{W_{n-1}}$ is reduced and spells W_{n-1}. So ends the proof. □

We then have the following.

Corollary 3.5.18. *In type D_n, for $l \in [n-2]$, the reduced word $\mathrm{W_l}$ flips the roots in* $\mathrm{Inv}\,(W_l)$ *in the order $\prec$ determined by the following conditions.*

1. *For fixed k, the roots $k\delta - \alpha$ are flipped in lexicographic order with respect to the alphabet $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.*
2. *For $l \in [2, n-2]$, $j \in [2, l]$, and $i \in [j-1]$,*

$$\delta - (\lambda_j + \lambda_l) \prec 2\delta - (\lambda_i + \lambda_j) \prec \delta - (\lambda_j + \lambda_{l+1})\,.$$

The reduced word $\mathrm{W_{n-1}}$ flips $\mathrm{Inv}\,(W_{n-1})$ *in lexicographic order with respect to the alphabet $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, -\lambda_n\}$, and the reduced word $\mathrm{W_n}$ flips* $\mathrm{Inv}\,(W_n)$ *in lexicographic order with respect to the alphabet $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.*

Example 3.5.19. For $n = 5$, and $l = 1$, we have

$$\begin{aligned} &\delta - (\lambda_1 + \lambda_2) \prec \delta - (\lambda_1 + \lambda_3) \prec \delta - (\lambda_1 + \lambda_4) \prec \delta - (\lambda_1 + \lambda_5) \\ &\quad \prec \delta - (\lambda_1 - \lambda_5) \prec \delta - (\lambda_1 - \lambda_4) \prec \delta - (\lambda_1 - \lambda_3) \prec \delta - (\lambda_1 - \lambda_2)\,, \end{aligned}$$

while for $n = 5$ and $l = 3$, we have

$$\begin{aligned} &\delta - (\lambda_1 + \lambda_2) \prec \delta - (\lambda_1 + \lambda_3) \prec \delta - (\lambda_1 + \lambda_4) \prec \delta - (\lambda_1 + \lambda_5) \\ &\quad \prec \delta - (\lambda_1 - \lambda_5) \prec \delta - (\lambda_1 - \lambda_4) \prec \\ &\quad \delta - (\lambda_2 + \lambda_3) \prec 2\delta - (\lambda_1 + \lambda_2) \prec \delta - (\lambda_2 + \lambda_4) \prec \delta - (\lambda_2 + \lambda_5) \\ &\quad \prec \delta - (\lambda_1 - \lambda_5) \prec \delta - (\lambda_1 - \lambda_4) \prec \\ &\quad 2\delta - (\lambda_1 + \lambda_3) \prec 2\delta - (\lambda_2 + \lambda_3) \prec \delta - (\lambda_3 + \lambda_4) \prec \delta - (\lambda_3 + \lambda_5) \\ &\quad \prec \delta - (\lambda_3 - \lambda_5) \prec \delta - (\lambda_3 - \lambda_4)\,. \end{aligned}$$

For $n = 5$ and $l = 4$, we have

$$\begin{aligned} &\delta - (\lambda_1 + \lambda_2) \prec \delta - (\lambda_1 + \lambda_3) \prec \delta - (\lambda_1 + \lambda_4) \prec \delta - (\lambda_1 - \lambda_5) \\ &\quad \prec \delta - (\lambda_2 + \lambda_3) \prec \delta - (\lambda_2 + \lambda_4) \prec \delta - (\lambda_2 - \lambda_5) \\ &\quad \prec \delta - (\lambda_3 + \lambda_4) \prec \delta - (\lambda_3 - \lambda_5) \prec \delta - (\lambda_4 - \lambda_5)\,. \end{aligned}$$

and for $n = 5$ and $l = 5$, we have

$$\begin{aligned} &\delta - (\lambda_1 + \lambda_2) \prec \delta - (\lambda_1 + \lambda_3) \prec \delta - (\lambda_1 + \lambda_4) \prec \delta - (\lambda_1 + \lambda_5) \\ &\quad \prec \delta - (\lambda_2 + \lambda_3) \prec \delta - (\lambda_2 + \lambda_4) \prec \delta - (\lambda_2 + \lambda_5) \\ &\quad \prec \delta - (\lambda_3 + \lambda_4) \prec \delta - (\lambda_3 + \lambda_5) \prec \delta - (\lambda_4 + \lambda_5) \,. \end{aligned}$$

As in B_n, there are certain commutation relations which will help us to combine the reduced words $\mathrm{W_l}$.

Lemma 3.5.20. *In type D_n, for odd numbers l, l' and even numbers k, k', we have*

$$\mathrm{W_k W_{k'}} = \mathrm{W_{k'} W_k} = t_{\Theta_k + \Theta_{k'}}, \qquad \mathrm{W_{l'}^{\sigma_0} W_l} = \mathrm{W_l^{\sigma_0} W_{l'}} = t_{\Theta_l + \Theta_{l'}},$$

$$\mathrm{W_k^{\sigma_0} W_l} = \mathrm{W_l W_k} = \sigma_0 \sigma_n t_{\Theta_k + \Theta_l}.$$

Proof. The commutation relation for even coweight elements is due to the fact that these elements are translations by elements of $\check{T}$. For the remaining relations, we first calculate the window notation for $\mathrm{W_l^{\sigma_0}}$. We have

$$\begin{aligned} &\mathrm{c_l c_l^{\sigma_0}}[1, \ldots, 2n] \\ &= \mathrm{c_l} s_l s_{l+1} \ldots s_{n-2} s_n [2, \ldots, n, \mathbf{1}, \mathbf{2n}, n+1, \ldots, 2n-1] \\ &= \mathrm{c_l} s_l s_{l+1} \ldots s_{n-2} [2, \ldots, n-1, \mathbf{2n}, \mathbf{n+1}, \mathbf{n}, \mathbf{1}, n+2, \ldots, 2n-1] \\ &= \mathrm{c_l} [2, \ldots, l, \mathbf{2n}, l+1, \ldots, n-1, n+1, n, n+2, \ldots, 2n-l, \mathbf{1}, 2n-l+1, \ldots, 2n-1] \\ &= s_l s_{l+1} \ldots s_{n-2} s_n s_{n-1} \ldots s_3 s_2 [\mathbf{-2}, \mathbf{-1}, 4, \ldots, l, 2n, l+1, \ldots, n-1, n+1, \\ &\qquad n, n+2, \ldots, 2n-l, 1, 2n-l+1, \ldots, 2n-3, \mathbf{2n+2}, \mathbf{2n+3}] \\ &= s_l s_{l+1} \ldots s_{n-2} s_n [\mathbf{-2}, 4, \ldots, l, 2n, l+1, \ldots, n-1, n+1, \mathbf{-1}, \\ &\qquad \mathbf{2n+2}, n, n+2, \ldots, 2n-l, 1, 2n-l+1, \ldots, 2n-3, \mathbf{2n+3}] \\ &= s_l s_{l+1} \ldots s_{n-2} s_n [\mathbf{-2}, 4, \ldots, l, 2n, l+1, \ldots, n-1, \mathbf{2n+2}, n, \\ &\qquad n+1, \mathbf{-1}, n+2, \ldots, 2n-l, 1, 2n-l+1, \ldots, 2n-3, \mathbf{2n+3}] \\ &= [-2, 4, \ldots, l, 2n, \mathbf{2n+2}, l+1, \ldots, 2n-l, \mathbf{-1}, 1, 2n-l+1, \ldots, 2n-3, 2n+3]. \end{aligned}$$

Thus, for $k = 2s$, we have

$$\begin{aligned}
&(\mathrm{c_k c_k^{\sigma_0}})^s [1, \ldots, 2n] \\
&= \mathrm{c_k c_k^{\sigma_0}}[-k+2, k, 2n, 2n+2, \ldots, 2n+k-2, k+1, \ldots \\
&\qquad \ldots, 2n-k, -k-1, \ldots, -1, 1, 2n+1-k, 2n+k-1] \\
&= \mathrm{c_k}[k, 2n, 2n+2, \ldots, \mathbf{2n+k-1}, k+1, \ldots, n-1, n+1, \\
&\qquad n, n+2, \ldots, 2n-k, \mathbf{-k+2}, \ldots, -1, 1, 2n+1-k] \\
&= s_k \ldots s_n \ldots s_2[\mathbf{-2n+1}, \mathbf{-k+1}, 2n+2, \ldots, 2n+k-1, k+1, \ldots, n-1, n+1, \\
&\qquad n, n+2, \ldots, 2n-k, -k+2, \ldots, -1, \mathbf{2n+k}, \mathbf{4n}] \\
&= [-2n+1, 2n+2, \ldots, 2n+k-1, \mathbf{2n+k}, k+1, \ldots \\
&\qquad \ldots, 2n-k, \mathbf{-k+1}, -k+2, \ldots, -1, 4n] \\
&= e + [-2n, 2n, \ldots, 2n, 0, \ldots, 0, -2n, \ldots, -2n, 2n] \\
&= t_{(-1,1,\ldots,1,0,\ldots,0,-1,\ldots,-1,1)} = t_{\sigma_0 \Theta_k}.
\end{aligned}$$

Now, let k be even and l be odd. Then we have

$$\begin{aligned}
\mathrm{W_k^{\sigma_0} W_l} &= t_{\sigma_0\Theta_k}\sigma_0\sigma_n t_{\Theta_l} = \sigma_0\sigma_n\left(\sigma_n\sigma_0 t_{\sigma_0\Theta_k}\sigma_0\sigma_n\right) t_{\Theta_l} \\
&= \sigma_0\sigma_n t_{\sigma_n\sigma_0\sigma_0\Theta_k} t_{\Theta_l} = \sigma_0\sigma_n t_{\Theta_k+\Theta_l},
\end{aligned}$$

while

$$\mathrm{W_l W_k} = \sigma_0\sigma_n t_{\Theta_l} t_{\Theta_k} = \sigma_0\sigma_n t_{\Theta_k+\Theta_l}.$$

calculations in the previous proof make it clear that if $l = 2s$, then $(\mathrm{c_l^{\sigma_0} c_l})^s$ is a translation, thus $\Theta_l \in \check{T}$. On the other hand, if $l = 2s+1 \geq 2$, then $\mathrm{c_l}\,(\mathrm{c_l^{\sigma_0} c_l})^s$ is not a translation, but $(\mathrm{c_l^{\sigma_0} c_l})^{2s+1}$ is To check the commutation relation for odd elements, let $l = 2s+1$ and $l' = 2s'+1$, and let $l' < l$. Then,

$$\begin{aligned}
&\mathrm{W_{l'}^{\sigma_0} W_l} \\
&= \mathrm{W_{l'}^{\sigma_0}}[-2n, 2n+2, \ldots, 2n+l, l+1, \ldots, n-1, n+1, \\
&\qquad n, n+2, 2n-l, -l+1, \ldots, -1, 4n+1] \\
&= (\mathrm{c_{l'}^{\sigma_0} c_{l'}})^{s'} [2n+2, \ldots, 2n+l', \mathbf{4n+1}, 2n+l'+1, \ldots, 2n+l, l+1, \ldots \\
&\qquad \ldots, 2n-l, -l+1, \ldots, -l', \mathbf{-2n}, -l'+1, \ldots, -1] \\
&= (\mathrm{c_{l'}^{\sigma_0} c_{l'}})^{s'-1} [2n+4, \ldots, 2n+l', 4n+1, \mathbf{4n+2}, \mathbf{4n+3}, 2n+l'+1, \ldots, 2n+l, l+1, \ldots \\
&\qquad \ldots, 2n-l, -l+1, \ldots, -l', \mathbf{-2n-2}, \mathbf{-2n-1}, -2n, -l'+1, \ldots, -3] \\
&= \mathrm{c_{l'}^{\sigma_0} c_{l'}}[2n+l'-1, 2n+l', 4n+1, \ldots, 4n+l'-2, 2n+l'+1, \ldots, 2n+l, l+1, \ldots \\
&\qquad \ldots, 2n-l, -l+1, \ldots, -l', -2n-l'+2, \ldots, -2n, -l', -l'+1] \\
&= [4n+1, \ldots, 4n+l', 2n+l'+1, \ldots, 2n+l, l+1, \ldots \\
&\qquad \ldots, 2n-l, -l+1, \ldots, -l', -2n-l', \ldots, -2n] \\
&= t_{(2,\ldots,2,1,\ldots,1,0,\ldots,0,-1,\ldots,-1,-2,\ldots,-2)} = t_{\Theta_l+\Theta_{l'}},
\end{aligned}$$

∎

while

$$
\begin{aligned}
&\mathrm{W}_1^{\sigma_0}\mathrm{W}_{l'}\\
&= \mathrm{W}_1^{\sigma_0}[-2n, 2n+2, \ldots, 2n+l', l'+1, \ldots, n-1, n+1,\\
&\qquad n, n+2, \ldots, 2n-l', -l'+1, \ldots, -1, 4n+1]\\
&= (\mathrm{c}_l^{\sigma_0}\mathrm{c}_l)^s\,[2n+2, \ldots, 2n+l', l'+1, \ldots, l, \mathbf{4n+1}, l+1, \ldots\\
&\qquad \ldots, 2n-l, \mathbf{-2n}, 2n-l+1, \ldots, 2n-l', -l'+1, \ldots, -1]\\
&= (\mathrm{c}_l^{\sigma_0}\mathrm{c}_l)^{s-1}\,[2n+4, \ldots, 2n+l', l'+1, \ldots, l, 4n+1, \mathbf{4n+2}, \mathbf{4n+3}, l+1, \ldots\\
&\qquad \ldots, 2n-l, \mathbf{-2n-2}, \mathbf{-2n-1}, -2n, 2n-l+1, \ldots, 2n-l', -l'+1, \ldots, -3]\\
&= \mathrm{c}_l^{\sigma_0}\mathrm{c}_l[l-1, l, 4n+1, \ldots, 4n+l', 2n+l'+1, \ldots, 2n+l-2, l+1, \ldots, 2n-l,\\
&\qquad -l+3, \ldots, -l', -2n-l'+1, \ldots, -2n, 2n-l+1, 2n-l+2]\\
&= [4n+1, \ldots, 4n+l', 2n+l'+1, \ldots, 2n+l, l+1, \ldots\\
&\qquad \ldots, 2n-l, -l+1, \ldots, -l', -2n-l'+1, \ldots, -2n]\\
&= t_{(2,\ldots,2,1,\ldots,1,0,\ldots,0,-1,\ldots,-1,-2,\ldots,-2)} = t_{\Theta_l+\Theta_{l'}}.
\end{aligned}
$$

□

We use these commutation relations to write down reduced words for the coweight translations in D_n.

