

PRESCRIBING CURVATURE FOR PIECEWISE FLAT TRIANGULATED 3-MANIFOLDS

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ABSTRACT. Here we study discrete notions of curvature for piecewise flat triangulated 3-manifolds. Of particular interest is the interplay between the combinatorics of a triangulation and its curvature. We present analytic and numerical results for prescribing curvature on the pentachoron, the boundary of the 4-simplex. We will also provide simulations of random pentachora to lend empirical support for conjectured bounds.

1. BACKGROUND AND MOTIVATION

In 1961, Regge [8] proposed a notion of curvature and the Einstein-Hilbert functional for piecewise flat manifolds. The study of such discrete geometric quantities has found widespread application, and for more information regarding their connection to smooth manifolds, we refer the reader to the introductions of [1] and [4]. Here we present analytic and numerical results related to discrete curvature and Regge's Einstein-Hilbert functional on the pentachoron.

Throughout this paper, $\mathcal{T} = (V, E, F, T)$ will refer to a compact, simplicial triangulation of a 3-manifold M , where V is the set of vertices, E the edges, F the faces, and T the tetrahedra. The valence of an n -simplex σ_n will refer to the number of $(n+1)$ -simplices adjacent to σ_n ; i.e. the vertex valence of $v \in V$ will be the number of edges adjacent to v , etc.

Definition 1.1. The *Cayley-Menger determinant* $CM_3(T)$ of a tetrahedron T with vertices labeled $1, 2, 3, 4$ is defined to be

$$\det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \ell_{12}^2 & \ell_{13}^2 & \ell_{14}^2 \\ 1 & \ell_{12}^2 & 0 & \ell_{23}^2 & \ell_{24}^2 \\ 1 & \ell_{13}^2 & \ell_{23}^2 & 0 & \ell_{34}^2 \\ 1 & \ell_{14}^2 & \ell_{24}^2 & \ell_{34}^2 & 0 \end{pmatrix}$$

where ℓ_{ij} denotes the length of the edge between the vertices i and j .

We remark that the volume of T is given by the quantity $\sqrt{\frac{CM_3(T)}{288}}$.

Definition 1.2. A *metric* ℓ is an assignment of edge lengths for $\mathcal{T}$ where the Cayley-Menger determinant for each tetrahedron in $\mathcal{T}$ is

strictly greater than zero, so that no tetrahedron is degenerate. The length that a metric ℓ assigns to edge e is denoted by ℓ_e , and the total volume of the triangulation, which is given by the sum of the volumes of the tetrahedra, is denoted by $\mathcal{V}$. We will refer to $\mathcal{T}$ equipped with a metric ℓ as a *triangulated piecewise flat manifold*, which we will denote by $(\mathcal{T}, \ell)$.

Definition 1.3. The *edge curvature* of an edge e is defined to be the quantity

$$K_e = (2\pi - \sum_{t \in T} \beta_{t>e}) \ell_e,$$

where $\beta_{t>e}$ is the dihedral angle at e in the tetrahedron t , and so the sum is taken over all tetrahedron t containing e . The *vertex curvature* for a vertex v is defined to be

$$K_v = \frac{1}{2} \sum_{e>v} K_e,$$

where $e > v$ similarly denotes edges containing v , so that the sum is taken over all edges containing v .

Definition 1.4. The *Einstein-Hilbert-Regge* ($\mathcal{EHR}$) functional is the sum of the edge curvatures

$$\mathcal{EHR}(T, \ell) = \sum_e K_e.$$

We will also consider the length-normalized and volume-normalized $\mathcal{EHR}$ ($\mathcal{LEHR}$ and $\mathcal{VEHR}$, respectively) functionals, which we define to be

$$\mathcal{LEHR}(T, \ell) = \frac{\sum_e K_e}{\sum_e \ell_e}, \quad \mathcal{VEHR}(T, \ell) = \frac{\sum_e K_e}{\mathcal{V}^{1/3}},$$

where $\mathcal{V}$ denotes the volume of the piecewise flat triangulated 3-manifold.

