Classification of Groups with Strong Symmetric Genus up to Twenty-Five

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Abstract

The strong symmetric genus of a finite group G is the minimum genus of a compact Riemann surface on which G acts as a group of automorphisms preserving orientation. May and Zimmerman have published papers detailing the classification of all groups with strong symmetric genus zero through four. Using the computer algebra system GAP, we have extended these classifications to all groups of strong symmetric genus up to twenty-five. In this paper, we give these classifications and present the theory and procedures behind our methods.

Contents

1 Introduction

The strong symmetric genus is a topological concept that can be explored using computational group theory. Suppose G is a finite group. The strong symmetric genus of G is the minimum genus of a compact Riemann surface on which G acts faithfully as a group of automorphisms while preserving orientation.

The strong symmetric genus has been analyzed extensively by May and Zimmermann in several previous papers. They have classified all groups with strong symmetric genus up to four. In [MZ00], May and Zimmermann classified all groups with strong symmetric genus zero, one, two, and three. They later classified all groups with strong symmetric genus four in [MZ05]. In the early stages of our research, these papers proved to be useful to us as they provided a means to check the initial results of our code.

Our research extends the classifications made by May and Zimmermann, using the computer algebra system GAP as our main resource. It is known that there are infinitely many groups of strong symmetric genus zero and one due to the inherent structural difference in the Riemann surfaces that the groups act on. However, there is a finite list of groups for each strong symmetric genus greater than or equal to two. Our code accounts for groups with strong symmetric genus zero and one, so we begin our classification starting with groups having strong symmetric genus two.

Using the approach developed by Breuer in [Bre00] as our foundation and the results generated by May and Zimmerman in [MZ00] and [MZ05] as our guide, we utilize GAP to create functions that ultimately output the complete list of groups with the prescribed strong symmetric genus. However, as we ascend to groups with higher strong symmetric genera, we encounter several restrictions in our code. The primary obstacle is the processing time of our functions. In order to bypass this difficulty, we incorporate various functions into our overall code that are designed specifically to optimize the efficiency of our procedure. Due to these modifications, we are able to gather data on groups of higher strong symmetric genera.

We begin by reviewing important definitions and results dealing with strong symmetric genera in Section 2. We then include in Section 3 a schematic of the functions in our code along with descriptions of each function. Data for groups of each strong symmetric genus up to twenty-four is given in Section 4. In the final sections, we present our conclusions and some open problems. An Appendix is included offering a proof of one of the theorems.

2 Review of Theory

To present our results in a thorough manner, it is necessary to provide terms and results from the theory of Fuchsian groups and Riemann surfaces. The following sections will cover the main definitions and theorems required, though omitting most proofs. We direct the reader to

[Bea01, Bre00, JS87, Kat92] for a much more in-depth study.

2.1 Hyperbolic Plane and Fuchsian Groups

Let $\mathbb{H}$ be the upper half-plane, $\mathbb{H} = \{x + iy : x, y \in \mathbb{R},\, y > 0\}$. Equipped with the metric

$$\mathrm{d}s = \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{y}$$

it becomes a model of the *hyperbolic* or *Lobachevski* plane and the group $\mathrm{PSL}(2, \mathbb{R})$ acts on $\mathbb{H}$ by isometries with respect to this metric [Kat92, p.1]. Recall that a subgroup Γ of a topological group G is said to be *discrete* if its subspace topology is the discrete topology. A *Fuchsian group* is then defined to be a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. A common example of a Fuchsian group is the modular group $\mathrm{PSL}(2, \mathbb{Z})$, which consists of the fractional linear transformations $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Additionally, a Fuchsian group that is torsion-free is called a *Fuchsian surface group.*

The following theorem allows us to represent certain Fuchsian groups in terms of a list of numbers.

Theorem 1 (Signature of a Fuchsian Group)**.** *If Γ is a Fuchsian group with compact orbit space $\mathbb{H}/\Gamma$ of genus g then there are elements $a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c_1, c_2, \ldots, c_r$ in $\mathrm{PSL}(2, \mathbb{R})$ such that the following hold.*

1. *We have $\Gamma = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c_1, c_2, \ldots..., c_r\rangle$.*
2. *Defining relations for Γ are given by*

$$c_1^{m_1}, c_2^{m_2}, \ldots, c_r^{m_r}, \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{r} c_j,$$

   where the m_i are integers with $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r$.

3. *Each nonidentity element of finite order in Γ lies in a unique conjugate of $\langle c_i\rangle$ for suitable i.*

Proof. See [Leh64, p.227 and p.234] or [Sah69, Appendix]. □

We call $(g;\, m_1, m_2, \ldots, m_r)$ the *signature* of Γ. The integer g is called the *orbit genus* of Γ, and the m_i's are called the *periods* of Γ [Bre00, p.8]. For a Fuchsian group Γ, we will sometimes denote the orbit genus by $g(\Gamma)$.

For a Fuchsian group as in Theorem 1, the numbers g, r, and $m_1, \ldots, m_r$ are uniquely determined. However, it is not always the case that given any orbit genus g and list of m_i's that the signature $(g;\, m_1, m_2, \ldots, m_r)$ defines a Fuchsian group. The following theorem allows us to determine precisely when this is the case.

Theorem 2 (Poincaré's Theorem). *If $g \geq 0$, $r \geq 0$, $m_i \geq 2$ $(1 \leq i \leq r)$ are integers and if*

$$(2g-2) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) > 0,$$

then there exists a Fuchsian group with signature $(g; m_1, m_2, \ldots, m_r)$.

Proof. For detailed sketches of the proof, see [JS87, pp.259-260] or [Kat92, pp.92-95] . For a rigorous proof, see [Mas71]. □

2.2 Compact Riemann Surfaces

There are several different ways in which one may define a Riemann surface. In analysis, it may be defined as a surface with a complex analytic structure. For our purposes, we may construct a Riemann surface as the orbit space of a suitable group action [Bea01, p.1]. In particular, we will only consider the action of certain Fuchsian groups on the upper half plane $\mathbb{H}$ which force the Riemann surfaces in consideration to be compact. Now, each compact Riemann surface X can be assigned a unique nonnegative integer g called the *genus* of X, which we will denote by $g(X)$. The genus is a topological invariant, and a compact Riemann surface with genus g is homeomorphic to the sphere with g handles.

Recall that a Fuchsian surface group is a torsion-free Fuchsian group. The next result, [Bre00, Theorem 3.11] shows that compact Riemann surfaces of genus ≥ 2 are in correspondence with Fuchsian surface groups.

