RESEARCH STATEMENT

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1. Background

In the last decade, analytic methods developed by a group of mathematicians including the late Oded Schramm have dramatically advanced our understanding of statistical mechanics on two-dimensional lattices. Physicists have long been interested in a wide variety of discrete, two-dimensional systems on the lattice thought to have meaningful scaling limits as the mesh size of the lattice goes to zero. Some of these systems are parameterized paths in the plane defined explicitly by a discrete random process, for example, self avoiding walks, loop-erased random walks and uniform spanning trees. Other systems, like site percolation on the triangular lattice and the 2-d Ising model (at criticality), can be used to implicitly define interfaces with nontrivial scaling limits. Numerical analysis and nonrigorous techniques developed by physicists predicted the existence of a class of these scaling limits, referred to as critical systems, with probability measures (on sets of curves) that were invariant with respect to conformal transformations in the plane in a certain well-defined sense, or *conformally invariant*.

Conformal invariance states that if the scaling limit of one of these systems is taken in two different simply connected domains, then the distribution of the scaling limits are invariant under conformal mappings from one domain to the other. Oded Schramm's idea was to utilize a classical method, called Loewner evolution, of generating curves in the upper-half plane with conformal maps. Loewner evolution is defined by the differential equation

$$\frac{d}{dt}g_t(z) = \frac{2}{g_t(z) - \psi(t)} \quad ; \quad g_0(z) = z \tag{1}$$

where the initial value, z, is a complex number in the upper half plane, $\mathbb{H}$, and $\psi : [0, \infty) \to \mathbb{R}$ is a continuous function. If one considers all initial conditions $z \in \mathbb{H}$, then, for each $t > 0$, g_t is defined on a subset $H_t \subset \mathbb{H}$ for which $g_t : H_t \to \mathbb{H}$ turns out to be a conformal map.

The amazing fact Schramm discovered was that every probability distribution of random curves, $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ exhibiting conformal invariance and a stochastic requirement called the domain Markov property is described by a solution of the Loewner differential equation with ψ given by a time-scaled Brownian motion, $\psi(t) = \sqrt{\kappa}B_t$. The models above were thought to have conformally invariant scaling limits, so there has been a campaign to rigorously prove that, for specified values of κ, the evolving sets $K_t := \mathbb{H} \setminus H_t$ given by solving (1) with $\psi(t) = \sqrt{\kappa}B_t$ were equivalent to scaling limits of interfaces described by a specific two-dimensional statistical system. The solutions to (1) are now denoted SLE_κ.

Today, it is known that the loop-erased random walk (LERW) corresponds to $\kappa = 2$, the percolation exploration process corresponds to $\kappa = 6$, uniform spanning trees correspond to

$\kappa = 8$, and critical Ising model interfaces correspond to $\kappa = 3$. It is conjectured that the self avoiding walk has a conformally invariant scaling limit corresponding to $\kappa = 8/3$.

Conformal invariance allows us to generalize the definition of SLE_κ to any simply connected domain by using the Riemann mapping theorem, a classic result that guarantees the existence of conformal maps between any two simply connected domains. Other versions of the process that have been studied allow us to take the initial value of the generating curve to be in the interior of a domain. Yet another way to generalize the process it to introduce a drift term into the driving function of the SLE_κ process, say $\psi(t) = \sqrt{\kappa}B_t + \xi(t)$. A number of recent studies including [4] and [1] have considered driving functions of this form in connection with so-called 'off-critical' scaling limits of certain 2-d statistical systems.

Perhaps the most basic drift term one can consider is the linear function $\xi(t) = \mu t$ where μ is a positive real number. I use the notation SLE_κ^μ to denote the solution to (1) corresponding to the driving function $\psi(t) = \sqrt{\kappa}B_t + \mu t$, the subject of my dissertation research. Although the curves defined by the SLE_κ^μ process do not exhibit full conformal invariance (for example, they are not scale invariant), they still retain conformal invariance with respect to a certain set of conformal maps.

2. Dissertation Research

I am interested in the solution, $g_t : H_t \to \mathbb{H}$, of the equation,

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \psi(t)} \quad ; \quad g_0(z) = z, \tag{2}$$

with $\psi(t) := \sqrt{\kappa}B_t + \mu t$. Note that each $z \in \mathbb{H}$ corresponds to a random time T_z, possibly infinity, at which $g_t(z) - \psi(t) = 0$; hence, the differential equation initiated at z is undefined at time T_z. If $T_z = \infty$ then the solution is always defined. The random set of points $z \in \mathbb{H}$ for which the solution, $g_t(z)$ is undefined by a fixed time t is exactly the set K_t referred to in the introduction, and the complement $H_t := \mathbb{H} \setminus K_t$ is the domain of the solution of the Loewner equation, g_t. It can be shown that for any driving function $\psi : [0, \infty) \to \mathbb{R}$, H_t is a well-defined set that grows with t and the map g_t is the unique conformal map from H_t to $\mathbb{H}$ with the expansion,