Proposition 3.5.21. *For $l \in [2, n]$, the l^{th} coweight translation has reduced word*

$$
\mathrm{T}_l = \begin{cases} (\mathrm{c}_l^{\sigma_0}\mathrm{c}_l)^s & l = 2s,\\ (\mathrm{c}_l^{\sigma_0}\mathrm{c}_l)^{2s+1} & l = 2s+1, \end{cases}
$$

while the first coweight translation has reduced words

$$
\mathrm{T}_1 = \mathrm{c}_1^2, \qquad \mathrm{T}_1' = \mathrm{W}_1^{\sigma_0}\mathrm{W}_1.
$$

Proof. The words for T_l when $l \in [2, n-2]$ follow from Proposition 3.5.20. When $l = 1$, we have

$$
\begin{aligned}
\mathrm{T}_1' &= s_1 s_2 \ldots s_{n-2} s_n s_{n-1} \ldots s_2 s_1 [-2n, 2, \ldots, n-1, n+1, n, n+2, \ldots, 2n-1, 4n+1]\\
&= s_1 \ldots s_{n-2} s_n [2, \ldots, n-1, n+1, \mathbf{-2n}, \mathbf{4n+1}, n, n+2, \ldots, 2n-1]\\
&= s_1 \ldots s_{n-2} [2, \ldots, n-1, \mathbf{4n+1}, \mathbf{n}, \mathbf{n+1}, \mathbf{-2n}, n+2, \ldots, 2n-1]\\
&= [\mathbf{4n+1}, 2, \ldots, n-1, n, n+1, n+2, \ldots, 2n-1, \mathbf{-2n}]\\
&= e + [4n, 0, \ldots, 0, -4n] = t_{2\Theta_1}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\mathrm{T}_1\\
&= \mathrm{c}_1 s_1[-1, 2n+1, 3, \ldots, n-1, n+1, n, n+2, \ldots, 2n-2, 0, 2n+2]\\
&= \mathrm{c}_1[2n+1, -1, 3, \ldots, n-1, n+1, n, n+2, \ldots, 2n-2, 2n+2, 0]\\
&= s_1 \ldots s_{n-2} s_n s_{n-1} \ldots s_2[2, -2n, 3 \ldots, n-1, n+1, n, n+2, \ldots, 2n-2, 4n+1, 2n-1]\\
&= s_1 \ldots s_{n-2} s_n[2, \ldots, n-1, n+1, -2n, 4n+1, n, n+2, \ldots, 2n-1]\\
&= s_1 \ldots s_{n-2}[2, \ldots, n-1, 4n+1, n, n+1, -2n, n+2, \ldots, 2n-1]\\
&= [4n+1, 2, \ldots, 2n-1, -2n] = t_{2\Theta_1}.
\end{aligned}
$$

□

Finding the coweight translations for Θ_{n-1} and Θ_n is not so simple. We provide the following proposition, whose proof is straightforward but tedious.

Proposition 3.5.22. *The* $(n-1)^{\mathrm{st}}$ *and* n^{th} *coweight translations have reduced words*

$$
\begin{aligned}
\mathrm{T_n} &= \left(\mathrm{W_n}^{\sigma_0 r_{n/2}} \mathrm{W_n}\right)^{\sigma_0 \sigma_{n-1}} \mathrm{W_n}^{\sigma_0 r_{n/2}} \mathrm{W_n}\\
&= \mathrm{W_n}^{\sigma_{n-1} r_{n/2}} \mathrm{W_n}^{\sigma_0 \sigma_{n-1}} \mathrm{W_n}^{\sigma_0 r_{n/2}} \mathrm{W_n} = \mathrm{t}_{4\Theta_\mathrm{n}}
\end{aligned}
$$

and $\mathrm{T_{n-1}} = \mathrm{T_n}^{\sigma_{n-1}} = \mathrm{t}_{4\Theta_{\mathrm{n-1}}}$.

The complexity of the above expressions motivates the following.

Proposition 3.5.23. *There are reduced words*

$$
\mathrm{W_{n-1,n}} = \mathrm{W_n}^{r_{n/2}} \mathrm{W_n}, \qquad \mathrm{W'_{n-1,n}} = \begin{cases} \left(\mathrm{c_n}^{\sigma_0} \mathrm{c_n}\right)^s & n = 2s+1\\ \mathrm{c_n} \left(\mathrm{c_n}^{\sigma_0} \mathrm{c_n}\right)^s & n = 2s+2 \end{cases}
$$

satisfying

$$
\mathrm{Inv}\left(\mathrm{W_{n-1,n}}\right) = \mathrm{Inv}\left(\mathrm{W'_{n-1,n}}\right) = \mathrm{Inv}\left(W_{n-1}\right) \cup \mathrm{Inv}\left(W_n\right),
$$

and there are reduced words for $t_{2\Theta_{n-1}+2\Theta_n}$ *given by*

$$
\mathrm{T_{n-1,n}} = \mathrm{W^2_{n,n-1}}, \qquad \mathrm{T'_{n-1,n}} = \left(\mathrm{c_n}^{\sigma_0} \mathrm{c_n}\right)^{n-1}.
$$

Proof. For the first reduced word, if n is even, we have

$$
\begin{aligned}
&\mathrm{W_{n-1,n}} \\
&= \mathrm{W_n}^{r_{n/2}}[-n,-n+2,-n+3,\dots,-1,2n+1,0,2n+2,\dots,3n-2,3n-1,3n+1] \\
&= \left(\prod_{l=0}^{\overleftarrow{\lfloor \frac{n-2}{2} \rfloor}} \mathrm{d_{2l}}\right) s_2\dots s_{n-1}s_1\dots s_{n-2}[-n,-n+2,-n+3,\dots,-2,\mathbf{0},\mathbf{2n}+\mathbf{2}, \\
&\quad -\mathbf{1},\mathbf{2n}+\mathbf{1},2n+3,\dots,3n-2,3n-1,3n+1] \\
&= \left(\prod_{l=0}^{\overleftarrow{\lfloor \frac{n-2}{2} \rfloor}} \mathrm{d_{2l}}\right) s_2\dots s_{n-1}[\mathbf{2n}+\mathbf{1},-n,-n+2,\dots,-2,2n+2, \\
&\quad -1,2n+2,\dots,3n-1,3n+1,\mathbf{0}] \\
&= \left(\prod_{l=0}^{\overleftarrow{\lfloor \frac{n-2}{2} \rfloor}} \mathrm{d_{2l}}\right) [2n+1,\mathbf{2n}+\mathbf{2},-n,-n+2,\dots,-2,2n+2, \\
&\quad 2n+2,\dots,3n-1,3n+1,-\mathbf{1},0] \\
&= s_n[2n+1,\dots,3n-2,-n,-n+2,3n-1,3n+1,-n+3,\dots,0] \\
&= [2n+1,\dots,3n-2,\mathbf{3n}-\mathbf{1},\mathbf{3n}+\mathbf{1},-\mathbf{n},-\mathbf{n}+\mathbf{2},-n+3,\dots,0] \\
&= \sigma_n\left(e+\sigma_n[2n,\dots,2n,-2n,\dots,-2n]\right) = \sigma_n t_{\sigma_n\Theta_n}.
\end{aligned}
$$

Similarly, if n is odd, we have

$$\mathrm{W_{n-1,n}} = [2n+1,\dots,3n-1,n,n+1,-n+2,\dots,-1,0] = t_{\Theta_{n-1}+\Theta_n}.$$

For the other reduced words, we calculate

$$
\begin{aligned}
&\mathrm{c_n^{\sigma_0}c_n}[1,\dots,2n] \\
&= \mathrm{c_n^{\sigma_0}} s_n s_{n-1}\dots s_3 s_2[-\mathbf{1},\mathbf{0},3,\dots,2n-2,\mathbf{2n}+\mathbf{1},\mathbf{2n}+\mathbf{2}] \\
&= \mathrm{c_1^{\sigma_0}} s_n[-1,3,\dots,n,\mathbf{0},\mathbf{2n}+\mathbf{1},n+1,\dots,2n-2,2n+2] \\
&= s_n s_{n-1}\dots s_2 s_1[-1,3,\dots,n-1,\mathbf{2n}+\mathbf{1},n+1,n,\mathbf{0},n+2,\dots,2n-2,2n+2] \\
&= s_n[3,\dots,n-1,2n+1,n+1,-\mathbf{1},\mathbf{2n}+\mathbf{2},n,0,n+2,\dots,2n-2] \\
&= [3,\dots,n-1,2n+1,\mathbf{2n}+\mathbf{2},\mathbf{n},\mathbf{n}+\mathbf{1},-\mathbf{1},0,n+2,\dots,2n-2].
\end{aligned}
$$

Thus, if $n=2s+1$,

$$\mathrm{W'_{n-1,n}} = [2n+1,\dots,3n-1,n,n+1,-n+2,\dots,0] = t_{\Theta_{n-1}+\Theta_n}.$$

On the other hand, if $n = 2s + 2$,

$$
\begin{aligned}
&\mathrm{W}'_{\mathrm{n-1,n}} \\
&= \mathrm{c_n}[n-1, 2n+1, \ldots, 3n-2, n, n+1, -n+3, \ldots, 0, n+2] \\
&= s_n s_{n-1} \ldots s_2[-\mathbf{2n}, -\mathbf{n}+\mathbf{2}, 2n+2, \ldots, 3n-2, n, n+1, -n+3, \ldots, -1, \mathbf{3n}-\mathbf{1}, \mathbf{4n}+\mathbf{1}] \\
&= s_n[-2n, 2n+2, \ldots, 3n-2, n, -\mathbf{n}+\mathbf{2}, \mathbf{3n}-\mathbf{1}, n+1, -n+3, \ldots, -1, 4n+1] \\
&= [-2n, 2n+2, \ldots, 3n-2, \mathbf{3n}-\mathbf{1}, \mathbf{n}+\mathbf{1}, \mathbf{n}, -\mathbf{n}+\mathbf{2}, -n+3, \ldots, -1, 4n+1] \\
&= \sigma_0 \sigma_n t_{\Theta_{n-1}+\Theta_n}.
\end{aligned}
$$

Now, if $n = 2s + 1$, then $\mathrm{T_{n-1,n}} = \mathrm{W}^2_{\mathrm{n-1,n}}$ and $\mathrm{T}'_{\mathrm{n-1,n}} = \left(\mathrm{W}'_{\mathrm{n-1,n}}\right)^2$ are reduced words for $t_{2(\Theta_{n-1}+\Theta_n)}$. If $n = 2s + 2$, then

$$
\begin{aligned}
\mathrm{W}^2_{\mathrm{n,n-1}} =& \sigma_n t_{\sigma_n \Theta_n} \sigma_n t_{\sigma_n \Theta_n} = t_{\sigma_n \sigma_n \Theta_n} t_{\sigma_n \Theta_n} = t_{\Theta_n + \sigma_n \Theta_n} \\
=& t_{(1,\ldots,1,-1,\ldots,-1)+(1,\ldots,1,-1,1,-1,\ldots,-1)} = t_{(2,\ldots,2,0,0,-2,\ldots,-2)} = t_{2(\Theta_n+\Theta_{n-1})},
\end{aligned}
$$

so that $\mathrm{T_{n-1,n}}$ is again the reduced word we seek, and

$$
\begin{aligned}
\mathrm{T}'_{\mathrm{n-1,n}} =& \left(\mathrm{c_n^{\sigma_0} c_n}\right)^s \mathrm{c_n^{\sigma_0}}[-2n, 2n+2, \ldots, 3n-1, n+1, n, -n+2, \ldots, -1, 4n+1] \\
=& \left(\mathrm{c_n^{\sigma_0} c_n}\right)^s s_n[2n+2, \ldots, 3n-1, n+1, -\mathbf{2n}, \mathbf{4n}+\mathbf{1}, n, -n+2, \ldots, -1] \\
=& \left(\mathrm{c_n^{\sigma_0} c_n}\right)^s [2n+2, \ldots, 3n-1, \mathbf{4n}+\mathbf{1}, \mathbf{n}, \mathbf{n}+\mathbf{1}, -\mathbf{2n}, -n+2, \ldots, -1] \\
=& [4n+1, \ldots, 5n-1, n, n+1, -3n+2, \ldots, -2n] = t_{2(\Theta_{n-1}+\Theta_n)}.
\end{aligned}
$$

□

For the following theorem, we use $\mathrm{W_{n-1}}$ to denote $\mathrm{W_{n-1,n}}$.

