1) The group $G := \mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices $S := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Now consider the subgroup

$$S := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \mid c \equiv 0 \pmod 3 \right\}$$

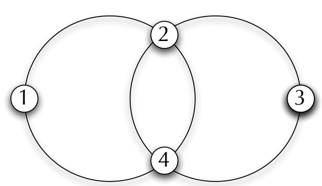
What is $[G : S]$? Determine coset representatives for S in G and a generating set for S.

2) Let $G = \mathrm{SL}_2(3)$. Write down permutations for the action of G on the nonzero vectors of $\mathrm{GF}(3)^2$. Create (without simply using `StabChain`) a stabilizer chain for the group generated by the permutations. How many Schreier generators arise? How many are really useful? GAP **Commands**: `Permutation` or `Action`.

3) (If you feel up to programming something already at this point) Implement a routine that calculates a stabilizer chain for a permutation group. (You do not need to implement the recursive version, but can do the easier "layer-by layer" version!)

4)
Consider a puzzle, as depicted on the side, which consists of two overlapping rings filled with balls. By moving the balls along either ring one can mix the balls and then try to restore the original situation.
In this problem we will consider a simplified version, that consists only of 4 balls, arranged as given in the second picture on the right.
We will describe this state by a scheme of the form:

Image: Egner,Püschel: Proc ISSAC 1998

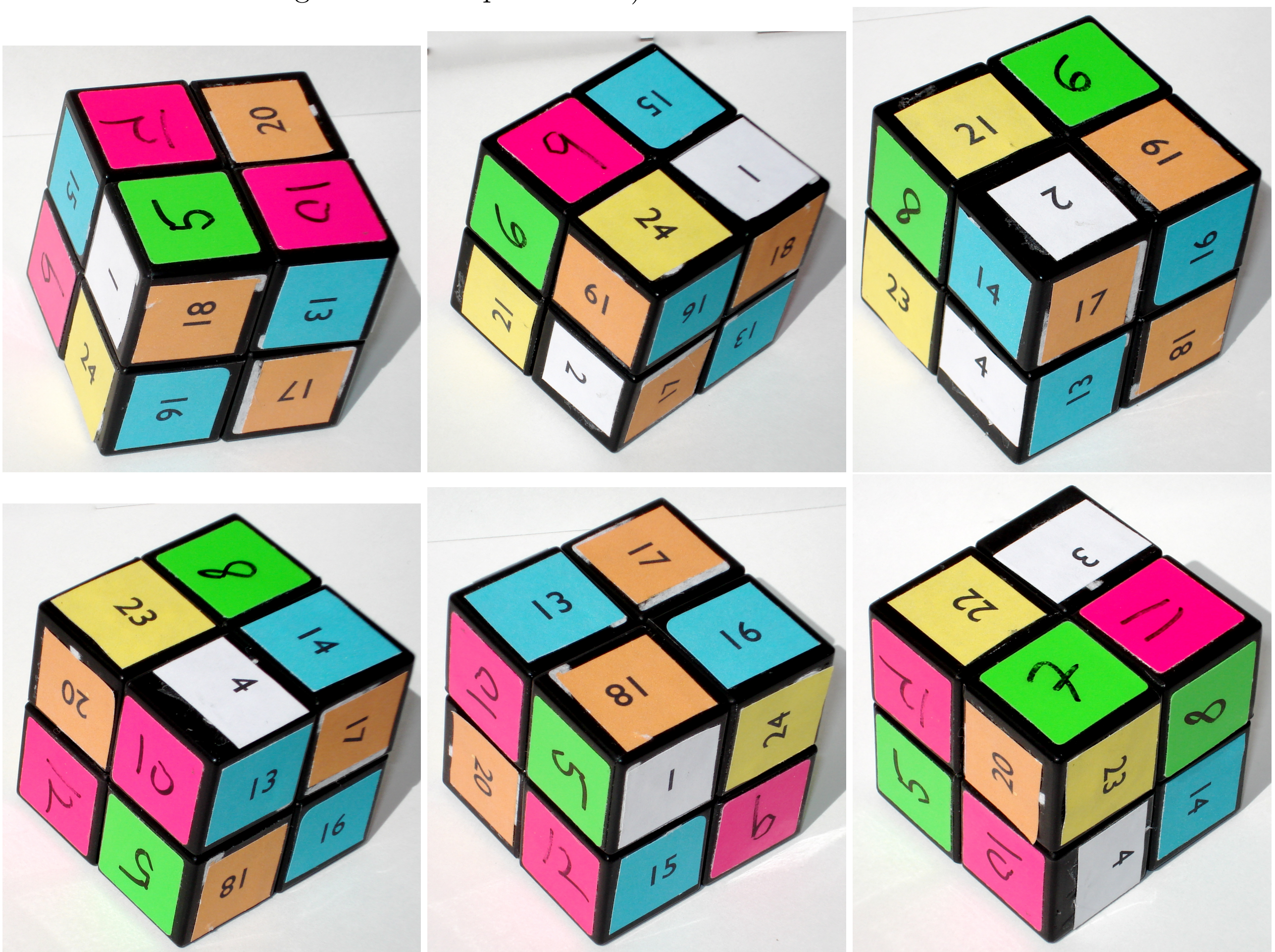


$$\begin{matrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{matrix}$$

Using permutations for both rotations, create (by hand or with GAP) a stabilizer chain for the group. Using this stabilizer chain, solve (i.e. bring back to the initial configuration) the following states, or show that it is not possible.

$$\begin{matrix} & 1 & \\ 4 & & 2 \\ & 3 & \end{matrix} \qquad\qquad \begin{matrix} & 1 & \\ 2 & & 4 \\ & 3 & \end{matrix}$$

5) The following six pictures depict all faces of a mixed-up $2 \times 2 \times 2$ Rubik's cube (with the same face labelling as the example in class):



a) Suppose you unfold the cube to a cross-shaped diagram (as on the handout), give the correct labeling of the faces.