We note that the $\mathcal{LEHR}$ and $\mathcal{VEHR}$ functionals are scale invariant with respect to edge lengths. The main question we investigate is to determine the extent to which the combinatorics of a triangulated 3-manifold determine its vertex and edge curvatures, and subsequently, the normalized $\mathcal{EHR}$ functionals. We will use analytic and numerical methods for prescribing curvature. We will also provide simulations of random pentachora to lend empirical support for conjectured bounds.

There are some results of this nature which have already been established. For example, any triangulation where each edge has valence less than or equal to five admits a metric of positive edge curvature, namely, the equal length metric, where $\ell_e = 1$ for all edges e . In this case, the dihedral angles are equal to $\arccos(1/3)$ for each edge and

hence $K_e = (2\pi - \sum_{e < t} \beta_e) \ell_e \geq (2\pi - 5 \arccos(1/3)) > 0$. In [6], Luo and Stong were able to show that any triangulation with average edge valence less than or equal to 4.5 must in fact be a triangulation of S^3 . Additionally, Walkup [9], proved the following bound: for every triangulation of a 3-manifold, $E \geq 4V - 10$. Moreover, this bound may be improved if the topological type of the manifold is known. In the following section, we provide results which were obtained mainly through considering the combinatorial structures of a (simplicial) triangulation.

2. COMBINATORIAL RESULTS

Proposition 2.1. *Consider a triangulated 3-manifold $\mathcal{T} = (V, E, F, T)$ and let n and z be the average edge valence and the average number of vertex valence, respectively. Then the following equalities hold*

$$\begin{aligned} 2|T| &= |F|, \\ 3|F| &= n|E|, \\ 2|E| &= z|V|. \end{aligned}$$

Proof. The first equality is clear since every face in a simplicial triangulation must be inside two distinct tetrahedra. The second equality follows from the fact that n is equal to the sum of the faces next to each edge divided by the number of edges, and each face is counted exactly three times, being adjacent to three edges. Establishing the last equality is entirely similar. $\square$

Theorem 2.2 (Euler-Poincaré). *For any triangulated 3-manifold, $|V| - |E| + |F| - |T| = 0$.*

Corollary 2.3. *For any triangulated 3-manifold, $|T| = |E| - |V|$.*

Proof. Substitute the first equality of Proposition 2.1 into the Euler-Poincaré Theorem. $\square$

To establish results relating to curvature, we first seek to find some bounds for the dihedral angles in a tetrahedron. The following theorem, proved in [2], is useful.

Theorem 2.4 (Gaddum). *For any tetrahedron, we have the following bound for the sum of the dihedral angles: $2\pi \leq \sum_e \beta_3 \leq 3\pi$. Moreover, equality occurs when the tetrahedron is degenerate.*

We are now ready to prove the following result:

Proposition 2.5. *If all edges have negative curvature in a triangulated 3-manifold $\mathcal{T} = (V, E, F, T)$, then $|E| > 3|V|$.*

Proof. If $K_e = (2\pi - \sum_{e < t} \beta_{e,t})\ell_e < 0$ for all e , then since edge lengths are nonnegative, it follows that $(2\pi - \sum_{e < t} \beta_{e,t}) < 0$ for all e and hence $\sum_e (2\pi - \sum_{e < t} \beta_{e,t}) < 0$. This implies that $2\pi E < \sum_e \sum_{e < t} \beta_{e,t}$, hence $2\pi E < 3\pi T$. By Corollary 2.3, $|T| = |E| - |V|$, and substituting yields $3|V| < |E|$. $\square$

We will now examine some simpler classes of triangulations. One example of a basic triangulation is the double tetrahedron, which is the triangulation obtained by identifying the boundaries of two congruent tetrahedra with each other. Any metric of the double tetrahedron must have positive edge curvatures for all edges, since $\beta_{e < t} < \pi$ for any non-degenerate tetrahedron t . Hence, it also follows that any metric on the double tetrahedron must have positive vertex curvature for each vertex. The paper [1] gives an in-depth study of the $\mathcal{EHR}$ functional on the double tetrahedron. Here, we would like our triangulation to be simplicial, so we restrict our attention to the pentachoron, which is defined to be the piecewise flat 3-manifold obtained from the boundary of the 4-simplex.