Theorem 3. *A finite group G acts as a group of automorphisms of some compact Riemann surface of genus $g \geq 2$, if and only if G is isomorphic to Γ/K where Γ is a Fuchsian group with compact orbit space, and K is a Fuchsian surface group with orbit genus g that is a normal subgroup of Γ.*

Proof. See [JS87, 5.7.5]. □

One result from the theory of Riemann surfaces which we use extensively is the Riemann-Hurwitz formula. This formula relates the periods and orbit genus of a Fuchsian group Γ to the genus of the Riemann surface X which is the quotient of $\mathbb{H}$ by a Fuchsian surface group that is normal in Γ.

Theorem 4 (Riemann-Hurwitz Formula). *Suppose Γ is a Fuchsian group with signature $(g; m_1, m_2, \ldots, m_r)$ and K is a torsion-free normal subgroup of Γ of finite index. Let G be a finite group with $G \cong \Gamma/K$ and X be the Riemann surface given by $\mathbb{H}/K$. Then the genus of X and the orbit genus of Γ are related by the equation*

$$g(X) - 1 = |G|\,(g(\Gamma) - 1) + \frac{|G|}{2} \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right).$$

Proof. See [Bre00, Chapter 1.4]. □

The usefulness of this formula is as follows: Suppose we are given groups G and Γ. Then, the genus of possible Riemann surfaces that are given by quotients of $\mathbb{H}$ by normal subgroups of Γ with factor group G is fixed [Bre00, p.14]. Also, the possible signatures of Γ are limited by the Riemann-Hurwitz formula to a finite set given a strong symmetric genus g and the order of G.

2.3 Surface Kernel Epimorphisms

Let Γ be a Fuchsian group and G be a finite group. Suppose there is an epimorphism $\Phi : \Gamma \to G$. We say that Φ is a *surface kernel epimorphism* if the kernel of Φ is a Fuchsian surface group. The following lemma allows us to bound the order of such a finite group G supposing the existence of a surface kernel epimorphism from a Fuchsian group Γ to G. We include the proof of the lemma below to illustrate the combinatorial nature of finding surface kernel epimorphisms.

Lemma 1. *Suppose there exists a surface kernel epimorphism Φ from the group Γ with signature $(g_0; m_1, m_2, \ldots, m_r)$ to a finite group G such that the kernel of Φ is torsion-free and $g(\ker \Phi) = g \geq 2$. Then we have the following:*

(a) If $g_0 > 0$ or $r \geq 5$, then $|G| \leq 4(g-1)$.

(b) If $r = 4$, then $|G| \leq 12(g-1)$.

(c) If $|G| \geq 24(g-1)$, then $g_0 = 0$ and $r = 3$, and Γ has one of the following signatures in Table 1 with the corresponding group size for G.

$(g_0; m_1, m_2, m_3)$	$\lvert G\rvert$
$(0; 2, 3, 7)$	$84(g-1)$
$(0; 2, 3, 8)$	$48(g-1)$
$(0; 2, 4, 5)$	$40(g-1)$
$(0; 2, 3, 9)$	$36(g-1)$
$(0; 2, 3, 10)$	$30(g-1)$
$(0; 2, 3, 11)$	$\frac{132}{5}(g-1)$
$(0; 2, 3, 12)$	$24(g-1)$
$(0; 2, 4, 6)$	$24(g-1)$
$(0; 3, 3, 4)$	$24(g-1)$

Table 1: Signatures for Large Group Orders

Proof. Using the Riemann-Hurwitz formula,

$$g - 1 = |G|(g_0 - 1) + \frac{|G|}{2} \sum_{i=1}^{r} \Big(1 - \frac{1}{m_i}\Big),$$

we see that if $g_0 \geq 2$, then $g - 1 \geq |G| + \frac{|G|}{2} \sum_{i=1}^{r} \big(1 - \frac{1}{m_i}\big) \geq |G|$. Now if $g_0 = 1$, then $g - 1 = \frac{|G|}{2} \sum_{i=1}^{r} \big(1 - \frac{1}{m_i}\big) \geq \frac{|G|}{4}$. Now assume $g_0 = 0$ and $r \geq 5$. Then $g - 1 \geq -|G| + \frac{|G|}{2} \cdot 5 \cdot \frac{1}{2} = \frac{|G|}{4}$. This proves part (a).

If $g_0 = 0$ and $r = 4$, then we see that at least one of the m_i must be greater than 2. If all the m_i's were 2, then we have $g - 1 = -|G| + \frac{|G|}{2} \sum_{i=1}^{4} \big(1 - \frac{1}{2}\big) = 0$ which contradicts our assumption that $g \geq 2$. Therefore, $g - 1 \geq -|G| + \frac{|G|}{2} \cdot \big(3 \cdot \frac{1}{2} + \frac{2}{3}\big) = \frac{|G|}{12}$, which proves part (b).

Now suppose $|G| \geq 24(g-1)$. Then from the first two parts, we can easily deduce that $g_0 = 0$ and $r = 3$ and $2 \leq m_1 \leq m_2 \leq m_3$. If $m_1 \geq 3$, then we see that $m_3 > 3$ because otherwise the right hand side of the Riemann-Hurwitz formula will be zero which contradicts our assumption that $g \geq 2$. The first signature that satisfies this condition is $(0; 3, 3, 4)$ according to the lexicographical ordering of the m_i's. Using the Riemann-Hurwitz formula for that signature, we obtain that $|G| = 24(g-1)$. Hence, all other possible signatures must have $m_1 = 2$. This implies that $m_2 \geq 3$ because the right hand side of the Riemann-Hurwitz formula must be nonnegative.

Now suppose $m_1 = 2$ and $m_2 = 3$. From manipulating the Riemann-Hurwitz formula, we conclude that only $7 \leq m_3 \leq 12$ satisfy the formula for $|G| \geq 24(g-1)$. If $m_1 = 2$ and $m_2 = 4$, then again from analyzing the Riemann-Hurwitz formula, we see that only $5 \leq m_3 \leq 6$ satisfy the requirement for $|G|$. Finally, we check to see if any signatures with $m_1 = 2$ and $m_2 \geq 5$ can occur. However, if $m_2 \geq 5$, then $m_3 \geq 5$ and this implies that $g - 1 \geq -|G| + \frac{|G|}{2}(\frac{1}{2} + \frac{4}{5} + \frac{4}{5}) = \frac{|G|}{20}$ which contradicts our assumption on $|G|$. This proves part (c).

□

3 Derivation of Functions

In the Appendix, Figure 1 is a schematic that illustrates how our functions interact with one another. The functions themselves are enclosed in boxes with arrows denoting the flow of information through the functions. The boxes are then partitioned into cells by dotted lines according to the functions' roles in compiling the end function, `mainFunction`. The "Generation Of Signatures" cell contains functions that determine the possible orders and signatures for groups. The "Existence Of Surface Kernel Epimorphisms" cell contains functions that check to see if a surface kernel epimorphism exists for certain signatures of Fuchsian groups. The "Checks For Groups Of Strong Symmetric Genus Zero And One" cell contains functions that eliminate groups with strong symmetric genus zero or one from a list of groups. All outputs from each cell are then sent to `mainFunction` for processing.