Theorem 3.5.24. *Let $w \in W_{\text{aff}}$ be such that*

$$
\text{center}\left(w^{-1}A_0\right) - \text{center}\left(A_0\right) = \sum_{l \in [n-2]} a_l \Theta_l + a_{n-1}\left(\Theta_{n-1} + \Theta_n\right),
$$

let $P = \sum_l a_l$, and let $J \in \mathbb{Z}^P$ be a vector of a_1 ones, followed by a_2 twos, a_3 threes, and so on. For any $\sigma \in S_P$ and $m \in [P]$, define $K_{\sigma,m} = \sum_{i \in [m]} J_{\sigma(i)}$. Then, for any $\sigma \in S_P$, w has the reduced word

$$
\mathrm{W}\left(\boldsymbol{a}, \sigma\right) = \mathrm{W}_{\mathrm{J}_{\sigma(\mathrm{P})}}^{\sigma_0^{K_{\sigma,P-1}}} \ldots \mathrm{W}_{\mathrm{J}_{\sigma(3)}}^{\sigma_0^{K_{\sigma,2}}} \mathrm{W}_{\mathrm{J}_{\sigma(2)}}^{\sigma_0^{J_{\sigma(1)}}} \mathrm{W}_{\mathrm{J}_{\sigma(1)}}.
$$

The proof is very similar to the proof of Theorem 3.5.7. As an application of this theorem, we have reduced words

$$
\mathrm{t_{h*}} = \begin{cases}
\left(\mathrm{W_{n-1,n}W_{n-2}}\right)^{\sigma_0} \mathrm{W_{n-3}} \ldots \mathrm{W_5W_4}\left(\mathrm{W_3W_2}\right)^{\sigma_0} \mathrm{W_1} & n = 0 \bmod 4 \\
\mathrm{W_{n-1,n}}\left(\mathrm{W_{n-2}W_{n-3}}\right)^{\sigma_0} \ldots \mathrm{W_5W_4}\left(\mathrm{W_3W_2}\right)^{\sigma_0} \mathrm{W_1} & n = 1 \bmod 4 \\
\left(\mathrm{W_{n-1,n}W_{n-2}}\right)^{\sigma_0} \mathrm{W_{n-3}} \ldots \left(\mathrm{W_5W_4}\right)^{\sigma_0} \mathrm{W_3W_2W_1^{\sigma_0}W_1} & n = 2 \bmod 4 \\
\mathrm{W_{n-1,n}}\left(\mathrm{W_{n-2}W_{n-3}}\right)^{\sigma_0} \ldots \left(\mathrm{W_5W_4}\right)^{\sigma_0} \mathrm{W_3W_2W_1^{\sigma_0}W_1} & n = 3 \bmod 4.
\end{cases}
$$

Example 3.5.25. For $n = 5$, we have

$$\begin{aligned} \mathrm{t}_{\mathrm{h}^*} =& \mathrm{W}_{4,5}\mathrm{W}_3^{\sigma_0}\mathrm{W}_2^{\sigma_0}\mathrm{W}_1 \\ =& s_5s_4s_3s_5s_2s_3s_4s_1s_2s_3s_5s_0s_1s_2s_0s_3s_2s_1s_4s_3s_2s_0s_3s_5s_4s_3s_2s_1s_3s_5s_4s_3s_2s_0s_3s_5s_4s_3s_2s_1 \\ & \times s_2s_3s_5s_4s_3s_2s_0s_2s_3s_5s_4s_3s_2s_1s_0s_2s_3s_5s_4s_3s_2s_0. \end{aligned}$$

3.6 Lam & Pylyavskyy's Method for Finding Reduced Words

In their work on total positivity for loop groups [24], Lam & Pylyavskyy perform an in-depth study of the infinite elements of the affine Weyl group of type A_n. They divide $\mathcal{W}$ into a finite number of equivalence classes, called blocks; they put a partial ordering, called the limit weak order, on $\mathcal{W}$; and they give a method for finding reduced words for the unique minimal elements of each blcok. In this section, we discuss their work and its relationship to the results appearing in the previous sections.

3.6.1 Blocks and the Limit Weak Order

Definition 3.6.1. ([24]) The blocks of $\mathcal{W}$ are the equivalence classes of the relation $\sim$, where $I \sim J$ if

$$\#\left(I \triangle J\right) = \#\left(\left(I \cup J\right) \setminus \left(I \cap J\right)\right) < 0.$$

Thus, the infinite values of m_α are the same for all I and J in the same block.

Definition 3.6.2. ([24]) For $I, J \in \mathcal{W}$, the limit weak order is the order defined by the relation $I \preceq J$ if and only if $I \subset J$.

The limit weak order induces an order on the blocks of $\mathcal{W}$, which we will also call the limit weak order.

We can restate these ideas in terms of biconvex subsets of $\mathfrak{h}_\mathbb{R}$ as follows: $I, J \in \mathcal{W}$ are in the same block if $B(I) \triangle B(J)$ is bounded, and $I \preceq J$ if and only if $B(I) \subset B(J)$.

Lam & Pylyavskyy give these ideas a tidy geometric interpretation in terms of the Artin arrangement of W, which we now define.

Intersecting the hyperplane arrangement

$$\mathcal{H}_0 = \cup_{\alpha \in \Phi^+} H_{0,\alpha}$$

with the sphere S^r in $\mathfrak{h}_\mathbb{R}$ carves S^r into a spherical simplicial complex (a simplicial complex with Euler characteristic zero), called the Coxeter complex. The top-dimensional simplices of the Coxeter complex are the intersections of the Weyl chambers with S^r, having dimension $r-1$. The zero-dimensional simplices are the intersections with S^r of lines obtained by intersecting $r-1$ hyperplanes $H_{0,\alpha}$. For example, each of the coweights Θ_l corresponds to a zero-dimensional simplex of the Coxeter complex.

The Artin arrangement $\mathcal{A}$ of W is the set of cones C_S spanned by simplices S of the Coxeter complex. We can think of it as a simplicial complex whose simplices are unbounded, where we have $C_S \subset C_{S'}$ if and only if $S \subset S'$. Following Lam & Pylyavskyy, we will call the simplices of $\mathcal{A}$ faces.

Using Proposition 3.2.12, we have the following (Remark 4.5 of [24]).

Proposition 3.6.3. *The elements $I, J \in \mathcal{W}$ are in the same block if and only if $\overline{\mathrm{dir}}\,(I)$ and $\overline{\mathrm{dir}}\,(J)$ belong to the same face of $\mathcal{A}$.*

Thus, the blocks of $\mathcal{W}$ are in correspondence with the faces of the Artin arrangement (Proposition 4.3 of [24]). Furthermore, the limit weak order on blocks is isomorphic to the inclusion order on the faces of $\mathcal{A}$. The maximal blocks for the limit weak order correspond to the Weyl chambers of W, while the minimal elements correspond to the one-dimensional faces described above. We denote the block corresponding to a face F of the Artin arrangement by B_F.

Using biconvex subsets of $\mathfrak{h}_{\mathbb{R}}$, we can make this correspondence more direct.

Proposition 3.6.4. *An element $w \in \mathcal{W}$ belongs to B_F if and only if $B(w) \supset F$.*

This makes it clear that the minimal element of B_F is the element $w \in \mathcal{W}$ having $B(w) = F$. We denote this minimal element by I_F. For example, $\mathbb{Z}\delta - \Phi^+ = I_{C_0}$ is the minimal element of the (maximal) block corresponding to C_0; and the minimal elements which are less than $\mathbb{Z}\delta - \Phi^+$ in the limit weak order are the elements $\mathbb{Z}\delta - \Phi_l = I_{\langle \Theta_l \rangle}$ for $l \in [r]$.

These facts are hinted at by results of Cellini & Papi [8], who give the following correspondence between elements of $\mathcal{RW}$ and infinite powers of elements of W.

Proposition 3.6.5. *If $I \in \mathcal{W}$, then there exist $w \in W$ and $\tau \in \check{T}$ such that*

$$I = \bigcup_{j \in \mathbb{N}} \mathrm{Inv}\left(vt_\tau^j\right) =: \mathrm{Inv}\left(vt_\tau^\infty\right).$$

If, further, for all $\alpha \in \Phi^+$, $m_\alpha(I) \in \{0, \pm\infty\}$, then $w = 1$, i.e. there exists $\tau \in \check{T}$ such that $I = \mathrm{Inv}\,(t_\tau^\infty)$.

Thus, the minimal elements of the limit weak order correspond to infinite translations.

3.6.2 Finding Explicit Reduced Words

Their results on the limit weak order motivated Lam & Pylyavskyy to give a method for finding reduced words for infinite translations in type A_{n-1}. In reviewing their work, we will use the representation of S_n on $\mathbb{R}^n$. Recall the positive roots $\alpha_{ij} = \lambda_i - \lambda_j$ for $1 \leq i < j \leq n$, and write

$$H_{0,i,j} := \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_i - x_j = 0 \right\}, \qquad \mathcal{H}_0 = \cup_{1 \leq i < j \leq n} H_{0,i,j}.$$

Lam & Pylyavskyy call the Artin arrangment for type A_{n-1} the braid arrangement.

Definition 3.6.6. A preorder on a set is a reflexive, transitive relation. A total preorder is a preorder whose equivalence classes are totally ordered.

Each face of the braid arrangement corresponds to a total preorder on the coefficients x_i of elements $\boldsymbol{x} \in \mathbb{R}^n$. For example, if $\langle \boldsymbol{x} \rangle$ denotes the line spanned by $\boldsymbol{x}$, then

$$\langle \Theta_l \rangle = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_1 = \ldots = x_l > x_{l+1} = \ldots = x_n \right\},$$

$$C_0 = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_1 > x_2 > \ldots > x_n \right\}.$$

Given a face F of the braid arrangement, there is a unique smallest vector $\boldsymbol{z}^F \in \mathbb{Z}^n \cap F$ whose coefficients take consecutive values starting with one. The coefficients z_i^F lie in $[n]$ for all F.

Given an element $\sigma \in \tilde{S}_n = S_n \propto \check{T}$, let $\pi(\sigma)$ denote its projection into the quotient group S_n. For example, $\pi(\sigma_0) = (1, n)$, while $\pi(\sigma_i) = (i, i+1)$ for $i \in [n-1]$. We will assume that $\pi(\sigma)$ acts on the coefficients of $\boldsymbol{z}^F \in \mathbb{Z}^n$ by permuting the indices of the coefficients (in other words, on positions rather than values). The following result is due to Lam & Pylyavskyy [24].

Proposition 3.6.7. *Let F be a face of the braid arrangement of A_{n-1}, and let the coefficients of $\boldsymbol{z}^F$ take the values $[m]$ for $m \leq n$. If $s_{i_p} \ldots s_{i_1}$ is such that*

1. $\pi\left(s_{i_p} \ldots s_{i_1}\right) \boldsymbol{z}^F = \boldsymbol{z}^F$
2. *each s_{i_j} creates a descent (for example $\pi\left(s_{i_j}\right)$ is applied to $\boldsymbol{z}^F$ only if $z_{i_j}^F < z_{i_j+1}^F$)*
3. *for each $i \in [m]$, the values i and $i+1$ are swapped at least once*

then $\mathrm{w} = s_{i_p} \ldots s_{i_1}$ is reduced, and $\mathrm{Inv}(w^\infty) = I_F$.