Proposition 2.6. *Every metric on the pentachoron must admit at least one edge with positive curvature.*

Proof. Suppose to the contrary that every edge e has negative curvature. Summing edge curvatures over the edges and discarding edge lengths as in the proof of 2.5, we have

$$\sum_e (2\pi - \sum_{e < t} \beta_{e,t}) = 20\pi - \sum_e \sum_{e < t} \beta_{e,t} < 0.$$

It follows from Theorem 2.4 that $10\pi \leq \sum_e \sum_t \beta_{e < t} \leq 15\pi$. Applying this bound to the inequality above, we obtain a contradiction. Hence, it must be the case that $(2\pi - \sum_t \beta_{e_0 < t})\ell_{e_0} > 0$ for at least one edge e_0 . $\square$

We conjecture that all metrics on the pentachoron admit at most one vertex with negative curvature. We first restrict our study to triangulations of the following types of convex polytopes, called cyclic polytopes, which exhibit a high degree of symmetry.

Definition 2.7. The convex hull of n distinct points on the curve $t \mapsto (t, t^2, \dots, t^4)$ is called a (4-dimensional) *cyclic polytope*.

Consider a 4-dimensional convex polytope P where each pair of distinct vertices are connected by an edge, then the combinatorics of the vertices and edges of P can be viewed as a complete graph K_P . The following theorem establishes a necessary and sufficient condition for realizing this polytope as a triangulation. In particular, these triangulations are able to be embedded in $\mathbb{R}^4$ by placing vertices appropriately on curve given in the above definition. For the proof of this theorem and more information on cyclic polytopes of an arbitrary dimension, see [5].

Theorem 2.8 (Gale’s Evenness Condition). *Let P be a cyclic polytope and K_P the associated complete graph. Then a triangulation $\mathcal{T}$ can be formed as follows: a 4-tuple V_4 of vertices in V forms a tetrahedron if and only if every two points of $V \setminus V_4$ are separated on the outer cycle of K_P by an even number of points of V_4 .*

Definition 2.9. Let $\mathcal{T}$ be a triangulation of a convex polytope. A *cyclic length metric* is a metric ℓ on $\mathcal{T}$ where the length of the edge between vertices v_i and v_j is a function of $|i - j|$ modulo $|V|$.

Note that the pentachoron is an example of a cyclic polytope. We now examine the pentachoron equipped with a cyclic length metric. Such metrics endow the pentachoron a high degree of symmetry and simplify our computations significantly.

Lemma 2.10. *For every cyclic length metric on the pentachoron, each vertex is incident to two pairs of edges with equal lengths.*

Proof. Note that the 1-skeleton of the pentachoron is K_5 , the complete graph of order five. Numbering the vertices $v_1, \dots, v_5$, it is straightforward to see that

$$|\{ |i - j| \pmod{5}, i \neq j : i, j \in \mathbb{Z}_5 \}| = 2.$$

□

Lemma 2.11. *For every cyclic length metric on the pentachoron, each tetrahedron is isometric.*

Proof. Consider K_5 with edges colored according to length as in the above lemma. Observe that this partitions the edges into Hamiltonian cycles: an outer cycle of one length, and an inner “star-shaped” cycle of the other. For any vertex v_i denote by Δ_i the tetrahedron formed by the vertices other than v_i . The graph automorphism specified by the cyclic shift $v_k \mapsto v_{k+(j-i)}$ sends Δ_i to Δ_j . □

3. NUMERICAL RESULTS

In this section we provide experimental data from Mathematica and the GEOCAM software. More information on GEOCAM can be found at the project page [3].