3.1 Basic Description

The basic idea of the code is as follows: first the user chooses a genus g. Since the code relies on previous data, the g chosen must be one greater than the previous g for which data has been collected. Next, using various rules which will be described below, the program computes all possible group orders associated with g. Using this list of group orders and g, we generate all possible signatures that satisfy the Riemann-Hurwitz formula. Then, using GAP library functions (some of which we modified), the program runs through the list of signatures, checking whether or not there exists a surface kernel epimorphism from the Fuchsian group represented by each signature onto a group of one of the predetermined group orders. Collecting the list of target groups satisfying the above condition, we check back to see if any of these groups previously appeared for a smaller genus g. Finally, we output this list and continue to a higher genus.

The GAP library of small groups contains representatives of isomorphism types of all groups of order up to 2000 (except 1024). Thus, this whole process is *a priori* finite. There are only finitely many group orders that may occur, so the set of signatures that can occur is finite. Therefore, one has to check only finitely many possibilities for establishing or disproving the existence of a surface kernel epimorphism. The only thing to do is to write the code for this task and to speed it up via suitable shortcuts, and this is where the algorithmic and mathematical ideas are needed.

3.2 Generation of Signatures

The first task is performed by the functions labeled in green boxes in Figure 1. These functions generate two lists. One list gives all possible group orders that can occur for the given genus and the other list gives all possible signatures for which Fuchsian groups can occur.

This section of our code is separated into two stages, in that we distinguish between "large" and "small" group orders. This distinction arises in Lemma 1, where we are given certain bounds and signatures for group orders greater than or equal to $24(g-1)$, and is a computational shortcut by which we can eliminate unnecessary group orders more efficiently. Thus, we denote groups orders greater than or equal to $24(g-1)$ as "large" and group orders smaller than $24(g-1)$ as "small", and deal with these group orders separately.

We first call the function `listOfSignaturesForSmallGroupOrders`, which in turn calls two other functions. The first is `possibleSmallGroupOrders`, which contains a series of conditions on groups of order less than $24(g-1)$. The first step is to create a list of numbers from 1 to $24(g-1)$ representing group orders which might be possible. We then proceed to throw out the impossible group orders. It is known that if $|G| = p$ is prime, then $p \leq g$ or $p \in \{g+1, 2g+1\}$. Thus, we can throw out all prime numbers that do not satisfy these criteria. Furthermore, for any group G of composite order, $|G|$ must not be divisible by any of those primes that we just eliminated. Finally, as every cyclic group has strong symmetric genus zero, we can exclude any group that is cyclic. Since there is a cyclic group of every order, we remove every number on our list for which the library of small groups indicates that there is only one group of that order, as that group must then be cyclic.

One further check incorporated into `possibleSmallGroupOrders` is performed only when $g \geq 8$. This check throws out group orders which have been completely classified into strong symmetric genera zero through seven. We are allowed to do this because in dealing with strong symmetric genus we care only about the smallest genus for which each group occurs. Once the list of groups of strong symmetric genus zero through seven is complete, there exist certain group orders for which all groups have already been classified, so that these group orders need not be considered for higher genera. We perform this check simply for purposes of optimization when looking at large strong symmetric genera and the selection of $g \geq 8$ is arbitrary.

Once all the possible "small" group orders have been calculated for a given g, the next task is to generate a list of all the signatures that satisfy the Riemann-Hurwitz formula for any group of each possible order. This is done by the function `listofSignaturesForSmallGroupOrders`, which calls `validSignatures` to perform the appropriate checks. Inside `validSignatures`, bounds on the number of periods that can occur in any signature are determined by the Riemann-Hurwitz formula and then passed into the function `generateAllPeriods`.

Since the code will later attempt to find an epimorphism from Γ, represented by some signature, onto G, we can establish certain restrictions on the values which can occur in any period. Each period in a given signature represents the order of one of the finite elements in the associated Fuchsian group Γ, and since a surface kernel epimorphism must preserve the orders of the generators, each period in the signature must divide the order of G. Thus, for each group order passed into `generateAllPeriods`, the function creates a list of all the factors of $|G|$ with which it creates lists of periods whose lengths are within the bounds established by `validSignatures`.

Once these lists of periods have been created in `generateAllPeriods`, they are passed back into `validSignatures` for further checks. By Theorem 4, for any pair of groups Γ and G that are candidates for a surface kernel epimorphism, the Riemann-Hurwitz formula must hold. Thus, for every list of periods generated and every orbit genus g_0 within the bounds, we employ the function `riemannHurwitzSum` to check that this equality holds. If it does, then the list of periods is appended to the appropriate orbit genus g_0 and sent back to `mainFunctions` for the next stage of checks.

We now proceed to the second stage of generating signatures: the generation of possible group orders and signatures for "large" group orders. This is much easier, due to Lemma 1, which provides a list of seven possible group orders for $|G| \geq 24(g-1)$. By this lemma, there is also a list of exactly nine possible signatures, each one corresponding to one of the group orders. Thus, the function `possibleLargeGroupOrders` generates a list of group orders based on calculations given by Lemma 1. `listofSignaturesForLargeGroupOrders` then creates a list of the possible signatures given in Table 1. These signatures are already known to be valid, as they were originally determined using the Riemann-Hurwitz formula, and are thus ready to be passed back to `mainFunctions`.

3.3 Existence of Surface Kernel Epimorphisms

The section of Figure 1 labeled "Existence of Surface Kernel Epimorphisms" consists of functions which will use the previously generated list of possible groups and signatures and determine whether there exists a surface kernel epimorphism from the Fuchsian group Γ represented by a given signature onto a group G.

For each signature given by the earlier functions `listOfSignaturesForSmallGroupOrders` and `listOfSignaturesForLargeGroupOrders`, the function `listOfEpiGroupsOfGamma` takes in the signature, strong symmetric genus, and a list of all the groups which have occurred for smaller strong symmetric genus. It then runs through the list of all possible groups, calling a series of functions which either eliminate a given group or find an epimorphism onto that group from the Fuchsian group Γ represented by the signature.

The first function called by `listOfEpiGroupsOfGamma` is `SigToGamma`, which produces a finitely presented group that is isomorphic to the Fuchsian group Γ represented by the signature. We then enter a series of tests which eliminate certain classes of groups from consideration. The first test that all groups undergo is in `abelianInvariantsCondition`.

