Prescribing Curvature on Triangulated 3-Manifolds

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Introduction

Let $\mathcal{T} = (V, E, F, T)$ be a triangulated 3-manifold. Definitions:

- Edge Curvature: $K_e = (2\pi - \Sigma_{e<t}\beta_{e,t})\ell_e$
- Vertex Curvature: $\frac{1}{2}\Sigma_{v<e}K_e$

where $\beta_{e,t}$ is the dihedral angle about edge e in tetrahedron t, ℓ_e is the length of e, and $e < t$ denotes iteration over tetrahedra containing e.

The Pentachoron

The pentachoron has the combinatorial structure of the boundary of the 4-simplex.

- The 1-skeleton of the pentachoron is the complete graph K_5
- The pentachoron has F vector $(5, 10, 10, 5)$ (i.e. 5 vertices, 10 edges, 10 faces, 5 tetrahedra).
- As a triangulation, the pentachoron is a compact 3-manifold homeomorphic to $\mathbb{S}^3$

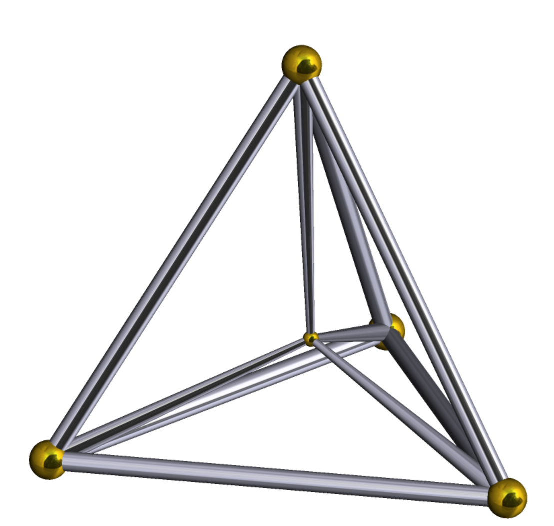

Figure: A projection of the pentachoron

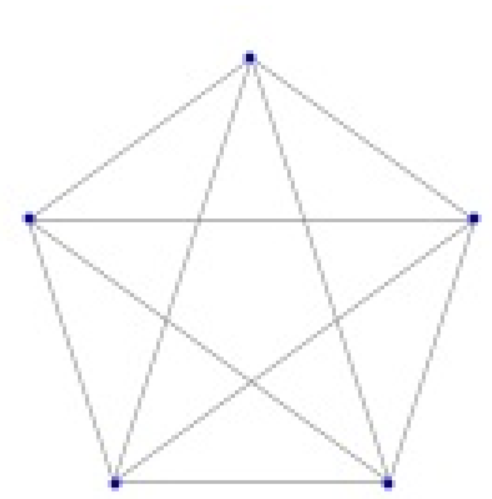

Figure: The pentachoron's 1-Skeleton K_5

Edge Curvature

Theorem: Let $\mathcal{T}(V, E, F, T)$ be a triangulated 3-manifold with negative edge curvature for all edges. Then

$$|E| > 3|V|.$$

Proof requires the following results:

- Euler-Poincare Theorem: |V|-|E|+|F|-|T|=0.
- Gaddum [1]: if $\beta_1, \ldots, \beta_6$ are the dihedral angles of a tetrahedron, then $2\pi \leq \Sigma_{i=1}^{6}\beta_i \leq 3\pi$.

Corollary: The pentachoron does not admit a metric with negative edge curvature at each edge.

Proof: The pentachoron has $|V| = 5$ and $|E| = 10$. □

Vertex Curvature

Conjecture: The pentachoron does not admit a metric with negative vertex curvature for each vertex.

- This question is more complicated than the corresponding result for edge curvature because some edges may have negative curvature.
- On the pentachoron, after normalizing by setting $\Sigma_i \ell_i = 1$, $\Sigma_v K_v$ is a nine variable function.

References

[1] J.W. Gaddum, *The Sums of the Dihedral and Trihedral Angles in a Tetrahedron*, Amer. Math. Month. (1952), **59**, no. 6, 70–371.

[2] D. W. Walkup, *The Lower Bound Conjecture for 3- and 4-manifolds*, Act Math. **125** (1970), no. 1, 75–107.

[3] Marsaglia, G, *Choosing a Point from the Surface of a Sphere*, Ann. Math. Stat. (1972) 43: 645Ð646

[4] Glickenstein, D. *Discrete conformal variations and scalar curvature on three dimensional manifolds* arXiv:0906.1560v1 [math.DG]

A Cyclic Length Metric

Definition: A *cyclic length metric* on the pentachoron is one where each vertex is incident to edges with the same set of lengths.

- A cyclic length metric imposes more symmetry, specifically:
  - Only two distinct lengths, with two edges of each length incident to each vertex
  - One type of tetrahedron up to isometry

Theorem: A cyclic length metric on the pentachoron has positive vertex curvature for each vertex.

- Simplifies $\Sigma_v K_v$ into a one variable problem after normalization.
- Minimum of $\Sigma_v K_v$ is well above 0.
- Minimum occurs at equal length metric

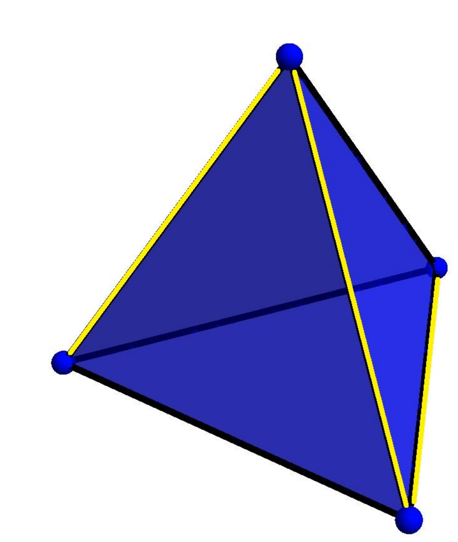

Figure: A Tetrahedron in the cyclic length metric pentachoron

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[geocam]

Simulations

We randomly sampled pentachora to find empirical bounds for edge and vertex curvatures. As the boundary of a 4-simplex, any pentachoron can be inscribed in a 3-sphere. Hence we took the 3-sphere as our sample space, picking points uniformly by using the following algorithm of Marsaglia [3]:

- Pick $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with each coordinate normally distributed.
- Normalize the vector $\mathbf{x}$ by the transformation $\mathbf{x} \mapsto \frac{x_i}{\|\mathbf{x}\|}$.
- Pick five points according to this method to create a pentachoron.

We sampled 10,000 pentachora using the above method. Below are frequency plots of all the associated edge and vertex curvatures.