Example 3.6.8. In A_3, let

$$F = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_2 > x_1 = x_3 = x_4 \right\},$$

and choose $\boldsymbol{z}^F = (1, 2, 1, 1)$. We have

$$\begin{aligned}\pi(s_2 s_3 s_0 s_1)(1,2,1,1) =& \pi(s_2 s_3 s_0)(2,1,1,1) \\ =& \pi(s_2 s_3)(1,1,1,2) = \pi(s_2)(1,1,2,1) \\ =& (1,2,1,1),\end{aligned}$$

so Proposition 3.6.7 says that $\mathrm{w} = s_2 s_3 s_0 s_1$ is reduced (which we can see by inspection), and $\mathrm{Inv}(\mathrm{w}^\infty) = I_S$.

For completeness, we include a proof.

Proof. First, we show that $w = s_{i_p} \dots s_{i_1}$ is reduced, by creating an affine permutation for which it is length-additive. For $i \in [m]$, let M_i be the number of coefficients of $\boldsymbol{z}^F$ taking value i. We assign a number $M\left(z_i^F\right)$ in $[n]$ to each coefficient of $\boldsymbol{z}^F$ as follows: we number the ones in $\boldsymbol{z}^F$ from left to right, up to M_1, then we number the twos in $\boldsymbol{z}^F$ from left to right, up to $M_1 + M_2$, and so on. Then, we let $w_F \in S_n \subset \tilde{S}_n$ be defined by the window notation

$$w_F = \left[M\left(z_1^F\right) \dots M\left(z_n^F\right)\right].$$

For example, for $\boldsymbol{z}^F$ as above, we have

$$w_F = [1, 4, 2, 3] = (\dots, -3, 0, -2, -1, 1, 4, 2, 3, 5, 8, 6, 7, \dots).$$

Now, for $1 \leq i < j \leq n$, we have $m_{\alpha_{ij}} \in \{0, 1\}$, with $m_{\alpha_{ij}} = 1$ implying that $z^F_{w_F^{-1}(j)} > z^F_{w_F^{-1}(i)}$. For our running example, we have

$$m_{\alpha_{12}} = m_{\alpha_{13}} = m_{\alpha_{14}} = m_{\alpha_{23}} = 0, \qquad m_{\alpha_{24}} = m_{\alpha_{34}} = 1,$$

and $z^F_{w_F^{-1}(4)} = z_2^F = 2$ is greater than either $z^F_{w_F^{-1}(2)} = z_3^F = 1$ or $z^F_{w_F^{-1}(3)} = z_4^F = 1$. Since $z_i^F < z_j^F$ implies that $w_F(i) < w_F(j)$, it is clear that the first condition guarantees that every reflection we apply increases some $m_{\alpha_{ij}}$ by one. Thus, w is reduced. For our example, we have

$$\begin{aligned} w w_F =& s_2 s_3 s_0 s_1 [1, 4, 2, 3] = s_2 s_3 s_0 [4, 1, 2, 3] \\ =& s_2 s_3 [-1, 1, 2, 8] = s_2 [-1, 1, 8, 2] = [-1, 8, 1, 2], \end{aligned}$$

so that

$$m_{\alpha_{12}} = m_{\alpha_{13}} = m_{\alpha_{23}} = 0, \qquad m_{\alpha_{14}} = m_{\alpha_{24}} = m_{\alpha_{34}} = 2.$$

Next, to show that w^∞ is in the block B_F, let V_k be all the values in w_F which correspond to k in $\boldsymbol{z}^F$,

$$V_k = \left\{i + qn | i \in [1, n], \#^{-1}(i) = k, q \in \mathbb{Z}\right\},$$

and let P_k be the set of positions occupied by elements of V_k. For our example,

$$V_1 = \{\dots, -3, -2, -1, 1, 2, 3, 5, 6, 7, \dots\}, \qquad V_2 = \{\dots, 0, 4, 8, \dots\},$$

while

$$P_1 = \{\dots, -3, -1, 0, 1, 3, 4, 5, 7, 8, \dots\}, \qquad P_2 = \{\dots, -2, 2, 6, \dots\}.$$

The second condition, and the fact that $w\boldsymbol{z}^F = \boldsymbol{z}^F$, means that in the window notation for ww_F, elements of V_k continue to occupy the positions P_k, but the set V_k

has rotated to the left by 1 with respect to the numbers V_{p-1}. This can be seen in our example:

$$w_F = [1, \mathbf{4}, 2, 3], \qquad ww_F = [-1, \mathbf{8}, 1, 2].$$

Thus, there is some finite number N so that for every k, the values of $w^N w_F$ at P_k are some integer multiple N_k of n larger than the values of w_F at P_k. In other words, if χ_{P_k} is the characteristic function of P_k, having ones at the positions P_k and zeros elsewhere, then

$$w^N w_F = w_F + \sum_{p=1}^{m} N_k n \chi_{P_k},$$

so that w^N has translated w_F by the vector $\tau = \sum_{k=1}^{m} N_k \chi_{P_k} \in \check{T}$. In our example,

$$\begin{aligned} w^3 w_F =& [-3, 16, -2, -1] \\ =& [1, 4, 2, 3] + [-4, 12, -4, -4] = t_{(-1,3,-1,-1)}[1, 4, 2, 3]. \end{aligned}$$

Furthermore, the second condition guarantees that $N_k > N_{k-1}$ for all k. Thus, τ lies in F, and $w^\infty = t_\tau^\infty$. □

We remark that in fact, the minimum value for N is the least common multiple of $\{M_1, \ldots, M_m\}$. We also have the following.

Corollary 3.6.9. *Let $w = s_{i_p} \ldots s_{i_1}$ act on $\boldsymbol{z}^F$ as in the previous proposition, and keep track of the "winding numbers" of the elements of $\boldsymbol{z}^F$, so that $k^{(-q)}$ indicates a k that has been moved from the n^{th} position to the first q times. If the winding numbers of the entries of $w\boldsymbol{z}^F$ depend only on the values of $w\boldsymbol{z}^F$, then w is a translation.*

In [24], Lam & Pylyavskyy show that if the dimension of F is one, then one of the reduced words produced by their technique is a Coxeter element c, a product of in which each of the simple reflections s_i appears once. Furthermore, all the other reduced words for the minimal element of B_F can be obtained from c^∞ by exchanging commuting simple reflections. In particular, we have the following. Recall that for $l \in [n]$,

$$\mathrm{c_l} = s_0 s_n s_{n-1} \ldots s_{l+1}, \qquad \mathrm{d_l} = s_0 s_1 \ldots s_{l-1}.$$

Proposition 3.6.10. *The words $\mathrm{d_l^{(1)} c_l}$ and $\mathrm{c_l^{(1)} d_l}$ are reduced, $\left(\mathrm{d_l^{(1)} c_l}\right)^{l(n-l)}$ and $\left(\mathrm{c_l^{(1)} d_l}\right)^{l(n-l)}$ are translations, and $\mathrm{Inv}\left(\left(\mathrm{d_l^{(1)} c_l}\right)^\infty\right) = \mathrm{Inv}\left(\left(\mathrm{c_l^{(1)} d_l}\right)^\infty\right) = \mathbb{Z}\delta - \Phi_l$.*

Proof. The face F of the braid arrangement corresponding to $\mathbb{Z}\delta - \Phi_l$ is

$$\langle \Theta_l \rangle = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_1 = \ldots = x_l > x_{l+1} = \ldots = x_n \right\},$$

so the unique element $\boldsymbol{z}^F \in \mathbb{Z}^n \cap \langle \Theta_l \rangle$ with coefficients taking consecutive values starting with 1 is

$$\boldsymbol{z}^F = (2, \ldots, 2, 1, \ldots, 1),$$

a vector with l twos followed by $n-l$ ones. Now c_l rotates the first two $n-l$ places to the left, so that it ends up in position $l+1$:

$$\mathrm{c_l}\boldsymbol{z}^F = (1, 2, \dots, \mathbf{2}, 1, \dots, 1).$$

Then, $\mathrm{d_l^{(1)}}$ moves the 1 in position 1 l places to the right:

$$\mathrm{d_l^{(1)}}\mathrm{c_l}\boldsymbol{z}^F = (2, \dots, 2, \mathbf{1}, \dots, 1) = \boldsymbol{z}^F.$$

On the other hand, applying $\mathrm{d_l}$ to $\boldsymbol{z}^F$ rotates the last one through the block of twos to position l, and following this with $\mathrm{c_l^{(1)}}$ rotates moves the two in position n to position l. Keeping track of the "winding numbers" of the ones and twos, we see that when either element has been applied $l(n-l)$ times, the twos have all been moved $n-l$ positions to the left, and the ones have all been moved l positions to the right. □

We also have the following.

Proposition 3.6.11. *The reduced words* $\mathrm{W_l^{[n-l]}} = \mathrm{c_l^{[l(n-l)]}}$ *and* $\mathrm{W_l'^{[n-l]}} = \mathrm{d_l^{[(n-l)^2]}}$ *spell translations.*

3.6.3 Other Types

We begin by establishing maps from the Artin arrangements of classical type to $\mathbb{Z}^{2n}$.

The Coxeter complexes of types B_n and C_n are isomorphic. They are obtained by intersecting S^n with the hyperplane arrangement

$$\mathcal{H}_0 = \bigcup_{1 \le i < j \le n} (\{x_i = x_j\} \cup \{x_i = -x_j\}) \cup \bigcup_{i \in [n]} \{x_i = 0\}.$$

Thus each face of the Artin arrangement for these types corresponds to a preorder on the set
$S = \{x_1, \dots, x_n, 0, -x_1, \dots, -x_n\}$ with the property that for all $s, s' \in S$, $s > s'$ if and only if $s < s'$. For example, we have

$$\langle \Theta_l \rangle = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_1 = \ldots = x_l > x_{l+1} = \ldots = x_n = -x_n = \ldots = -x_{l+1} > -x_l = \ldots = -x_1 \right\},$$
$$C_0 = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_1 > x_2 > \ldots > x_n > -x_n > \ldots > -x_2 > -x_1 \right\}.$$

The Coxeter complex of type D_n is obtained by intersecting S^n with the hyperplane arrangement

$$\mathcal{H}_0 = \bigcup_{1 \le i < j \le n} (\{x_i = x_j\} \cup \{x_i = -x_j\}).$$

Thus each face of the braid arrangement for this type corresponds to a preorder on the set $\{x_1, \ldots, x_n, 0, -x_1, \ldots, -x_n\}$ with the property that for all $s, s' \in S$, $s > s'$ if and only if $-s < -s'$, and $x_n = -x_n$. For example, we have

$$\langle \Theta_l \rangle = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_1 = \ldots = x_l > x_{l+1} = \ldots = x_n = -x_n = \ldots = -x_{l+1} > -x_l = \ldots = -x_1 \right\},$$

$$C_0 = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_1 > x_2 > \ldots > x_n = -x_n > \ldots > -x_2 > -x_1 \right\}.$$

Alternatively, the faces of $\mathcal{A}$ are in correspondence with the preorders on

Given a face F of the Artin arrangement for type B_n, C_n, or D_n, let $\boldsymbol{w}^F$ be the unique element of $\mathbb{Z}^n \cap F$ having smallest norm, and let $\boldsymbol{z}^F = (\boldsymbol{w}^F, -\boldsymbol{w}^F) \in \mathbb{Z}^{2n}$. The coefficients z_i^F lie in $[-n, n]$.

Since the permutation representations of the affine Weyl groups of classical type express these groups as subgroups of $\tilde{S}_n$, we can apply the projection π to the elements of these groups. We will again assume that $\pi(\sigma)$ acts on the coefficients of $\boldsymbol{z}^F \in \mathbb{Z}^n$ by permuting their indices. We then have the following.

Proposition 3.6.12. *Let $\boldsymbol{z}^F$ correspond to an open face F of the Artin arrangement for a Lie algebra of classical type, as above. Let the coefficients of $\boldsymbol{z}^F$ have values $\{v_1 < v_2 < \ldots < v_m\}$. If the word $\mathrm{w} = s_{i_p} \ldots s_{i_1}$ is such that*

1. $\pi(\mathrm{w})\, \boldsymbol{z}^F = \boldsymbol{z}^F$

2. *each s_{i_j} creates a descent, and in particular for types $\tilde{B}_n$ and $\tilde{D}_n$, s_0 is applied at step m only if*

$$\min\left(z^F_{\mathrm{w}^{-1}_{m-1}(1)}, z^F_{\mathrm{w}^{-1}_{m-1}(2)}\right) \geq \max\left(z^F_{\mathrm{w}^{-1}_{m-1}(2n-1)}, z^F_{\mathrm{w}^{-1}_{m-1}(2n)}\right)$$

and

$$\max\left(z^F_{\mathrm{w}^{-1}_{m-1}(1)}, z^F_{\mathrm{w}^{-1}_{m-1}(2)}\right) > \min\left(z^F_{\mathrm{w}^{-1}_{m-1}(2n-1)}, z^F_{\mathrm{w}^{-1}_{m-1}(2n)}\right)$$

3. *for each $k \in [1, m-1]$, the values v_k and v_{k+1} are swapped at least once*

then w *is reduced, and* $\mathrm{Inv}(\mathrm{w}^\infty) = I_F$.