Theorem 3.1. *Consider the pentachoron with a cyclic length metric. Then every vertex has positive curvature.*

Proof. A tetrahedron incident to a vertex v is determined by three edges incident to v . As there are two edges of each length incident to v , there are $\binom{4}{3} = 4$ isometrically distinct ‘corners’ corresponding to choosing three edges incident to v . By the lemma above, there is only one type of tetrahedron up to isometry, call it Δ . So v is incident to each of the four “corners” exactly once. This is true for every vertex,

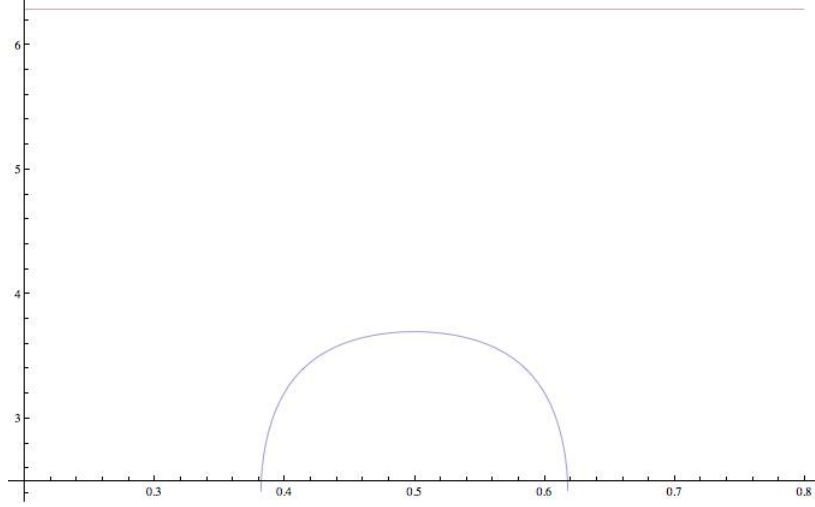


FIGURE 1. Comparison of $\ell_1(2\beta_{11} + \beta_{12}) - \ell_2(2\beta_{22} + \beta_{21})$ and 2π in Mathematica.

so K_v is constant. We now argue that K_v is in fact positive. Suppose v is incident to edges e_1, e_2, e_3, e_4 with lengths $\ell_1, \ell_2, \ell_1, \ell_2$ respectively. Then

$$\begin{aligned} K_v &\equiv \frac{1}{2}(K_{e_1} + K_{e_2} + K_{e_3} + K_{e_4}) \\ &= K_{e_1} + K_{e_2} \\ &= 2\pi(\ell_1 + \ell_2) - \ell_1(2\beta_{11} + \beta_{12}) - \ell_2(2\beta_{22} + \beta_{21}) \end{aligned}$$

where $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$ are the four distinct dihedral angles in the tetrahedron Δ . Now normalize $\ell_1 + \ell_2 = 1$. By numerical optimization in Mathematica (see Figure 1), the quantity $\ell_1(2\beta_{11} + \beta_{12}) - \ell_2(2\beta_{22} + \beta_{21})$ has a maximum at approximately 3.9. Comparing to 2π , we conclude that $K_v > 0$. $\square$

Interesting to note is that the maximum occurs at the equal length metric and that a degeneracy occurs when one of the edge lengths is set to the golden ratio $\frac{1+\sqrt{5}}{2}$. The Mathematica code that we used to perform the optimization in the above theorem can be found in the appendix of this paper.

