1. Let $G = A_5 \times A_5$, where A_5 is the alternating group om five points. Find two elements in G that generate G.

   (Try the function `Random`.)

2. Show that $SL(2,4) \cong A_5$.

   (Consider the action of $SL(2,4)$ on the right cosets of the normalizer of a Sylow 2-subgroup.)

3. Study the groups returned by the GAP function `ShuffleGroup`, for small values of the argument n.

```
ShuffleGroup:= function( n )
    local outShuffle, inShuffle;

    outShuffle:= [];
    outShuffle{[1,3..2*n-1]} := [1..n];
    outShuffle{[2,4..2*n]}   := [n+1..2*n];

    inShuffle:= [];
    inShuffle{[1,3..2*n-1]} := [n+1..2*n];
    inShuffle{[2,4..2*n]}   := [1..n];

    return Group( PermList( outShuffle ),
                  PermList( inShuffle ) );
end;
```

   Possible questions (in alphabetical order): abelian, abelian invariants, number of conjugacy classes, order compared with that of $Sym(2n)$, perfect, simple, solvable? Try also `DisplayCompositionSeries`.

4. Define the following group of symmetries of a cube.

```
     2 ___________  1             _ y
      /|         /|              /|
     / |        / |             /
  3 /__|_______/ 4|          -/----> x
    |  |       |  |
    |8 |_______|__| 7
    | /        | /
    |/         |/
  5 /__________/ 6
```

```
gap> x:= (1,4,6,7)(2,3,5,8);;
gap> y:= (1,7,8,2)(4,6,5,3);;
gap> g:= Group( x, y );;
```

The action on the edges can be constructed as the action of `g` on the orbit of the set `[ 1, 2 ]` via `OnSets`.

Consider the actions of `g` on the faces, face diagonals, diagonals, pairs of opposite edges, and pairs of opposite faces, with appropriate action functions in GAP.

Compute orbit length, point stbilizer, and the order of the image of the action homomorphism.

What does the action on the orbit of `[ 1, 3, 6, 8 ]` via `OnSets` describe?

5. Consider the cube in the above exercise, embedded into the 3-dimensional space such that the coordinates of the vertices are $(\pm 1, \pm 1, \pm 1)$. Then the symmetries can be represented by 3×3 matrices, via

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, y \mapsto \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Show that this map really defines a group isomorphism

(a) using `GroupHomomorphismByImages`.

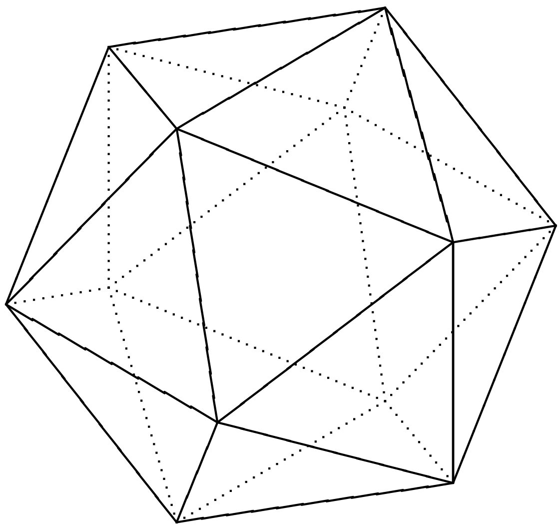
(b) via the action of the matrix group on the orbit of the row vector `[ 1, 1, 1 ]`.

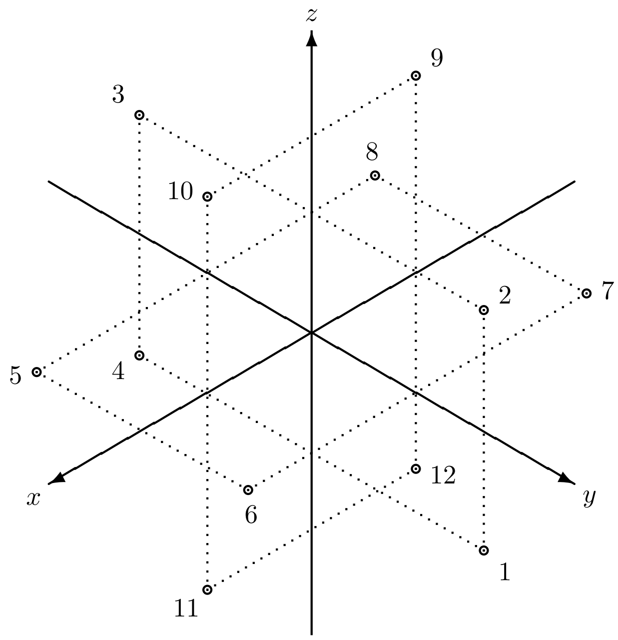
Construct the various actions mentioned in the above exercise (on vertices, edges, faces, etc.) as actions of the matrix group on orbits of suitable sets of row vectors.

6. Let $b = (1+\sqrt{5})/2$. (There is a GAP function `Sqrt`.) The twelve points with the coordinates $(0, \pm b, \pm 1)$, $(\pm b, \pm 1, 0)$, $(\pm 1, 0, \pm b)$ are the vertices of a regular icosahedron.

   The following matrices are symmetries of this icosahedron.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, 1/2 \cdot \begin{bmatrix} b-1 & 1 & -b \\ -1 & b & b-1 \\ b & b-1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$





   Consider the action of the group G generated by these matrices on the vertices, on the edges, on the faces. (Compute the orbit of the row vector describing one vertex by multiplication from the right. An edge can be represented by its mid point or by the sum of the vectors of its two end points. A face can be represented by its mid point or by the sum of the vectors of its three vertices.)

   What does the set

```
[ [ -1, -1, -1 ], [ -1, 1, 1 ], [ 1, -1, 1 ], [ 1, 1, -1 ] ]
```

   describe? Compute its stabilizer in G and the action of G on its G-orbit. Consider also the action of the subgroup generated by the first two matrices on this orbit.

7. Compute, for small values of n, the number of fixed point free permutations on n points.

   Compute the proportion `x(n)` of these permutations in $Sym(n)$. Compute also the first decimal digits of `x(n)` and `1/x(n)`, using `Int( 10^10 * x(n) )` and `Int( 10^10 / x(n) )`.

   Do you have a conjecture what $\lim_{n\to\infty}$ `x(n)` is?

   (For very small values of n, one can run over all elements of $Sym(n)$; for larger values of n, one can run over conjugacy class representatives in $Sym(n)$; what can one do for large n?)