The purpose of `abelianInvariantsCondition` is to serve as an efficient way to determine whether or not epimorphisms from a Fuchsian group Γ onto a finite group G can exist. We would like to have such a check to immediately rule out the possibility of such an epimorphism existing from a particular Γ onto a particular G, without needing to call more costly functions.

The function takes a group Γ and a finite group G and returns a boolean value. Whenever `abelianInvariantsCondition` returns false there can be no epimorphism from Γ to G, and so we can immediately eliminate G as a possible epimorphic image of Γ.

Let Γ' and G' be the derived subgroups of Γ and G respectively. If there is an epimorphism ϕ: $\Gamma \to G$ then we can show that there is an epimorphism ϕ': $\Gamma/\Gamma' \to G/G'$. By Theorem 5 below, this implies that for each distinct prime p_i dividing $|G/G'|$ there exists an epimorphism ϕ_{p_i}: $\text{Syl}_{p_i}(H) \oplus \mathbb{Z}^r \to \text{Syl}_{p_i}(G/G')$. This implies that we can consider G/G' as a quotient group of Γ/Γ'. Now we consider the abelian invariants of Γ and G and some prime p_i dividing $|G/G'|$. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the abelian invariants of Γ where γ_1 has the highest power of p_i as a divisor, γ_2 has the second highest power of p_i as a divisor, and so forth. Let $g_1, g_2, \ldots, g_m$ be the abelian invariants of G sorted in the same way. Then the power of p_i dividing γ_j is larger than the power of p_i dividing g_j for all $1 \leq j \leq n$ (when $\gamma_j = 0$ we consider it to have a higher power of p_i as a divisor than g_j). `abelianInvariantsCondition` checks this condition for all primes dividing $|G/G'|$. If this condition fails for any divisor of $|G/G'|$, then we know that G cannot be an epimorphic image of Γ, so the function `abelianInvariantsCondition` returns false.

Theorem 5. *If Γ and G are groups and $\Phi : \Gamma \to G$ is an epimorphism, then there is an epimorphism $\tilde{\Phi} : \Gamma/\Gamma' \to G/G'$*

Proof. For $\gamma \in \Gamma$, define $\tilde{\Phi}(\gamma \cdot \Gamma') = \Phi(\gamma) \cdot G'$. This is well-defined because $\ker(\Phi) \supseteq \Gamma'$. □

Theorem 6. *If Γ is a finitely-generated, abelian group and G is a finite abelian group, then $\Gamma \cong \mathbb{Z}^r \oplus H$ for some $r \geq 0$ and a finite group H, and there is an epimorphism $\Phi : \Gamma \to G$ if and only if, for each prime divisor p of $|G|$, there is an epimorphism $\Phi_p : \mathbb{Z}^r \oplus H_p \to G_p$, where H_p and G_p are the unique Sylow p-subgroups of H and G, respectively.*

Proof. See Appendix. □

Theorem 7. *If $\Gamma = \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{p^{N_i}}$ and $G = \bigoplus_{i=1}^{k} \mathbb{Z}_{p^{n_i}}$ with $0 \leq N_1 \leq N_2 \leq \ldots \leq N_k$ and $0 \leq n_1 \leq n_2 \leq \ldots n_k$, then there is an epimorphism $\phi : \Gamma \to G$ if and only if $r + N_k \geq n_k$, $r + N_k + N_{k-1} \geq n_k + n_{k-1}$, $\ldots$, $r + N_k + N_{k-1} + \ldots + N_1 \geq n_k + n_{k-1} + \ldots + n_1$.*

Proof. We can map successive generators of the highest order in Γ to generators of the highest order in G. This is clearly a homomorphism. In addition, by construction it is surjective since the generators of G appear as images of generators of Γ.

Now suppose the condition is not satisfied. Then Γ does not have enough generators of order at least p^k, for some k, in order to map onto the elements of order at least p^k in G.

□

After eliminating groups based on these conditions on their abelian invariants, we next turn to eliminating those groups from consideration which have occurred for previously calculated strong symmetric genus. Groups for genus g are then run through a list of all groups that have strong symmetric genus 2 to $g-1$ and eliminated. Next, groups of strong symmetric genus 0 are eliminated through the function `GroupChecks`, which checks for cyclic groups, dihedral groups, A_4, A_5, S_4. Note that we have not yet checked for groups of strong symmetric genus 1; this check occurs later in the code in the function `genusOneCheck`.

In the case of abelian G, the function `abelianInvariantsCondition` has guaranteed the existence of an epimorphism $\phi : \Gamma/\Gamma' \to G$. From this epimorphism, one can construct an epimorphism $\Phi : \Gamma \to G$ by composition of ϕ with the natural homomorphism which maps $\Gamma \to \Gamma/\Gamma'$. Every abelian group which has passed both `abelianInvariantsCondition` and `groupChecks` now enters the function `largeAbelianGroupsCheck` which contains a set of four additional criteria that are necessary and sufficient for the existence of a surface kernel epimorphism. For these criteria, observe the following:

Theorem 8. *Let Γ be a Fuchsian group as defined above and let $M = \operatorname{lcm}(m_1, m_2, \ldots, m_r)$, where the m_i belong to the signature of Γ. Then there is a surface kernel epimorphism from Γ onto a finite abelian group G if and only if the following conditions are satisfied.*

(o) *There exists an epimorphism from Γ onto G.*

(i) $\operatorname{lcm}(m_1, m_2, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r) = M$ *for all i.*

(ii) *M divides the exponent $\exp(G)$ of G, and if $g = 0$, then $M = \exp(G)$.*

(iii) $r \neq 1$*, and if* $g = 0$*, then* $r \geq 3$*.*

(iv) If M *is even and only one of the abelian invariants of* G *is divisble by the maximum power of* 2 *that divides* M*, then the number of* m_i *divisible by that maximum power of* 2 *is even.*

Proof. See [Bre00], pp. 30-31. □

The next check is performed by `primeOrderCheckByConjugacyClasses`, which tests every signature of the form $(0; p, p, \ldots, p)$ where p is some prime. This negative check is based on the idea that a surface kernel epimorphism must preserve the order of the generators. The function obtains the conjugacy classes of the group entered and selects those classes which consist of elements of order p. Then the normal closure of the selected conjugacy classes is calculated. If the normal closure is equal to the entire group, then the signature remains in consideration; if not, the signature can be eliminated.

Finally, groups that have passed all of these tests are now sent to `SpecializedGQuotients`, which in turn calls `SpecializedMorClassLoop`. Both of these functions are based on GAP library functions, although we have modified them for purposes of optimization in our situation. This is the final stage in determining whether there exists a surface kernel epimorphism from Γ onto G, and it is done by construction. If such an epimorphism is determined to exist, then we exit the loop and send the group ID of G and signature of Γ back into the main call function, `mainFunction`. If no such epimorphism is found, the group is discarded.