b) Write down a permutation that describes the mixup of this cube from its original position.
c) Using GAP, determine a sequence of moves to bring the cube back to its original position.

6) (due to Gene Luks) Let H be a group of order $n-2$ acting regularly on $\{1, \ldots, n-2\}$ and let $G = \langle H, (n-1, n) \rangle$. Show that $B = [1, n-1]$ is a nonredundant base for G but the probability that a randomly chosen Schreier Generator detects that $G^{(1)} \neq 1$ is only $1/(n-1)$.

7) In this problem we want to see how the selection of base points changes the "shape" (i.e. the subsequent indices of stabilizers in each other) of a stabilizer chain.

Let G be a permutation group (or a stabilizer in the stabilizer chain) that has orbits $\Delta_1, \ldots, \Delta_n$ on the permutation domain. We want to select the next base point.

a) Let $\beta, \gamma \in \Delta_i$ for some i. Show that either choice as first base point will yield a chain with same stabilizer indices.

b) Suppose that $|\Delta_1| < |\Delta_2|$. Why is it beneficial to chose a base point from Δ_1 rather than from Δ_2?

8) Let $G = \langle \mathbf{g} \rangle$ with $\mathbf{g} = \{g_1, \ldots, g_m\}$ and H a group. We define a map $\varphi: \mathbf{g} \to H$ by prescribing images $h_i := g_i^{\varphi}$ for $g_i \in \mathbf{g}$.
a) Show that φ extends to a homomorphism on G, if and only if

$$h_{i_1} \cdot \cdots \cdot h_{i_k} = 1 \quad \text{whenever} \quad g_{i_1} \cdot \cdots \cdot g_{i_k} = 1$$

b) If G and H are permutation groups, we form $G \times H$ as group on $\Omega_G \cup \Omega_H$.

Let $S = \langle (g_i, h_i) = (g_i, g_i^{\varphi}) \mid g_i \in \mathbf{g} \rangle$. Show that φ extends to a homomorphism on G, if and only if $\mathrm{Stab}_S(\Omega_G) = \langle 1 \rangle$.

9) Let $G = \langle (2,4,6), (1,5)(2,4), (1,4,5,2)(3,6) \rangle$. Using a stabilizer chain for G (which you may get from GAP) find an element x such that

$$\begin{array}{ll} a) & (1,2)(3,4,5,6)^x = (1,4,3,6)(2,5) \\ b) & (1,2)(3,4,5,6)^x = (1,2)(3,5,6,4) \end{array}$$

or show that no such element exists.