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{|z|^2}\right),\ z \to \infty. \tag{3}$$

In general, the family of sets, H_t, can be extremely complicated; a deep result about standard SLE_κ states that for all $\kappa > 0$, there exists a continuous generating curve $\gamma(t)$ that coincides with K_t in the sense that, for all $t > 0$, H_t is the unbounded component of $\mathbb{H} \setminus \gamma(t)$, see for example [2]. In the special case that $0 < \kappa \leq 4$, it can be shown that $K_t = \{\gamma(s)\}|_{0<s\leq t}$.

A classical result regarding solutions to stochastic differential equations named after Igor Girsanov gives us that the probability measure on paths given by the distribution of generating curves corresponding to (2) must be mutually absolutely continuous with respect to a similar measure on curves given by standard SLE_κ. The property of mutual absolute continuity of these measures allows us to carry over certain properties of standard SLE_κ to SLE_κ^μ. Foremost, we can conclude that a generating curve for SLE_κ^μ exists with probability one; moreover, we know that the generating curve will be a simple curve if $\kappa < 4$ and the fractal dimension of SLE_κ^μ will be the same as that of SLE_κ. The goal of my dissertation research is to characterize the distribution of SLE_κ^μ generating curves by identifying and proving qualitative properties of the process.

Many techniques that have already been developed for SLE_κ can also be used to derive properties of SLE_κ^μ. For example, by relating (1) to solutions of a classical stochastic differential equation called the stochastic Bessel-d equation, one can prove that for $0 < \kappa \leq 4$, γ is almost surely a simple curve and for $\kappa > 4$, γ almost surely has self-intersections. A similar analysis of (2) reveals a relationship with a process very similar to the d-Bessel process, that can be analyzed to produce the same qualitative dependence of γ on κ that we have for SLE_κ and detailed information concerning asymmetry of the new process. One can also use simple (random) bounds on $\psi(t)$, to show that for SLE_κ^μ, the real part of $\gamma(t)$, denoted $\Re\gamma(t)$, is almost surely bounded below; this is in contrast to the situation with SLE_κ in which $\Re\gamma(t)$ is recursive.

The basic observations above and further estimates that utilize differential inequalities derived from the Loewner equation reveal that the SLE_κ^μ generating curve has an imaginary component that stays close to the real line in a distributional sense and a real component that almost surely diverges to infinity. Also, by exact solution, it can be shown that the generating curve corresponding the Loewner equation driven by a linear (non-random) driving function has a strictly increasing imaginary component with an asymptotic limit. These observations lead us to conjecture that the SLE_κ^μ generating curve converges distributionally to a temporally stationary process as $t \to \infty$. We call this distributional limit $\hat{\gamma} : \mathbb{R} \to \mathbb{H}$. A large portion of my dissertation research has been dedicated to establishing the existence of this limit. My main result is a proof that $\hat{\gamma}$ is invariant with respect to both temporal shifts in the parametrization of the generating curve and horizontal translations in the spatial range, $\mathbb{H}$, of the process.

My hope is that a thorough understanding of this special generalization of SLE_κ can be used as a tool to further the contemporary use of the Loewner equation as an object for creating two-dimensional, continuous-time processes from one-dimensional driving processes. My research will also provide insight for mathematical physicists working to extend the range of scaling limits of discrete, 2-dimensional systems that SLE-type processes can be used to describe. It is currently an open question as to how effective generalizations of SLE_κ are in describing such scaling limits, and this research also brings us closer to understanding the limitations of this powerful tool.

3. Future Research

In the near future, as I finish my dissertation, I intend to further understand features of SLE_κ^μ by generalizing other methodologies that have been used developed SLE_κ literature. There are also open questions about SLE_κ^μ I am interested in that are not likely to be resolved by known methods; one of these is the question of whether the stationary limit of the SLE_κ^μ generating curve is invariant under reflections about a vertical axis. I expect that this does not hold, but even this result would yield an important feature of the generating curve.

A more ambitious future goal of mine is to identify specific examples of 2-dimensional systems that have off-critical scaling limits that can be related to SLE_κ^μ or the process stationary process given by $\hat{\gamma}$. I also need to determine whether some of the techniques I have used to control the effect of the linear drift term in the $\mathrm{SLE}_{\kappa}^{\mu}$ can be applied to more dynamic generalizations where the driving function has a drift term that is random, depends on t or both.

REFERENCES

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