Example 3.6.13. In B_4, let

$$F = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; -x_3 > x_1 = -x_2 > x_4 > 0 > -x_4 > -x_1 = x_2 > x_3 \right\}.$$

Then, we have $\boldsymbol{w}^F = (2, -2, -3, 1)$ and $\boldsymbol{z}^F = (2, -2, -3, 1, -1, 3, 2, -2)$. Now,

$$\pi\left(s_1 s_2 s_3 s_4 s_3 s_0 s_3 s_4 s_3 s_2 s_1\right)(2, -2, -3, 1, -1, 3, 2, -2) = (2, -2, -3, 1, -1, 3, 2, -2),$$

so the proposition says that $\mathrm{w} = s_1 s_2 s_3 s_4 s_3 s_0 s_3 s_4 s_3 s_2 s_1$ is reduced, and that $\mathrm{Inv}\left(\mathrm{w}^\infty\right) = I_F$.

Proof. In the case of C_n, since we can express every m_α in terms of some $\hat{m}_{ij}$, we need only keep track of the latter. Thus, the proof for type A_n carries over to this case. The proof for case A_n goes through for B_n and D_n as well, provided that we show that applying s_0 (in both cases) and s_n (in the case of D_n) to w_F increases some m_α by one.

Just as in A_n, we construct w_F, this time by numbering all the v_1s from left to right, and then all the v_2s, and so on. Since $z_{2n}^F = -z_1^F$ and $z_{2n-1}^F = -z_2^F$, if both $\min\left(z_1^F, z_2^F\right) \geq \max\left(z_{2n-1}^F, z_{2n}^F\right)$ and $\max\left(z_1^F, z_2^F\right) > \min\left(z_{(2n-1)}^F, z_{(2n)}^F\right)$, then either

$$z_1^F > z_2^F = 0 = z_{2n-1}^F > z_{2n}^F \quad \text{or} \quad z_1^F = z_2^F > z_{2n-1}^F = z_{2n}^F$$

$$\text{or} \quad z_1^F > z_2^F > z_{2n-1}^F > z_{2n}^F,$$

or one of these possibilities occurs with 1 and 2 (and $2n-1$ and $2n$) swapped. In the last two cases, since $z_i^F > z_j^F$ implies $w_F(i) > w_F(j)$, and $w_F(i-2n) = w_F(i) - 2n$, acting with s_0 will be increasing two m_αs from one to two, and decreasing no m_α. In the first case, we have $w_F(2) < w_F(2n-1)$; however, $w_F(2n-1) - w_F(2) < 2n$, so acting with s_0 will increase $\hat{m}_{w_F(2),w_F(2n-1)}$ from zero to negative one, while decreasing no m_α.

□

Unlike in type A_{n-1}, using this result to find reduced words for the minimal infinite elements $I_{\langle\Theta_l\rangle}$ leads us neatly to our coweight elements. We will not reproduce the proofs here.

Chapter 4

Factorization in Unitary Loop Groups

We continue to use the notation for finite-dimensional Lie algebras and groups introduced in Chapter 2. This results in a departure from the notation in [34], where finite-dimensional objects were denoted with dots. The only place we use dots is in distinguishing pointwise Iwasawa and triangular factorizations from their loop group analogs - we hope the dots will bring to mind the “point” in pointwise. When confusion is possible, such as when referring to the nilpotent subalgebras and subgroups of a loop group, we will denote infinite-dimensional analogues of finite-dimensional structures with rings, which we hope will remind the reader of loops. Thus, the nilpotent subalgebras and subgroups of LG are denoted $\mathring{\mathfrak{n}}^{\pm}$ and $\mathring{N}^{\pm}$. Otherwise, as a general principle, given an object X, we will use $LX = C^{\infty}(S^1, X)$ to denote the smooth loop space of X, $L_{\text{fin}}X = X[z, z^{-1}]$ to denote the subspace of Laurent polynomial functions on S^1 with coefficients in X, and $\hat{X}$ and $\tilde{X}$ to denote (central) extensions of X. Finally, $H^0(D; X)$ denotes the space of holomorphic sections on D taking values in X, and $H^0(D, p; X, Y)$ denotes the subset of $H^0(D; X)$ taking values in $Y \subset X$ at $p \in D$.

4.1 Affine Lie Algebras

We let $L\mathfrak{k} = C^{\infty}(S^1, \mathfrak{g})$ and $L\mathfrak{g} = C^{\infty}(S^1, \mathfrak{g})$, viewed as Lie algebras with pointwise bracket $[X, Y](z) = [X(z), Y(z)]$. We may view elements of $L\mathfrak{k}$ and $L\mathfrak{g}$ as either functions or Laurent series. These are Fréchet spaces, with the topology of uniform convergence on compact sets. The subalgebras $L\mathfrak{h}$ and $L\mathfrak{t}$ are maximal abelian subalgebras for $L\mathfrak{g}$ and $L\mathfrak{k}$, respectively. However, in a departure from the finite-dimensional case, they are not the only ones, even up to conjugacy [35]. We identify $\mathfrak{g}$ with the set of constant loops in $L\mathfrak{g}$. We let $L_{\text{fin}}\mathfrak{k} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{k} z^p$, $L_{\text{fin}}\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g} z^p$, and similarly for other subgroups of $\mathfrak{g}$.

4.1.1 Central Extensions

The smooth completion of the universal central extension of $L\mathfrak{g}$ is the vector space

$$\tilde{L}\mathfrak{g} = L\mathfrak{g} \oplus \mathbb{C}c,$$

and the smooth completion of the affine Kac-Moody Lie algebra corresponding to $L\mathfrak{g}$ is the vector space

$$\hat{L}\mathfrak{g} = L\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Both are equipped with the bracket $[\cdot,\cdot]$, where

$$[X,Y](z) = [X(z),Y(z)] + \frac{i}{2\pi}\int_{S^1}\left\langle X(z), \frac{dY}{dz}(z)\right\rangle c,$$

and

$$[X,c] = [c,X] = [X,d] = [d,c] = [c,d] = 0, \qquad [d,X] = \frac{d}{dz}X$$

for all $X \in L\mathfrak{g}$. There are corresponding central extensions of $L\mathfrak{k}$, and real forms

$$\tilde{L}\mathfrak{k} = L\mathfrak{k} \oplus i\mathbb{R}c, \qquad \hat{L}\mathfrak{k} = L\mathfrak{k} \oplus i\mathbb{R}c \oplus i\mathbb{R}d$$

for $\tilde{L}\mathfrak{g}$ and $\hat{L}\mathfrak{g}$, respectively. We define $\hat{\mathfrak{t}} = \mathfrak{t} \oplus i\mathbb{R}c \oplus i\mathbb{R}d$.

For each $\chi, \xi \in \hat{L}\mathfrak{g}$, we define $\mathrm{ad}_\chi\, \xi = [\chi, \xi]$. The eigenvalues of $\mathrm{ad}_{\hat{\mathfrak{h}}_\mathbb{R}} = \left\{ \mathrm{ad}_h \;\middle|\; h \in \hat{\mathfrak{h}}_\mathbb{R} \right\}$ are linear functionals α on $\hat{\mathfrak{h}}_\mathbb{R}$. The set of these eigenvalues is the extended root system of $\mathfrak{g}$, $\hat{\Phi}$.

4.1.2 Triangular Decomposition

We define

$$\mathring{\mathfrak{n}}^{\pm}_{\mathrm{pol}} = \mathfrak{n}^{\pm} \oplus \bigoplus_{p \in \mathbb{Z}_{>0}} \mathfrak{g} z^{\pm p},$$

and we define $\mathring{\mathfrak{n}}^{\pm}$ to be the smooth completion of $\mathring{\mathfrak{n}}^{\pm}_{\mathrm{pol}}$. We then have the triangular decompositions

$$\tilde{L}\mathfrak{g} = \mathring{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}}_\mathbb{R} \oplus \mathring{\mathfrak{n}}^+, \qquad \hat{L}\mathfrak{g} = \mathring{\mathfrak{n}}^- \oplus \hat{\mathfrak{h}}_\mathbb{R} \oplus \mathring{\mathfrak{n}}^+.$$

4.1.3 Root Homomorphisms

In addition to the root homomorphisms ι_i for $i \in [r]$, we define the root homomorphism

$$\iota_0\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = e_\theta z^{-1}, \qquad \iota_0\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = e_{-\theta} z, \tag{4.1.1}$$

where $\{e_{-\theta}, h_\theta, e_\theta\}$ satisfy the $\mathfrak{sl}(2,\mathbb{C})$-commutation relations, and e_θ is a highest root vector for $\mathfrak{g}$.

4.2 Loop Groups and Extensions

Let $\Pi : \tilde{L}G \to LG$ ($\Pi : \tilde{L}K \to LK$) denote the universal central $\mathbb{C}^*$ ($\mathbb{T}$) extension of the smooth loop group LG (LK, respectively), as in [35]. The Lie algebra of $\tilde{L}G$ is $\tilde{L}\mathfrak{g}$, and the Lie algebra of $\tilde{L}K$ is $\tilde{L}\mathfrak{k}$. Let $\mathring{N}^{\pm}$ denote the subgroups corresponding to $\mathring{\mathfrak{n}}^{\pm}$ (they are not, strictly speaking, the exponentials of $\mathring{\mathfrak{n}}^{\pm}$). Since the restriction of Π to $\mathring{N}^{\pm}$ is an isomorphism, we will always identify $\mathring{N}^{\pm}$ with its image. In other words, $l \in \mathring{N}^+$ is identified with a smooth loop having a holomorphic extension to Δ satisfying $l(0) \in N^+$. Also set $\tilde{H} = \exp\left(\tilde{\mathfrak{h}}\right)$, $\tilde{T} = \exp\left(\tilde{\mathfrak{t}}\right)$, and $\tilde{A} = \exp\left(\tilde{\mathfrak{h}}_\mathbb{R}\right)$, and $\mathring{B}^{\pm} = H\mathring{N}^{\pm}$.

4.2.1 The Weyl Group

The Weyl group of LG is W_{aff}, which is isomorphic to the quotient groups $N_{LG}\left(\tilde{H}\right) = N_{LK}\left(\tilde{T}\right)$. As in the finite-dimensional case, a representative for w will be denoted with a bold type $\mathbf{w}$, and these representatives act on $g \in LG$ by conjugation, and on $X \in L\mathfrak{g}$ by $\mathrm{Ad}_{\mathbf{w}}$.

Recall that $W_{\text{aff}} = W \propto \check{T}$, and that the kernel of $\exp : \mathfrak{t} \longrightarrow T$ is $2\pi i$ times the coroot lattice $\check{T}$. Thus, there is a natural identification of $\check{T}$ with $\mathrm{Hom}\left(S^1, T\right)$, under which we identify $\tau \in \check{T}$ with the homomorphism

$$S^1 \longrightarrow T : e^{2\pi i x} \longmapsto \exp\left(2\pi i x \tau\right)$$

for $\theta \in \mathbb{R}$. This identification provides us with representatives for the elements of W_{aff}. For example, given a choice of logarithm for $z \in S^1$, s_0 is represented by the loop

$$\mathbf{s_0} = \exp\left(2\pi i \log(z) h_\theta\right) \iota_0 \left(\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \right).$$

As in the finite-dimensional case, a reduced word $\mathrm{w} = s_{i_{l(w)}} \dots s_{i_1}$ for $w \in W_{\text{aff}}$, along with a set of representatives $\mathbf{s_i}$ for the simple reflections, gives a set of representatives for w_m, namely $\mathbf{w}_m = \mathbf{s_{i_m}} \dots \mathbf{s_{1_1}}$. Given $\mathbf{w} = \mathbf{s_{i_{l(w)}}} \dots \mathbf{s_{i_1}}$, we define the root homomorphisms $\iota_{\tau_j(\mathbf{w})} : \mathfrak{sl}(2,\mathbb{C}) \longrightarrow L\mathfrak{g}$ by

$$\iota_{\tau_j(\mathbf{w})}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) = \mathrm{Ad}_{\mathbf{w_{j-1}}} \iota_{i_j} \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right), \qquad \iota_{\tau_j(\mathbf{w})}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) = \mathrm{Ad}_{\mathbf{w_{j-1}}} \iota_{\tau_j(\mathbf{w})} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right),$$

$$\iota_{\tau_j(\mathbf{w})}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = \mathrm{Ad}_{\mathbf{w_{j-1}}} \iota_{\tau_j(\mathbf{w})} \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right),$$

where Ad denotes the derivative of conjugation at the identity. We define $\iota_{\tau_j(\mathbf{w})} : SL(2,\mathbb{C}) \longrightarrow LG$ by $\iota_{\tau_j(\mathbf{w})} \exp\left(X\right) = \exp\left(\iota_{\tau_j(\mathbf{w})}(X)\right)$ for each $X \in \mathfrak{sl}(2,\mathbb{C})$.