3.4 Checks for Groups of Strong Symmetric Genus Zero and One

The function `groupChecks` takes a genus g and a list of group IDs that were found to be the images of surface kernel epimorphisms from Fuchsian groups, have been checked against a list of the groups of strong symmetric genus $2, 3, \ldots, g-1$, and are suspected to have strong symmetric genus g. `groupChecks` eliminates from this list all groups that belong to the class of groups having strong symmetric genus 0. These are C_n and D_{2n} for all n, S_4, A_4 and A_5. [Bre00, Section 3.4]. It returns the list of group IDs with those representing groups of strong symmetric genus 0 removed.

The function `genusOneChecks` takes a list of group IDs representing groups that are suspected to have strong symmetric genus some $g > 2$ and checks to see whether any of the group IDs represent groups that are known to have strong symmetric genus 1. We know that if a group has strong symmetric genus 1 then there is a surface kernel epimorphism onto it from a group with signature either $(1; -)$, $(0; 2, 2, 2, 2)$, $(0; 3, 3, 3)$, $(0; 2, 4, 4)$ or $(0; 2, 3, 6)$. For each of these signatures and each group in the input list, this function first calls `abelianInvariantsCondition` to see whether an epimorphism can possibly exist from the group given by the signature to the group in the input list. If `abelianInvariantsCondition` returns true, then `genusOneChecks` uses `SpecializedGQuotiennts` to try to find an epimorphism from the Fuchsian group with that signature onto the group from the list. Any group in the input list for which `SpecializedGQuotients` does not succeed in finding an epimorphism is returned by `genusOneChecks`, because those are the groups in the input list that do not have strong symmetric genus 1.

4 GAP Results and Conclusions

We have successfully classified all groups of strong symmetric genus up to twenty-five. Below are tables of our results for groups with strong symmetric genus two to twenty-four. We have not included the table for groups with strong symmetric genus twenty-five due to its length. The first column of the tables contains the structure description of the groups as given by the GAP library function `StructureDescription`, and the second column gives the GAP group ID for those groups. Note that in general the structure given by `StructureDescription` does not uniquely determine the isomorphism type of the group. We have also manually replaced certain structure descriptions with their proper dicyclic and quasiabelian representations. For definitions of dicyclic and quasiabelian groups, see [MZ05, pp. 25-27]. In our tables, the dicyclic group of order $4n$ is denoted by DC_n, the quasidihedral group of order 2^n is denoted by QD_n and the quasiabelian group of order p^n is denoted by $QA_n(p)$.

Structure Description	ID
Q_8	$[8, 4]$
DC_3	$[12, 1]$
QD_{16}	$[16, 8]$
$SL(2, 3)$	$[24, 3]$
$(C_6 \times C_2) \rtimes C_2$	$[24, 8]$
$GL(2, 3)$	$[48, 29]$

Table 2: Groups with Strong Symmetric Genus 2

Structure Description	ID
$C_4 \rtimes C_4$	$[16, 4]$
$QA_4(2)$	$[16, 6]$
$(C_4 \times C_2) \rtimes C_2$	$[16, 13]$
$C_4 \times S_3$	$[24, 5]$
$(C_8 \times C_2) \rtimes C_2$	$[32, 9]$
$(C_4 \times C_4) \rtimes C_2$	$[32, 11]$
$SL(2, 3) \rtimes C_2$	$[48, 33]$
$C_2 \times S_4$	$[48, 48]$
$((C_4 \times C_4) \rtimes C_3) \rtimes C_2$	$[96, 64]$
$PSL(3, 2)$	$[168, 42]$

Table 3: Groups with Strong Symmetric Genus 3

Structure Description	ID
Q_{16}	$[16, 9]$
DC_5	$[20, 1]$
$C_3 \times D_8$	$[24, 10]$
QD_{32}	$[32, 19]$
$S_3 \times S_3$	$[36, 10]$
$C_6 \times S_3$	$[36, 12]$
$(C_{10} \times C_2) \rtimes C_2$	$[40, 8]$
$S_3 \wr S_2$	$[72, 40]$
$C_3 \times S_4$	$[72, 42]$
S_5	$[120, 34]$

Table 4: Groups with Strong Symmetric Genus 4

Structure Description	ID
$C_4 \times C_2 \times C_2$	$[16, 10]$
$C_2 \times C_2 \times C_2 \times C_2$	$[16, 14]$
$C_2 \times (C_3 \rtimes C_4)$	$[24, 7]$
$C_3 \times D_{10}$	$[30, 2]$
$(C_4 \times C_2) \rtimes C_4$	$[32, 2]$
$(C_8 \times C_2) \rtimes C_2$	$[32, 5]$
$(C_8 \rtimes C_2) \rtimes C_2$	$[32, 7]$
$(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$	$[32, 27]$
$(C_4 \times C_2 \times C_2) \rtimes C_2$	$[32, 28]$
$(C_2 \times D_8) \rtimes C_2$	$[32, 43]$
$C_4 \times D_{10}$	$[40, 5]$
$(C_{12} \times C_2) \rtimes C_2$	$[48, 14]$
$A_4 \rtimes C_4$	$[48, 30]$
$C_2 \times C_2 \times A_4$	$[48, 49]$
$((C_8 \times C_2) \rtimes C_2) \rtimes C_2$	$[64, 8]$
$((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$	$[64, 32]$
$(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5$	$[80, 49]$
$((C_4 \times C_2) \rtimes C_4) \rtimes C_3$	$[96, 3]$
$(C_2 \times C_2 \times A_4) \rtimes C_2$	$[96, 195]$
$C_2 \times A_5$	$[120, 35]$
$((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_2$	$[160, 234]$
$(((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2$	$[192, 181]$

Table 5: Groups with Strong Symmetric Genus 5

Structure Description	ID
$C_3 \rtimes C_8$	$[24, 1]$
DC_6	$[24, 4]$
DC_7	$[28, 1]$
$C_5 \times S_3$	$[30, 1]$
$(C_2 \times C_2) \rtimes C_9$	$[36, 3]$
$C_{24} \rtimes C_2$	$[48, 6]$
$(C_3 \times D_8) \rtimes C_2$	$[48, 15]$
$C_5 \times D_{10}$	$[50, 3]$
$(C_{14} \times C_2) \rtimes C_2$	$[56, 7]$
$((C_2 \times C_2) \rtimes C_9) \rtimes C_2$	$[72, 15]$
$((C_5 \times C_5) \rtimes C_3) \rtimes C_2$	$[150, 5]$