We now consider a more general case, namely, pentachora with metrics that allow them to be inscribed inside a 3-sphere. Here, we provide numerical support for establishing bounds on these particular metrics by simulating random pentachora. This is done by picking five points on the 3-sphere at random according to the following algorithm:

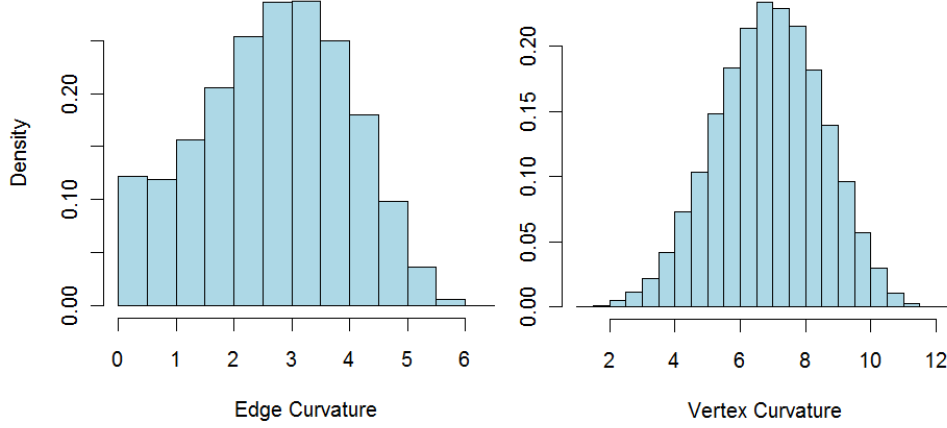


FIGURE 2. Histograms of the edge curvatures and vertex curvatures of 10,000 randomly chosen pentachora.

- (1) Pick $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with each coordinate normally distributed
- (2) Normalize the vector $\mathbf{x}$ by dividing it by its magnitude $\|\mathbf{x}\|$.

This gives a uniform distribution on the 3-sphere [7]. Now the convex hull of these randomly chosen five points forms a pentachoron. We use the GEOCAM software to choose 10,000 random pentachora in this manner and calculate their edge and vertex curvatures (see Figure 2).

Note that the vertex curvatures are roughly twice as large as the the edge curvatures. This is clear from the definition of vertex curvature and the fact that each vertex of the pentachoron is inside exactly four edges. Also interesting is that the equal length metric results in an edge curvature of approximately 2.6, which is the computed mean of edge curvatures in our random sample of pentachora. The data also clearly suggests certain bounds for the edge and vertex curvatures of these types pentachora, in particular, that the curvatures are bounded below by zero.

4. FUTURE WORK

We have shown that pentachora equipped with a cyclic length metric must have positive curvature at each vertex. More generally, we have previously shown that any metric on the pentachoron must admit at least one edge with positive curvature. Additionally, numerical simulations suggest that pentachora that can be inscribed inside a 3-sphere

must have positive curvature for each edge, and hence for each vertex as well. Thus, we make the following conjecture:

Conjecture 4.1. *Every pentachoron with a metric that is embeddable in $\mathbb{R}^4$ cannot have negative curvature at any vertex.*

Future work on establishing analytic results may benefit from using more sophisticated techniques, such as considering some type of gradient flow. Numerical simulations should focus on investigating more degenerate cases, such as sampling from points that are close to each other, or points that are colinear. We expect that negative curvature should only be able to occur at the boundary of admissible metrics, i.e. degenerate pentachora, and it would be perhaps be helpful for future studies to consider these particular cases.