For $w \in W_{\text{aff}}$, we define the subgroups

$$\mathring{N}_w^+ = \mathring{N}^+ \cap \mathbf{w}^{-1} \mathring{N}^- \mathbf{w}, \qquad \mathring{N}_w^- = \mathring{N}^- \cap \mathbf{w}^{-1} \mathring{N}^+ \mathbf{w},$$

as well as the "opposite" subgroups

$$\mathring{\mathring{N}}_w^+ = \mathring{N}^+ \backslash \mathring{N}_w^+ = \mathring{N}^+ \cap \mathbf{w}^{-1} \mathring{N}^+ \mathbf{w}, \qquad \mathring{\mathring{N}}_w^- = \mathring{N}^- \backslash \mathring{N}_w^- = \mathring{N}^- \cap \mathbf{w}^{-1} \mathring{N}^- \mathbf{w}.$$

These do not depend on the choice of representative $\mathbf{w}$. Note that $\mathring{N}_w^\pm$ has finite dimension $l(w)$, while $\mathring{\mathring{N}}_w^\pm$ is infinite dimensional, having codimension $l(w)$.

4.2.2 Iwasawa Decomposition

There are *two* Iwasawa decompositions for LG. The one induced by the Iwasawa decomposition for G will be referred to as the pointwise Iwasawa decomposition

$$LG = LK \cdot LA \cdot LN^{+},$$

so for $g \in LG$, we have

$$g = \dot{\mathbf{k}}(g)\dot{\mathbf{a}}(g)\dot{\mathbf{n}}(g),$$

or, more briefly, $g = \dot{k}\dot{a}\dot{n}$. There is also the Iwasawa decomposition appropriate to LG considered as a loop group,

$$LG = LK \cdot A \cdot \mathring{N}^{+},$$

where for $g \in LG$ we have

$$g = \mathbf{k}(g)\mathbf{a}(g)\mathbf{n}(g),$$

for a constant, diagonal loop $\mathbf{a}(g) \in A$, and a smooth loop $\mathbf{n}(g)$ having holomorphic extension to the open unit disk Δ and satisfying $\mathbf{n}(g)(0) \in N^{+}$. We will occasionally write $g = kan$ to denote such a decomposition.

4.2.3 Triangular Decomposition

We will also deal with two triangular decompositions on LG. The first results from pointwise application of triangular factorization for G. Thus, if $g \in LN^{-} \cdot LH \cdot LN^{+} \subset LG$, we have the pointwise triangular factorization

$$g = \dot{\mathbf{l}}(g)\dot{\mathbf{d}}(g)\dot{\mathbf{u}}(g) = \dot{\mathbf{l}}(g)\dot{\mathbf{m}}(g)\dot{\mathbf{a}}(g)\dot{\mathbf{u}}(g),$$

or, more briefly, $g = \dot{l}\dot{d}\dot{u} = \dot{l}\dot{m}\dot{a}\dot{u}$.

Alternatively, for $\tilde{g} \in \mathring{N}^{-} \cdot \tilde{H} \cdot \mathring{N}^{+} \subset \tilde{L}G$, there is a unique triangular decomposition

$$\tilde{g} = \mathbf{l}(\tilde{g})\, \mathbf{d}(\tilde{g})\, \mathbf{u}(\tilde{g}), \qquad \text{where } \mathbf{d}(\tilde{g}) = \mathbf{m}(\tilde{g})\, \mathbf{a}(\tilde{g}) = \prod_{j \in [0,r]} \sigma_j(\tilde{g})^{h_j}, \tag{4.2.1}$$

and σ_j is the fundamental matrix coefficient for the highest weight vector corresponding to Λ_j. We will sometimes write $g = ldu = lmau$ to denote such a triangular factorization.

Since we will mostly be working with $g \in LG$, we would like to know how this triangular factorization restricts to LG. If $\Pi(\tilde{g}) = g$, then because $\sigma_0^{h_0} = \sigma_0^{c-h_\theta}$ projects to $\sigma_0^{-h_\theta}$, $g = \mathbf{l}(\tilde{g})\, \Pi\left(\mathring{\mathbf{d}}(\tilde{g})\right) \mathbf{u}(\tilde{g})$, where

$$\Pi(d)(\tilde{g}) = \sigma_0(\tilde{g})^{-h_\theta} \prod_{j \in [r]} \sigma_j(\tilde{g})^{h_j} = \prod_{j \in [r]} \left(\frac{\sigma_j(\tilde{g})}{\sigma_0(\tilde{g})^{\check{a}_j}} \right)^{h_j}, \tag{4.2.2}$$

for positive integers $\check{a}_j$ such that $h_\theta = \sum_{j\in[r]} \check{a}_j h_j$.

If $\tilde{k} \in \tilde{L}K$, then $|\sigma_j\left(\tilde{k}\right)|$ depends only on $k = \Pi\left(\tilde{k}\right)$. We will indicate this by writing $|\sigma_j\left(\tilde{k}\right)| = |\sigma_j|(k)$. This also implies $\mathbf{a}\left(\tilde{k}\right) = \mathbf{a}(k)$. Thus, we have the following.

Lemma 4.2.3. *For $\tilde{k} \in \tilde{L}K$ and $k = \Pi\left(\tilde{k}\right)$, $\tilde{k}$ has a triangular factorization if and only if k has a triangular factorization. The restriction of the projection $\Pi : \tilde{L}K \longrightarrow LK$ to elements $k \in \tilde{L}K$ with $\mathbf{m}\left(\tilde{k}\right) = 1$ is injective.*

We will also use the following proposition.

Proposition 4.2.4. *The subgroup LN^+ is contained in $\mathring{N}^- \cdot \mathring{N}^+$. In other words, if $g \in LN^+$, then g has a triangular factorization $g = \mathbf{l}(g)\mathbf{u}(g)$.*

Proof. Let $m = \mathrm{ht}(\theta)$, so that $\mathfrak{n}^+$ is an m-step nilpotent Lie algebra. We will proceed by induction on m.

If $m = 1$, then N^+ is an additive abelian group, and triangular factorization is trivial. In particular, if G is simple, then $G = SU(2)$, and if $n \in LN^+$, then

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_- \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_+ \\ 0 & 1 \end{pmatrix}.$$

for some $x(z) \in C^\infty(S^1, \mathbb{C})$, where x_- is the singular part of x and x_+ is the holomorphic part.

Now, assume we can find a triangular factorization for $n \in LN^+$ when N^+ is m-step. Let N^+ be $m+1$-step, and let

$$LN^+ \ni n = \exp\left(\sum_{\alpha\in\Phi^+} x_\alpha e_\alpha\right),$$

for $x_\alpha \in C^\infty(S^1, \mathbb{C})$, and some choice of root vectors $\{e_\alpha\}_{\alpha\in\Phi^+}$. Now, let

$$l_1 = \exp\left(\sum_{\mathrm{ht}(\alpha)=1} (x_\alpha)_- e_\alpha\right), \qquad u_1 = \exp\left(\sum_{\mathrm{ht}(\alpha)=1} (x_\alpha)_+ e_\alpha\right).$$

Then,

$$l_1^{-1} n u_1^{-1} \in L\left[N^+, N^+\right],$$

and $[N^+, N^+]$ is m-step. Let $l_1^{-1} n u_1^{-1} = l_2 u_2$ be a triangular factorization. Then, $n = (l_1 l_2)(u_2 u_1)$ is a triangular factorization for n. □

Proposition 4.2.4 has the following consequences, which we will also use below.

Corollary 4.2.5.

1. *The subgroups LN^- and $LB^\pm$ are contained in the top stratum of LG.*

2. *If the singular (holomorphic) part of $g \in LB^\pm$ is polynomial, then $\mathbf{l}(g) \in \mathring{N}^-$ (respectively, $\mathbf{u}(g) \in \mathring{N}^+$ is a polynomial loop.*

4.2.4 Birkhoff and Bruhat Decompositions

The Birkhoff decompositions for LG and LK [35] are

$$LG = \bigcup_{w \in W_{\mathrm{aff}}} \Sigma_w^{LG}, \qquad LK = \bigcup_{w \in W_{\mathrm{aff}}} \Sigma_w^{LK},$$

where

$$\Sigma_w^{LG} = \mathring{N}^- \mathbf{w} H \mathring{N}^+, \qquad \Sigma_w^{LK} = \mathring{N}^- \mathbf{w} H \mathring{N}^+ \cap LK.$$

The multiplication map induces a diffeomorphism

$$\mathring{\bar{N}}_{w^{-1}}^- \times \{\mathbf{w}\} \times A \times \mathring{N}^+ \longrightarrow \Sigma_w^{LG}.$$

The Bruhat decompositions for $L_{\mathrm{fin}}G$ and $L_{\mathrm{fin}}K$ are

$$L_{\mathrm{fin}}G = \bigcup_{w \in W_{\mathrm{aff}}} C_w^{L_{\mathrm{fin}}G}, \qquad L_{\mathrm{fin}}K = \bigcup_{w \in W_{\mathrm{aff}}} C_w^{L_{\mathrm{fin}}K},$$

where

$$C_w^{L_{\mathrm{fin}}G} = \mathring{N}_{\mathrm{fin}}^+ \mathbf{w} H \mathring{N}_{\mathrm{fin}}^+, \qquad C_w^{L_{\mathrm{fin}}K} = \mathring{N}_{\mathrm{fin}}^+ \mathbf{w} H \mathring{N}_{\mathrm{fin}}^+ \cap L_{\mathrm{fin}}K.$$

The multiplication map induces a diffeomorphism

$$\mathring{N}_{w^{-1}}^+ \times \{\mathbf{w}\} \times A \times \mathring{N}_{\mathrm{fin}}^+ \longrightarrow C_w^{L_{\mathrm{fin}}G}.$$

4.2.5 The Flag Manifold

The flag manifold for LG is $LG/\mathring{B}^+ = LK/T$. The projections of the Birkhoff strata Σ_w^{LK} to LK/T will be denoted $\mathring{\Sigma}_w$; the projections of the Bruhat cells C_w^{LK} to LK/T will be denoted $\mathring{C}_w$. We have diffeomorphisms

$$\mathring{\bar{N}}_{w^{-1}} \longrightarrow \Sigma_w \longrightarrow \mathring{\bar{N}}_{w^{-1}}^- \mathbf{w} A \mathring{N}^+ \cap LK, \qquad \mathring{N}_{w^{-1}}^+ \longrightarrow C_w \longrightarrow \mathring{N}_{w^{-1}}^+ \mathbf{w} A \mathring{N}_{\mathrm{fin}}^+ \cap LK.$$

The cell C_w has dimension $l(w)$, while the stratum $\mathring{\Sigma}_w$ has codimension $l(w)$; thus, there is clearly no isomorphism between Bruhat cells and Birkhoff strata.

