Some GAP functions for dealing with finitely presented groups

- `AbelianInvariants`
- `AbelianInvariantsSubgroupFpGroup`
- `IsomorphismPermGroup`
- `IsomorphismFpGroup`
- `GQuotients`
- `LowIndexSubgroupsFpGroup`
- `PQuotient`
- `EpimorphismNilpotentQuotient`

Note that GAP distinguishes between **presentations** and **finitely presented groups**.

Examples for Finitely Presented Groups

1. Consider the following GAP function.

```
SomeGroupAsFpGroup:= function( n )
    local F, gens, rels, i, j;

    # There are n-1 generators.
    F:= FreeGroup( n-1 );
    gens:= GeneratorsOfGroup( F );

    # The generators shall have order (at most) two.
    rels:= List( gens, s -> s^2 );

    # The product of adjacent generators shall have order (at most)
```

```
    Append( rels, List( [ 1 .. n-2 ], i -> ( gens[i]*gens[i+1] )^3 )

    # Non-adjacent generators shall commute.
    for i in [ 1 .. n-1 ] do
      for j in [ i+2 .. n-1 ] do
        Add( rels, ( gens[i] * gens[j] )^2 );
      od;
    od;

    # This defines the group.
    return F / rels;
end;
```

Ask GAP to compute the order of the groups for small values of n.

Theoretical question: Can you find –without using GAP– an epimorphism onto the symmetric group on n points, using suitable involutions as images of the generators?

2. The dihedral group of order $2n$ can be written as

$$\langle a, b | a^2, b^2, (ab)^n \rangle.$$

Show –without using GAP– that this group is isomorphic with

$$\langle c, d | c^2, d^n, cd = d^{-1}c \rangle.$$

(Construct epimorphisms in both directions.)

3. Use GAP to show that the group

$$\langle a, b | a^2, b^2 \rangle$$

is infinite.

(Try `Size` and `IsFinite`; try `LowIndexSubgroupsFpGroup` and apply `AbelianInva` to the subgroups that are returned.)

Use the previous exercise to prove the claim without GAP.

4. Find out whether `FibonacciGroup( n )`, for $1 \leq n \leq 6$, is finite.

   (You can use the same idea as in the previous exercise.)

5. Consider the groups

   $$\langle a, b | a^2, b^3, (ab)^n \rangle,$$

   for $2 \leq n \leq 7$.

   Which are finite, which are not? Does the idea used in the previous exercises work?

6. Compute the abelian invariants of `FibonacciGroup(5)` using GAP.

   Write down the matrix of abelianized relators and compute its Smith normal form both using GAP and by hand.

   (GAP functions `RelatorsOfFpGroup` and `SmithNormalFormIntegerMat`)