Table 6: Groups with Strong Symmetric Genus 6

Structure Description	ID
$C_2 \times Q_8$	$[16, 12]$
$QA_3(3)$	$[27, 4]$
$Q_8 \rtimes C_4$	$[32, 10]$
$C_8 \rtimes C_4$	$[32, 13]$
$C_8 \rtimes C_4$	$[32, 14]$
$QA_5(2)$	$[32, 17]$
$(C_8 \times C_2) \rtimes C_2$	$[32, 42]$
$DC_3 \times C_3$	$[36, 6]$
$C_3 \times D_{14}$	$[42, 4]$
$C_2 \times SL(2, 3)$	$[48, 32]$
$D_8 \times S_3$	$[48, 38]$
$(C_4 \times S_3) \rtimes C_2$	$[48, 41]$
$C_3 \times D_{18}$	$[54, 3]$
$(C_9 \rtimes C_3) \rtimes C_2$	$[54, 6]$
$C_4 \times D_{14}$	$[56, 4]$
$(C_2 \times C_2 \times C_2) \rtimes C_7$	$[56, 11]$
$(C_{16} \times C_2) \rtimes C_2$	$[64, 38]$
$(C_{16} \rtimes C_2) \rtimes C_2$	$[64, 41]$
$C_3 \times SL(2, 3)$	$[72, 25]$
$(C_3 \times SL(2, 3)) \rtimes C_2$	$[144, 127]$
$PSL(2, 8)$	$[504, 156]$

Table 7: Groups with Strong Symmetric Genus 7

Structure Description	ID
$C_3 \times Q_8$	$[24, 11]$
Q_{32}	$[32, 20]$
DC_9	$[36, 1]$
$C_5 \times D_8$	$[40, 10]$
$C_2 \times (C_7 \rtimes C_3)$	$[42, 2]$
$(C_3 \times Q_8) \rtimes C_2$	$[48, 17]$
$C_3 \times D_{16}$	$[48, 25]$
$C_2.S_4$	$[48, 28]$
$S_3 \times D_{10}$	$[60, 8]$
QD_{64}	$[64, 53]$
$(C_{18} \times C_2) \rtimes C_2$	$[72, 8]$
$C_2 \times ((C_7 \rtimes C_3) \rtimes C_2)$	$[84, 7]$
$PSL(3, 2) \rtimes C_2$	$[336, 208]$

Table 8: Groups with Strong Symmetric Genus 8

Structure Description	ID
$C_6 \times C_2 \times C_2$	$[24, 15]$
$C_8 \rtimes C_4$	$[32, 4]$
$(C_2 \times C_2).(C_4 \times C_2)$	$[32, 8]$
$C_4 \rtimes C_8$	$[32, 12]$
$C_2 \times ((C_4 \times C_2) \rtimes C_2)$	$[32, 22]$
$C_4 \times D_8$	$[32, 25]$
$(C_4 \times C_2 \times C_2) \rtimes C_2$	$[32, 30]$
$(C_4 \times C_4) \rtimes C_2$	$[32, 31]$
$C_2 \times QD_{16}$	$[32, 40]$
$C_2 \times C_2 \times D_8$	$[32, 46]$
$(C_2 \times D_8) \rtimes C_2$	$[32, 49]$
$(C_2 \times Q_8) \rtimes C_2$	$[32, 50]$
$C_2 \times (C_5 \rtimes C_4)$	$[40, 7]$
$C_7 \times S_3$	$[42, 3]$
$C_8 \times S_3$	$[48, 4]$
$C_{24} \rtimes C_2$	$[48, 5]$
$(C_2 \times (C_3 \rtimes C_4)) \rtimes C_2$	$[48, 19]$
$C_3 \times ((C_4 \times C_2) \rtimes C_2)$	$[48, 21]$
$C_4 \times A_4$	$[48, 31]$
$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	$[48, 43]$
$(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3$	$[48, 50]$
$((C_8 \times C_2) \rtimes C_2) \rtimes C_2$	$[64, 4]$
$(C_8 \times C_4) \rtimes C_2$	$[64, 6]$
$(C_8 \rtimes C_4) \rtimes C_2$	$[64, 10]$
$(C_4 \rtimes C_8) \rtimes C_2$	$[64, 12]$
$(C_4 \times C_2 \times C_2) \rtimes C_4$	$[64, 23]$
$(C_4 \times C_4) \rtimes C_4$	$[64, 35]$
$((C_2 \times C_2).(C_4 \times C_2)) \rtimes C_2$	$[64, 36]$
$(C_2 \times C_2 \times D_8) \rtimes C_2$	$[64, 73]$
$(C_2 \times C_2 \times D_8) \rtimes C_2$	$[64, 128]$
$(C_4 \times C_4) \rtimes C_2) \rtimes C_2$	$[64, 134]$
$(C_4 \times C_4) \rtimes C_2) \rtimes C_2$	$[64, 135]$
$(((C_4 \times C2) \rtimes C_2) \rtimes C_2) \rtimes C_2$	$[64, 138]$
$(C_4 \times D_8) \rtimes C_2$	$[64, 140]$
$(C_2 \times D_{16}) \rtimes C_2$	$[64, 177]$
$(C_2 \times D_{16}) \rtimes C_2$	$[64, 190]$

Table 9: Groups with Strong Symmetric Genus 9

Structure Description	ID
$C_4 \times D_{18}$	$[72, 5]$
$(C_{20} \times C_2) \rtimes C_2$	$[80, 14]$
$(C_3 \times ((C_4 \times C_2) \rtimes C_2)) \rtimes C_2$	$[96, 13]$
$SL(2,3) \rtimes C_4$	$[96, 67]$
$((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2$	$[96, 70]$
$C_4 \times S_4$	$[96, 186]$
$(C_2 \times S_4) \rtimes C_2$	$[96, 187]$
$(SL(2,3) \rtimes C_2) \rtimes C_2$	$[96, 193]$
$(C_2 \times SL(2,3)) \rtimes C_2$	$[96, 202]$
$((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2$	$[96, 227]$
$((C_4 \times C_2 \times C_2) \rtimes C_4) \rtimes C_2$	$[128, 75]$
$((C_4 \rtimes C_8) \rtimes C_2) \rtimes C_2$	$[128, 134]$
$((C_8 \rtimes C_4) \rtimes C_2) \rtimes C_2$	$[128, 136]$
$((C_4 \times C_4) \rtimes C_4) \rtimes C_2$	$[128, 138]$
$((C_2 \times Q_8) \rtimes C_2) \rtimes C_5$	$[160, 199]$
$(((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2$	$[192, 194]$
$(((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2) \rtimes C_2$	$[192, 955]$
$((SL(2,3) \rtimes C_2) \rtimes C_2) \rtimes C_2$	$[192, 990]$
$(((C_2 \times Q_8) \rtimes C_2) \rtimes C_5) \rtimes C_2$	$[320, 1582]$