We define injections of C_w^{LK} and Σ_w^{LK} into Σ_1^{LK} by

$$\phi : C_w^{L_{\mathrm{fin}}K} \longmapsto \Sigma_1^{LK} : u_w \mathbf{w} du \longmapsto (u_w)^* \, du, \qquad \psi : \Sigma_w^{LK} \longmapsto \Sigma_1^{LK} : l_w \mathbf{w} du \longmapsto l_w du.$$

4.2.6 Birkhoff Strata and Bruhat Cells for Infinite Elements

For an infinite reduced word $\mathrm{w} = \left(s_{i_j}\right)_{j \in \mathbb{N}}$ for some $I \in \mathcal{W}$, we define

$$\mathring{N}_{I,\mathrm{fin}}^{\pm} = \bigcup_{m \in \mathbb{N}} \mathring{N}_{\mathrm{w}_m}^-, \qquad \mathring{C}_I^{L_{\mathrm{fin}}G} = \bigcup_{m \in \mathbb{N}} \mathring{C}_{\mathrm{w}_m}^{L_{\mathrm{fin}}G}, \qquad \mathring{C}_I^{L_{\mathrm{fin}}K} = \bigcup_{m \in \mathbb{N}} \mathring{C}_{\mathrm{w}_m}^{L_{\mathrm{fin}}K}$$

and we define $\mathring{N}_I^{\pm}$, $\mathring{C}_w^{LG}$ and $\mathring{C}_w^{LK}$ to be the smooth completions of these spaces. We also define $I^c = \hat{\Phi}_{\mathcal{R}}^+ \backslash I$, and

$$\mathring{\mathring{N}}_I^{\pm} = \mathring{N}_{I^c}^{\pm} = \bigcap_{m \in \mathbb{N}} \mathring{\mathring{N}}_{\mathrm{w}_m}^{\pm}, \qquad \Sigma_I^{LG} = \bigcap_{m \in \mathbb{N}} \Sigma_{\mathrm{w}_m}^{LG}, \qquad \Sigma_I^{LK} = \bigcap_{m \in \mathbb{N}} \Sigma_{\mathrm{w}_m}^{LK}.$$

Finally, we define

$$\phi\left(C_I^{LK}\right) = \mathring{N}_I^- A \mathring{N}^+ \cap LK \subset \Sigma_1^{LK}, \qquad \psi\left(\Sigma_I^{LK}\right) = \mathring{\mathring{N}}_I^- A \mathring{N}^+ \cap LK \subset \Sigma_1^{LK},$$

and note that if $I, I^c \in \mathcal{W}$, then

$$\psi\left(\Sigma_I^{LK}\right) = \phi\left(C_{I^c}^{LK}\right).$$

We will be interested in

$$I = \mathbb{N}\delta - \Phi^+, \qquad I^c = \mathbb{Z}_{\geq 0}\delta + \Phi^+.$$

Note that

$$\mathring{N}_{\mathbb{N}\delta - \Phi^+}^- = \mathring{N}^- \cap LN^+, \qquad \mathring{\mathring{N}}_{\mathbb{N}\delta - \Phi^+}^- = \mathring{N}_{\mathbb{Z}_{\geq 0}\delta + \Phi^+}^- = \mathring{N}^- \cap LN^-,$$

so that

$$\phi\left(C_{\mathbb{N}\delta-\Phi^+}^{LK}\right) = \psi\left(\Sigma_{\mathbb{Z}_{\geq 0}\delta+\Phi^+}^{LK}\right) = \left(\mathring{N}^- \cap LN^+\right) A \mathring{N}^+ \cap LK,$$
$$\phi\left(C_{\mathbb{Z}_{\geq 0}\delta+\Phi^+}^{LK}\right) = \psi\left(\Sigma_{\mathbb{N}\delta-\Phi^+}^{LK}\right) = \left(\mathring{N}^- \cap LN^-\right) A \mathring{N}^+ \cap LK.$$

4.3 Parametrizations of Birkhoff Strata for Infinite Elements

The proof of Theorem 2.4.13 can be repeated verbatim to prove the following.

Proposition 4.3.1. *For every $w \in W_{\mathrm{aff}}$ and each reduced word* w*, there is a diffeomorphism*

$$\mathbb{C}^{l(w)} \longrightarrow \mathring{\psi}\left(C_w^{L_{\mathrm{fin}}K}\right) = N_w^- A \mathring{N}_{\mathrm{fin}}^+ \cap L_{\mathrm{fin}}K \subset \Sigma_1^{LK}$$
$$\boldsymbol{\zeta} \longmapsto k(\boldsymbol{\zeta}) := \overleftarrow{\prod_{i=1}^{l(w)}} \iota_{\tau_j(\mathrm{w})}\left(N(\zeta_j)\begin{pmatrix} 1 & -\bar{\zeta}_j \\ \zeta_j & 1 \end{pmatrix}\right)$$

and

$$\mathbf{a}(k(\boldsymbol{\zeta})) = \overleftarrow{\prod_{j=1}^{l(w)}} \iota_{\tau_j(\mathrm{w})}\left(\begin{pmatrix} N(\zeta_i) & 0 \\ 0 & N(\zeta_i)^{-1} \end{pmatrix}\right).$$

The following result extends this proposition to the infinite element $\mathbb{N}\delta - \Phi^{+} \in \mathcal{W}$, and identifies the image of the resulting diffeomorphism. Recall that $N(\zeta_j) = \frac{1}{\sqrt{1+|\zeta_j|^2}}$.

Theorem 4.3.2. *For a loop $k \in L_{\text{fin}}K$, the following are equivalent.*

1. *The antiholomorphic factor $\mathbf{l}(k)$ in the triangular factorization of k takes values in N^{+}, and $\mathbf{m}(k) = 1$. In other words, $k \in \psi\left(C^{L_{\text{fin}}K}_{\mathbb{N}\delta-\Phi^{+}}\right) = \left(\mathring{N}^{-}_{\text{fin}} \cap L_{\text{fin}}N^{+}\right) A\mathring{N}^{+}_{\text{fin}} \cap L_{\text{fin}}K$.*

2. *For any irreducible representation π of K with highest-weight vector v, $\pi\left(k^{-1}\right)v$ is holomorphic in the open unit disk Δ and a positive multiple of v at zero.*

3. *For any $\mathrm{w} \in \mathcal{RW}$ with $\mathrm{Inv}(\mathrm{w}) = \mathbb{N}\delta - \Phi^{+}$, there is a factorization*

$$k = \overleftarrow{\prod_{j=1}^{M}} \iota_{\tau_j(\mathbf{w})}\left(N(\zeta_j)\begin{pmatrix} 1 & -\bar{\zeta}_j \\ \zeta_j & 1 \end{pmatrix}\right), \tag{4.3.3}$$

   where $\zeta_j \in \mathbb{C}$ and $\mathbb{N} \ni M < \infty$.

We remark that the factors in (4.3.3) are analogous to those appearing in, say, Theorem 2.4.13, since

$$\iota_{\tau_i(\mathbf{w})}\left(N(\zeta_j)\begin{pmatrix} 1 & -\bar{\zeta}_j \\ \zeta_j & 1 \end{pmatrix}\right) = \mathbf{k}\left(\iota_{\tau_i(\mathbf{w})}\left(\begin{pmatrix} 1 & \zeta_j \\ & 1 \end{pmatrix}\right)\right).$$

Proof. The equivalence of (3) and (1) follows from Proposition 4.3.1. We will show the equivalence of (1) and (2).

To show that (1) implies (2), let (π, V_π) be an irreducible representation for K, let Λ be the corresponding highest weight, and let v be the corresponding highest-weight vector. Then,

$$\begin{aligned}
\pi\left(k^{-1}\right)\phi =& \pi\left(\mathbf{u}(k)^{-1}\right)\pi\left(\mathbf{a}(k)^{-1}\right)\pi\left(\mathbf{l}(k)^{-1}\right)v \\
=& \pi\left(\mathbf{u}(k)^{-1}\right)\pi\left(\mathbf{a}(k)^{-1}\right)v \\
=& a^{-\Lambda}\pi\left(\mathbf{u}(k)^{-1}\right)v \in H^{0}\left(\Delta, 0; V_\pi, \mathbb{R}^{+}v\right).
\end{aligned}$$

Next, we will show that (2) implies (1), by constructing the required triangular factorization for k satisfying (2). By taking an irreducible unitary representation of K, we may assume that $k \in LSU(n)$. The fundamental highest-weight representations[1] for $SU(n)$ are explicit and familiar. They are the representations of $SU(n)$ on $\wedge^{l}\mathbb{C}^{n}$

[1]Those whose highest weights are the fundamental weights.

for $l \in [n]$, given by $\pi_{\Lambda_l}(k)\left(v_1 \wedge \ldots \wedge v_l\right) = (kv_1) \wedge \ldots \wedge (kv_n)$. The matrix coefficients of k in π_{Λ_l} are the determinants of the $l \times l$ minors of k, and we will use the notation

$$k_J^I = \det\left(M_J^I(k)\right).$$

The first step is to perform a pointwise triangular factorization of k^*,

$$k^* = \dot{\mathbf{l}}(k)\dot{\mathbf{m}}(k)\dot{\mathbf{a}}(k)\dot{\mathbf{u}}(k).$$

Recalling (1.1.1) from the distant past, we have for $n \geq i > j \geq 1$,

$$\dot{\mathbf{l}}(k)_j^i = (-1)^{i+j}\,\frac{k_{[j,i-1]\cup[i+1,n]}^{[j+1,n]}}{k_{[j+1,n]}^{[j+1,n]}},$$

which is by assumption a ratio of two holomorphic functions with a denominator that is real and positive at zero. Then,

$$\begin{aligned}\left(k\dot{\mathbf{l}}(k)\right)_j^i &= \sum_{m\in[j-1]} k_m^i\,(-1)^{m+j}\,\frac{k_{[j,m-1]\cup[m+1,n]}^{[j+1,n]}}{k_{[j+1,n]}^{[j+1,n]}} + k_j^i \\ &= \frac{1}{k_{[j+1,n]}^{[j+1,n]}}\sum_{m\in[j]}(-1)^{m+i}\,k_m^i k_{[j,m-1]\cup[m+1,n]}^{[j+1,n]} = \frac{k_{[j,n]}^{\{i\}\cup[j+1,n]}}{k_{[j+1,n]}^{[j+1,n]}}.\end{aligned}$$

As expected, this is zero if $i > j$; note that if $i = j$, it is a ratio of two holomorphic functions which are real and positive at zero.

This is the factorization we are after, but we must check whether and where it is defined. Since the functions $k_{[j+1,n]}^{[j+1,n]}$ are nonzero at zero, the functions $\dot{\mathbf{l}}(k)_j^i$ are defined and holomorphic in some small open disk around zero, say $\epsilon\Delta$. Since k has a finite Fourier transform, it can be extended holomorphically to $\mathbb{C}^*$. Thus, the factorization $k = \dot{\mathbf{l}}(k)\dot{\mathbf{m}}(k)\dot{\mathbf{a}}(k)\dot{\mathbf{u}}(k)$ holds on any loop in the annulus $\epsilon\overline{\Delta}\backslash\{0\}$, and we have

$$\dot{\mathbf{l}}(k) \in H^0\left(\epsilon\Delta, 0; N^-, 1\right), \qquad \dot{\mathbf{m}}(k) \in H^0\left(\epsilon\Delta, 0; \dot{\mathbf{m}}(k), 1\right), \qquad \dot{\mathbf{a}}(k) \in H^0\left(\epsilon\Delta; \dot{\mathbf{a}}(k)\right).$$

Next, we use this pointwise triangular factorization to construct a loop-wise one. Defining

$$a := \left(\dot{\mathbf{a}}(k)(0)\right)^{-1},$$

we write

$$k = \dot{\mathbf{u}}(k)^{-1} a\left(\dot{\mathbf{m}}(k)\dot{\mathbf{a}}(k)a\right)^{-1}\dot{\mathbf{l}}(k)^{-1}.$$

Since $\dot{\mathbf{u}}(k)^{-1}$ takes values in N^+, it has a loop-wise triangular factorization on ϵS^1,

$$\dot{\mathbf{u}}(k)^{-1} =: l\tilde{u},$$

where

$$l \in H^0\left(\epsilon^{-1}\Delta^*, 0; N^+, 1\right), \qquad \tilde{u} \in H^0\left(\epsilon\Delta; N^+\right).$$

Define

$$u := \left(a^{-1}\tilde{u}a\right)\left(\dot{\mathbf{m}}(k)\dot{\mathbf{a}}(k)a\right)^{-1}\dot{\mathbf{l}}(k)^{-1} \in H^0\left(\epsilon\Delta, 0; \dot{G}, N^+\right).$$

Then

$$k = lau$$

is the desired triangular factorization of k on ϵS^1.

To see that this decomposition can be extended to S^1, we must examine the entries of l more closely. Since

$$\left(\dot{\mathbf{u}}(k)^{-1}\right)^i_j = \frac{k^{\{i\}\cup[j+1,n]}_{[j,n]}}{k^{[j,n]}_{[j,n]}}$$

for $i < j$, we see that l has a pole of finite order at zero, and can be extended holomorphically to $\hat{\mathbb{C}}\backslash\{0\}$. Then the product $(la)^{-1}k$ is holomorphic on $\mathbb{C}^*$, and equals u on $\epsilon\Delta$. Thus $(la)^{-1}k$ analytically continues u to all of $\mathbb{C}$, including S^1. □

The previous theorem is extended to smooth loops in Theorem 3.2 of [34], which we state below.