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6. APPENDIX

The following is the Mathematica code used in the proof of Theorem 3.1.

```
Remove["Global`*"]
FaceAngle[x_, y_, z_] = ArcCos[(z^2 + y^2 - x^2) / (2 * y * z)];
```

```
SphericalAngle[x_, y_, z_] =
ArcCos[(Cos[x] - Cos[y] * Cos[z]) / (Sin[y] * Sin[z])];
```

```
(* This formula gives the dihedral angle of
the tetrahedron at the edge e12
(corresponding to L12) *)
```

```
DA[LL12_, LL13_, LL14_, LL23_, LL24_, LL34_] =
SphericalAngle[FaceAngle[LL34, LL13, LL14],
FaceAngle[LL24, LL12, LL14],
FaceAngle[LL23, LL13, LL12]];
```

```
F[l12_, l13_, l14_, l15_, l23_, l24_,
l25_, l34_, l35_, l45_] :=
l12(DA[l12, l13, l14, l23, l24, l34] +
DA[l12, l13, l15, l23, l25, l35] +
DA[l12, l14, l15, l24, l25, l45] ) +
l13(DA[l13, l12, l14, l23, l34, l24] +
DA[l13, l12, l15, l23, l35, l25] +
DA[l13, l14, l15, l34, l35, l45] ) +
l14 (DA[l14, l12, l13, l24, l34, l23] +
```



```

DA[114, 112, 115, 124, 145, 125]+
DA[114, 113, 115, 134, 145, 135] )+
115( DA[115, 112, 113, 125, 135, 123 ]+
DA[115, 112, 114, 125, 145, 124]+
DA[115, 113, 114, 135, 145, 134])+
123 ( DA[123, 112, 124, 113, 134, 114 ]+
DA[123, 112, 125, 113, 135, 115]+
DA[123, 124, 125, 134, 135, 145])+
124(DA[124, 112, 123, 114, 134, 113 ]+
DA[124, 112, 125, 114, 145, 115]+
DA[124, 123, 125, 134, 145, 135] )+
125(DA[125, 112, 123, 115, 135, 113 ]+
DA[125, 112, 124, 115, 145, 114]+
DA[125, 123, 124, 135, 145, 134] )+
134(DA[ 134, 113, 123, 114, 124, 112]+
DA[134, 113, 135, 114, 145, 115]+
DA[134, 123, 135, 124, 145, 125] )+
135(DA[135, 113, 123, 115, 125, 112 ]+
DA[135, 113, 134, 115, 145, 114]+
DA[135, 123, 134, 125, 145, 124] )+
145(DA[145, 114, 124, 115, 125, 112 ]+
DA[145, 114, 134, 115, 135, 113]+
DA[145, 124, 134, 125, 135, 123] )

```

```

FindMaximum[{F[a, b, c, d, e, f, g, h, i, j]
/Sqrt[a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2+i^2+j^2]
&&{a,b,c,d,e,f,g,h,i,j}
\[Element]Reals&&Det[{{0,1,1,1,1,1},
{1,0, a, b, c, d}, {1,a,0,e,f,g},{1,b,e,0,h,i},
{1,c,f,h,0,j}, {1,d,g,i,j,0}}]>0},
{a, b,c,d,e,f,g,h,i,j}]
FindMaximum::nrnum: The function value
\[NoBreak]-False\[NoBreak] is not a real number at
\[NoBreak]{a,b,c,d,e,f,g,h,i,j}\[NoBreak] =
\[NoBreak]{1.,1.,1.,1.,1.,1.,1.,1.,1.}\[NoBreak]. >>
FindMaximum[{F[a,b,c,d,e,f,g,h,i,j]/
Sqrt[a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2+i^2+j^2]
&&{a,b,c,d,e,f,g,h,i,j}\[Element]
Reals&&Det[{{0,1,1,1,1,1},{1,0,a,b,c,d},{1,a,0,e,f,g},
{1,b,e,0,h,i},{1,c,f,h,0,j},{1,d,g,i,j,0}}]>0},
{a,b,c,d,e,f,g,h,i,j}] F[1,1,1,1,1,1,1,1,1,1]
30 ArcCos[1/3]
Det[{{0, 1, 1, 1, 1, 1}, {1, 0, a, b, c, d},
{1, a, 0, e, f, g}, {1, b, e, 0, h, i},

```

```
{1, c, f, h, 0, j}, {1, d, g, i, j , 0}]]
Clear[a]
```

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