Table 10: Groups with Strong Symmetric Genus 9 Continued

Structure Description	ID
$C_3 \times C_3 \times C_3$	$[27, 5]$
DC_{10}	$[40, 4]$
DC_{11}	$[44, 1]$
$C_3 \times QD_{16}$	$[48, 26]$
$C_9 \times S_3$	$[54, 4]$
$((C_3 \times C_3) \rtimes C_3) \rtimes C_2$	$[54, 8]$
$C_2 \times ((C_3 \times C_3) \rtimes C_3)$	$[54, 10]$
$C_3 \times C_3 \times S_3$	$[54, 12]$
$C_3 \times ((C_3 \times C_3) \rtimes C_2)$	$[54, 13]$
$C_6 \times D_{10}$	$[60, 10]$
$(C_6 \times S_3) \rtimes C_2$	$[72, 23]$
$C_3 \times D_{24}$	$[72, 28]$
$C_3 \times ((C_6 \times C_2) \rtimes C_2)$	$[72, 30]$
$(C_3 \times C_3) \rtimes C_8$	$[72, 39]$
$(C_3 \times C_3) \rtimes Q_8$	$[72, 41]$
$(C_3 \times A_4) \rtimes C_2$	$[72, 43]$
$C_{40} \rtimes C_2$	$[80, 6]$
$(C_3 \times C_3 \times C_3) \rtimes C_3$	$[81, 7]$
$(C_{22} \times C_2) \rtimes C_2$	$[88, 7]$
$((C_3 \times C_3) \rtimes C_3) \rtimes C_4$	$[108, 15]$
$(((C_3 \times C_3) \rtimes C_3) \rtimes C_2) \rtimes C_2$	$[108, 17]$
$C_2 \times (((C_3 \times C_3) \rtimes C_3) \rtimes C_2)$	$[108, 25]$
$(C_3 \times C_3 \times C_3) \rtimes C_4$	$[108, 37]$
$C_3 \times S_3 \times S_3$	$[108, 38]$
$(C_3 \times ((C_3 \times C_3) \rtimes C_2)) \rtimes C_2$	$[108, 40]$
$C_3 \times GL(2, 3)$	$[144, 122]$
$((C_3 \times C_3) \rtimes C_8) \rtimes C_2$	$[144, 182]$
$((C_9 \times C_3) \rtimes C_3) \rtimes C_2$	$[162, 14]$
$GL(2, 4)$	$[180, 19]$
$(((C_3 \times C_3) \rtimes C_3) \rtimes C_4) \rtimes C_2$	$[216, 87]$
$((C_6 \times C_6) \rtimes C_3) \rtimes C_2$	$[216, 92]$
$((C_3 \times C_3) \rtimes Q_8) \rtimes C_3$	$[216, 153]$
$(C_3 \times S_3 \times S_3) \rtimes C_2$	$[216, 158]$
$((C_3 \times ((C_3 \times C_3) \rtimes C_2)) \rtimes C_2) \rtimes C_3$	$[324, 160]$
A_6	$[360, 118]$
$(((C_3 \times C_3) \rtimes Q_8) \rtimes C_3) \rtimes C_2$	$[432, 734]$

Table 11: Groups with Strong Symmetric Genus 10

Structure Description	ID
$C_4.D_8$	$[32, 15]$
$(C_2 \times Q_8) \rtimes C_2$	$[32, 44]$
$C_5 \rtimes C_8$	$[40, 3]$
$DC_3 \times C_4$	$[48, 11]$
$(C_3 \rtimes C_4) \rtimes C_4$	$[48, 12]$
$C_{12} \rtimes C_4$	$[48, 13]$
$QA_4(2) \times C_3$	$[48, 24]$
$(C_{12} \times C_2) \rtimes C_2$	$[48, 37]$
$C_3 \times (C_5 \rtimes C_4)$	$[60, 6]$
$C_{15} \rtimes C_4$	$[60, 7]$
$(C_{16} \times C_2) \times C_2$	$[64, 40]$
$(C_{16} \rtimes C_2) \times C_2$	$[64, 42]$
$C_3 \times D_{22}$	$[66, 2]$
$(C_5 \rtimes C_8) \rtimes C_2$	$[80, 28]$
$(C_5 \rtimes C_8) \rtimes C_2$	$[80, 29]$
$C_4 \times (C_5 \rtimes C_4)$	$[80, 30]$
$C_{20} \rtimes C_4$	$[80, 31]$
$D_8 \times D_{10}$	$[80, 39]$
$(C_4 \times D_{10}) \rtimes C_2$	$[80, 42]$
$C_4 \times D_{22}$	$[88, 4]$
$(C_{24} \times C_2) \rtimes C_2$	$[96, 28]$
$(C_3 \times (C_8 \rtimes C_2)) \rtimes C_2$	$[96, 32]$
$C_2 \times GL(2, 3)$	$[96, 189]$
$(C_2 \times SL(2, 3)) \rtimes C_2$	$[96, 190]$
$S_3 \times (C_5 \rtimes C_4)$	$[120, 36]$
$(C_{20} \rtimes C_4) \rtimes C_2$	$[160, 82]$
$(C_4 \times (C_5 \rtimes C_4)) \rtimes C_2$	$[160, 85]$
$C_2 \times S_5$	$[240, 189]$

Table 12: Groups with Strong Symmetric Genus 11

Structure Description	ID
$C_5 \rtimes C_8$	$[40, 1]$
DC_{12}	$[48, 8]$
$(C_3 \rtimes C_8) \rtimes C_2$	$[48, 16]$
DC_{13}	$[52, 1]$
$C_{11} \rtimes C_5$	$[55, 1]$
$C_7 \times D_8$	$[56, 9]$
$C_5 \times A_4$	$[60, 9]$
$C_{10} \times S_3$	$[60, 11]$
$(C_5 \times D_8) \rtimes C_2$	$[80, 15]$
$S_3 \times D_{14}$	$[84, 8]$
$C_{48} \rtimes C_2$	$[96, 7]$
$(C_{26} \times C_2) \rtimes C_2$	$[104, 8]$
$(C_{11} \rtimes C_5) \rtimes C_2$	$[110, 1]$
$(C_5 \times A_4) \rtimes C_2$	$[120, 38]$