Theorem 4.3.4. *For a loop $k \in LK$, the following are equivalent.*

1. *The antiholomorphic factor $\mathbf{l}(k)$ in the triangular factorization of k takes values in N^+, and $\mathbf{m}(k) = 1$. In other words, $k \in \psi\left(C^{LK}_{\mathbb{N}\delta-\Phi^+}\right) = \left(\mathring{N}^- \cap LN^+\right)A\mathring{N}^+ \cap LK$.*

2. *For any irreducible representation π of K with highest-weight vector v, $\pi\left(k^{-1}\right)v$ is holomorphic and nonvanishing in the open unit disk Δ, and a positive multiple of v at zero.*

3. *For any $\mathrm{w} \in \mathcal{RW}$ with $\mathrm{Inv}(\mathrm{w}) = \mathbb{N}\delta - \Phi^+$, there is a factorization*

$$k = \lim_{M\to\infty} \overleftarrow{\prod_{j=1}^{M}} \iota_{\tau_j(\mathbf{w})}\left(N\left(\zeta_j\right)\begin{pmatrix} 1 & \zeta_j \\ -\bar{\zeta}_j & 1 \end{pmatrix}\right),$$

where the sequence $\{\zeta_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ is rapidly decreasing.

Theorems 4.3.2 and 4.3.4 imply their analogues for $\psi\left(C^{LK}_{\mathbb{Z}_{\geq 0}\delta+\Phi^+}\right)$. We state the latter.

Theorem 4.3.5. *For a loop $k \in LK$, the following are equivalent.*

1. *The antiholomorphic factor $\mathbf{l}(k)$ in the triangular factorization of k takes values in N^-, and $\mathbf{m}(k) = 1$. In other words, $k \in \psi\left(C^{LK}_{\mathbb{Z}_{\geq 0}\delta+\Phi^+}\right) = \left(\mathring{N}^- \cap LN^-\right)A\mathring{N}^+ \cap LK$.*

2. *For any irreducible representation* π *of* K *with lowest-weight vector* u, $\pi\left(k^{-1}\right)u$ *is holomorphic and nonvanishing in the open unit disk* Δ, *and a positive multiple of* u *at zero.*

3. *For any* $\mathrm{w} \in \mathcal{RW}$ *with* $\mathrm{Inv}(\mathrm{w}) = \mathbb{Z}_{\geq 0}\delta + \Phi^+$, *there is a factorization*

$$k = \lim_{M \to \infty} \overleftarrow{\prod_{i=1}^{M}} \iota_{\tau_i(\mathbf{w})} \left(N\left(\eta_i\right) \begin{pmatrix} 1 & \eta_i \\ -\bar{\eta}_i & 1 \end{pmatrix} \right),$$

where the sequence $\{\eta_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ *is rapidly decreasing.*

Extending these theorems to arbitrary infinite elements $I \in \mathcal{W}$ is a subject for future research, and crucial to the alternative factorizations for Σ_1^{LK} hinted at below.

4.4 Factorizations for Σ_1^{LK}

Now we are in a position to prove that our factorization really is a refinement of triangular factorization. We show that every element of Σ_1^K can be written as a product of the conjugate transpose of an element of $\psi\left(C^{LK}_{\mathbb{Z}_{\geq 0}\delta+\Phi^+}\right)$, an element of $L\dot{T}$, and an element of $\psi\left(C^{LK}_{\mathbb{N}\delta-\Phi^+}\right)$.

Theorem 4.4.1. *A loop* $k \in LK$ *has a triangular factorization* k *if and only if it has a factorization* $k = k_1^* \lambda k_2$, *where* $k_1 \in \psi\left(C^{LK}_{\mathbb{Z}_{\geq 0}\delta+\Phi^+}\right)$, $\lambda \in LT$, *and* $k_2 \in \psi\left(C^{LK}_{\mathbb{N}\delta-\Phi^+}\right)$. *In other symbols, there is a diffeomorphism*

$$\Sigma^{LK}_{\mathbb{N}\delta-\Phi^+} \times LT \times \Sigma^{LK}_{\mathbb{Z}_{\geq 0}\delta+\Phi^+} \longrightarrow \Sigma_1^{LK}.$$

Proof. If $k = k_1^* \lambda k_2$, where k_1 and k_2 have the triangular factorizations

$$k_1 =: l_1 a_1 u_1, \qquad k_2 =: l_2 a_2 u_2$$

for

$$l_1 \in \mathring{N}^- \cap LN^-, \qquad l_2 \in \mathring{N}^- \cap LN^+,$$

then we also have

$$k = u_1^* a_1^* l_1^* \lambda l_2 a_2 u_2.$$

Here, $u_1^* \in N^- = H^0\left(\Delta^*, \infty; \dot{G}, N^-\right)$. Now,

$$b := a_1^* l_1^* \lambda l_2 a_2 \in LB^+,$$

so it has a triangular factorization

$$b = \tilde{l} m \tilde{u}.$$

Then k has the triangular factorization

$$\mathbf{l}(k) = u_1^* \tilde{l}, \qquad \mathbf{m}(k) = m, \qquad \mathbf{a}(k) = a, \qquad \mathbf{u}(k) = \tilde{u} u_2.$$

Now assume $k = lmau$, and perform the *pointwise* Iwasawa decompositions

$$l =: \dot{k}_1^* \dot{a}_1 \dot{n}_1, \qquad u =: \dot{n}_2 \dot{a}_2 \dot{k}_2,$$

where $\dot{a}_1, \dot{a}_2 \in LA$, and $\dot{n}_1, \dot{n}_2 \in LN^+$. Now,

$$\dot{a}_1 =: e^{\chi_1^* + X_1 + \chi_1}, \qquad \dot{a}_2 =: e^{\chi_2^* + X_2 + \chi_2}$$

for $X_1, X_2 \in \mathfrak{a}$ and $\chi_1, \chi_2 \in H^0(\Delta; \mathfrak{a})$. Then, $\chi_i - \chi_i^* \in \dot{\mathfrak{t}}$ for $i = 1, 2$, so we define the unitary loops

$$k_1 := e^{-\chi_1 + \chi_1^*} \dot{k}_1, \qquad k_2 := e^{-\chi_2 + \chi_2^*} \dot{k}_2.$$

Now,

$$k_1 = e^{-X_1 - 2\chi_1} \dot{n}_1^{-*} l^*, \qquad k_2 = e^{-X_2 - 2\chi_2} \dot{n}_2^{-1} u.$$

Since $e^{-X_1 - 2\chi_1} \dot{n}_1^{-*} \in LB^-$ and $e^{-X_2 - 2\chi_2} \dot{n}_2^{-1} \in LB^+$, both have triangular factorizations. Write

$$\dot{n}_1^{-*} =: \tilde{l}_1 \tilde{u}_1, \qquad \dot{n}_2^{-1} =: \tilde{l}_2 \tilde{u}_2.$$

Next write the triangular factorizations

$$\mathrm{Ad}_{e^{-X_1 - 2\chi_1}} \tilde{l}_1 =: l_1 \hat{u}_1, \qquad \mathrm{Ad}_{e^{-X_2 - 2\chi_2}} \tilde{l}_2 =: l_2 \hat{u}_2.$$

Then

$$\begin{array}{lll} l_1, & a_1 := e^{-X_1}, & u_1 := \mathrm{Ad}_{e^{X_1}}(\hat{u}_1)\, e^{-2\chi_1} \tilde{u}_1 l^*, \\ l_2, & a_2 := e^{-X_2}, & u_2 := \mathrm{Ad}_{e^{X_2}}(\hat{u}_2)\, e^{-2\chi_2} \tilde{u}_2 u \end{array}$$

are triangular factorizations of k_1 and k_2, and $l_1 \in N^- \cap L\dot{N}^-$, while $l_2 \in N^- \cap L\dot{N}^+$. Finally,

$$\lambda = k_1 k k_2^* = e^{\bar{X}_1 + 2\chi_1^*} \dot{n}_1 m a \dot{n}_2 e^{X_2 + 2\chi_2} \in LB^+ \cap LK = L\dot{T}.$$

This completes the proof. □

We might ask whether we can continue to decompose Σ_1^{LK}, or even $\Sigma_{\mathbb{N}\delta - \Phi^+}^{LK}$. For example, we might ask for something like a diffeomorphism

$$\Sigma_{\mathbb{N}\delta - \Phi_1}^{LK} \times \Sigma_{\mathbb{N}\delta - (\Phi_2 \setminus \Phi_1)}^{LK} \times \ldots \times \Sigma_{\mathbb{N}\delta - (\Phi_r \setminus \Phi_{r-1})}^{LK} \longrightarrow \Sigma_{\mathbb{N}\delta - \Phi^+}^{LK}.$$

A step in this direction, and a loop group analogue of Proposition 2.4.14, is the following result for $SU(n)$. Recall that in the proof of Proposition 2.4.9, we defined the subalgebra

$$\mathfrak{n}_{C,j}^+ = \oplus_{i \in [j-1]} \mathfrak{n}_{\alpha_{ij}}^+ \subset \mathfrak{n}^+ \subset \mathfrak{sl}(n, \mathbb{C}),$$

which contains matrices with nonzero entries only in the superdiagonal part of the j^{th} column. We also showed that it and the subgroup $N^+_{C,j} = \exp\left(\mathfrak{n}^+_{C,j}\right)$ are abelian. We now define the subalgebra and subgroup

$$\mathfrak{n}^+_{R,i} = \oplus_{j\in[i,n]}\mathfrak{n}^+_{\alpha_{ij}},$$

containing matrices with nonzero entries only in the superdiagonal part of the i^{th} row, and note that it and the corresponding subgroup $N^+_{R,i} = \exp\left(\mathfrak{n}^+_{R,i}\right)$ are abelian, and that $N^+_{R,i} = \oplus_{\alpha\in\Phi_i\setminus\Phi_{i-1}}\mathfrak{n}^+_\alpha$.

Proposition 4.4.2. *Let $k \in LSU(n)$. Then $k \in \psi\left(C^{LK}_{\mathbb{N}\delta-\Phi^+}\right)$ if and only if there is a factorization $k = k_{n-1}\dots k_1$, where for each $i \in [n-1]$, $\mathbf{l}(k_i) \in \mathring{N}^- \cap LN^+_{R,i}$.*

Proof. We use induction on n. Assume the proposition is true for $k < n$, and that $k \in \psi\left(C^{LK}_{\mathbb{N}\delta-\Phi^+}\right) \subset LSU(n)$, so that we have $\mathbf{l}(k) \in \mathring{N}^- \cap LN^+$. For $k \in [2,n]$ and any $i,j \in [n]$, $e_{ik}e_{1j} = 0$, so

$$\begin{aligned}\left(I_n + \sum_{i\in[2,n-1]}\sum_{j\in[i,n]} a_{ij}e_{ij}\right)&\left(I_n + \sum_{j\in[2,n-1]} b_{1j}e_{1j}\right)\\ &= I_n + \sum_{i\in[2,n-1]}\sum_{j\in[i,n]} a_{ij}e_{ij} + \sum_{j\in[2,n-1]} b_{1j}e_{1j}.\end{aligned}$$

We can thus write $\mathbf{l}(k) = l'l_1$, for $l' \in \mathring{N}^- \cap L\left(N^+\setminus N^+_{R,1}\right)$ and $l_1 \in \mathring{N}^- \cap LN^+_{R,1}$, say

$$l_1 = \begin{pmatrix} 1 & \boldsymbol{L}^t \\ & I_{n-1}\end{pmatrix}$$

for some $\boldsymbol{L} \in \mathbb{C}^{n-1}$. Take the Iwasawa decomposition $l' = \mathbf{k}(l')\mathbf{a}(l')\mathbf{n}(l')$. Then $\mathbf{k}(l') = l'b$ for some block-diagonal matrix

$$b = \begin{pmatrix} 1 & \\ & M\end{pmatrix} \in H^0\left(\Delta, 0; SL(n,\mathbb{C}), B^+\right)$$

where $M \in SL(n-1,\mathbb{C})$. Thus,

$$\begin{aligned}\left(\mathbf{k}(l')\right)^{-1}k =& b^{-1}\left(l'\right)^{-1}l'l_1\mathbf{a}(k)\mathbf{u}(k)\\ =& \begin{pmatrix} 1 & \\ & M^{-1}\end{pmatrix}\begin{pmatrix} 1 & \boldsymbol{L}^t \\ & I_{n-1}\end{pmatrix}\mathbf{a}(k)\mathbf{u}(k)\\ =& \begin{pmatrix} 1 & \left(M^t\boldsymbol{L}\right)^t \\ & I_{n-1}\end{pmatrix}\begin{pmatrix} 1 & \\ & M^{-1}\end{pmatrix}\mathbf{a}(k)\mathbf{u}(k).\end{aligned}$$

Defining $k_1 := \left(\mathbf{k}(l')\right)^{-1}k$, we see that k_1 has a triangular factorization with

$$\mathbf{l}\left(k_1\right) = \mathbf{l}\left(\begin{pmatrix} 1 & \left(M^{-t}\boldsymbol{l}\right)^t \\ & I_{n-1}\end{pmatrix}\right) \in N^- \cap LN^+_{R,i},$$

and furthermore that $kk_1^{-1} \in LSU(n-1) \subset LSU(n)$. ∎

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