Table 13: Groups with Strong Symmetric Genus 12

Structure Description	ID
$(C_2 \times C_4 \times S_3) \rtimes C_2$	$[96, 102]$
$(C_2 \times D_{24}) \rtimes C_2$	$[96, 115]$
$D_8 \times A_4$	$[96, 197]$
$C_2 \times (SL(2,3) \rtimes C_2)$	$[96, 200]$
$C_2 \times C_2 \times S_4$	$[96, 226]$
$C_4 \times D_{26}$	$[104, 5]$
$(C_{28} \times C_2) \rtimes C_2$	$[112, 13]$
$((C_{16} \rtimes C_2) \rtimes C_2) \rtimes C_2$	$[128, 71]$
$((C_8 \times C_2) \rtimes C_4) \rtimes C_2$	$[128, 79]$
$(C_2 \times ((C_3 \times C_3) \rtimes C_4)) \rtimes C_2$	$[144, 115]$
$S_3 \times S_4$	$[144, 183]$
$A_4 \times A_4$	$[144, 184]$
$C_2 \times A_4 \times S_3$	$[144, 190]$
$A_4 \times S_4$	$[288, 1024]$
$A_5 \times S_3$	$[360, 121]$

Table 14: Groups with Strong Symmetric Genus 13

Structure Description	ID
$C_4 \times C_4 \times C_2$	$[32, 21]$
$C_2 \times (C_4 \rtimes C_4)$	$[32, 23]$
$(C_4 \times C_4) \rtimes C_2$	$[32, 24]$
$(C_2 \times Q_8) \rtimes C_2$	$[32, 29]$
$(C_4 \times C_4) \rtimes C_2$	$[32, 33]$
$C_8 \times C_2 \times C_2$	$[32, 36]$
$QA_4(2) \times C_2$	$[32, 37]$
$(C_8 \times C_2) \rtimes C_2$	$[32, 38]$
$C_2 \times ((C_4 \times C_2) \rtimes C_2)$	$[32, 48]$
$C_2 \times C_4 \times S_3$	$[48, 35]$
$C_2 \times C_2 \times C_2 \times S_3$	$[48, 51]$
$C_2 \times (C_7 \rtimes C_4)$	$[56, 6]$
$(C_2 \times Q_8) \rtimes C_4$	$[64, 9]$
$(C_8 \times C_2) \rtimes C_4$	$[64, 18]$
$(C_4 \times C_4) \rtimes C_4$	$[64, 20]$
$(C_8 \times C_2) \rtimes C_4$	$[64, 21]$
$(C_{16} \times C_2) \rtimes C_2$	$[64, 29]$
$(C_{16} \rtimes C_2) \rtimes C_2$	$[64, 30]$
$(C_{16} \times C_2) \rtimes C_2$	$[64, 31]$
$(C_4 \times C_2 \times C_2) \rtimes C_4$	$[64, 33]$
$(C_2 \times D_{16}) \rtimes C_2$	$[64, 130]$
$(C_8 \times C_2 \times C_2) \rtimes C_2$	$[64, 147]$
$(C_2 \times (C_8 \rtimes C_2)) \rtimes C_2$	$[64, 150]$
$(C_2 \times D_{16}) \rtimes C_2$	$[64, 153]$
$C_2 \times ((C_2 \times C_2) \rtimes C_9)$	$[72, 16]$
$(C_3 \times (C_3 \rtimes C_4)) \rtimes C_2$	$[72, 21]$
$C_{12} \times S_3$	$[72, 27]$
$C_2 \times S_3 \times S_3$	$[72, 46]$
$C_6 \times A_4$	$[72, 47]$
$C_3 \times D_{26}$	$[78, 4]$
$C_3 \times D_{30}$	$[90, 7]$
$C_2 \times ((C_4 \times C_4) \rtimes C_3)$	$[96, 68]$
$((C_4 \times C_4) \rtimes C_3) \rtimes C_2$	$[96, 71]$
$(C_2 \times C_2 \times C_2 \times S_3) \rtimes C_2$	$[96, 89]$

Table 15: Groups with Strong Symmetric Genus 13 Continued

Structure Description	ID
$C_5 \times Q_8$	$[40, 11]$
$C_3 \rtimes C_{16}$	$[48, 1]$
$C_3 \rtimes Q_{16}$	$[48, 18]$
DC_{14}	$[56, 3]$
DC_{15}	$[60, 3]$
$C_5 \times D_{14}$	$[70, 2]$
$C_2 \times (C_{13} \rtimes C_3)$	$[78, 2]$
$(C_5 \times Q_8) \rtimes C_2$	$[80, 17]$
$C_6 \times D_{14}$	$[84, 12]$
$(C_3 \times D_{16}) \rtimes C_2$	$[96, 33]$
$C_{56} \rtimes C_2$	$[112, 5]$
$SL(2, 5)$	$[120, 5]$
$(C_{30} \times C_2) \rtimes C_2$	$[120, 30]$
$C_2 \times ((C_{13} \rtimes C_3) \rtimes C_2)$	$[156, 8]$
$PSL(2, 13)$	$[1092, 25]$

Table 16: Groups with Strong Symmetric Genus 14

Structure Description	ID
$C_2 \times Q_{16}$	$[32, 41]$
$C_2 \times (C_3 \rtimes C_8)$	$[48, 9]$
$(C_3 \rtimes C_8) \rtimes C_2$	$[48, 10]$
$C_3 \times (C_4 \rtimes C_4)$	$[48, 22]$
$(C_2 \times (C_3 \rtimes C_4)) \rtimes C_2$	$[48, 39]$
$Q_{16} \rtimes C_4$	$[64, 39]$
$C_{16} \rtimes C_4$	$[64, 47]$
$C_{16} \rtimes C_4$	$[64, 48]$
$QA_6(2)$	$[64, 51]$
$(C_{16} \times C_2) \rtimes C_2$	$[64, 189]$
$C_{11} \times S_3$	$[66, 1]$
$C_7 \times D_{10}$	$[70, 1]$
$C_8 \times D_{10}$	$[80, 4]$
$C_{40} \rtimes C_2$	$[80, 5]$
$(C_7 \rtimes C_4) \rtimes C_3$	$[84, 1]$
$(C_{12} \times C_4) \rtimes C_2$	$[96, 12]$
$(C_2 \times (C_3 \rtimes C_8)) \rtimes C_2$	$[96, 16]$
$SL(2, 3) \rtimes C_4$	$[96, 66]$
$(C_2.S_4) \rtimes C_2$	$[96, 192]$
$C_7 \times D_{14}$	$[98, 3]$
$D_8 \times D_{14}$	$[112, 31]$
$(C_4 \times D_{14}) \rtimes C_2$	$[112, 34]$
$C_4 \times D_{30}$	$[120, 27]$
$(C_{32} \times C_2) \rtimes C_2$	$[128, 147]$
$(C_{32} \rtimes C_2) \rtimes C_2$	$[128, 150]$
$C_2 \times (((C_2 \times C_2) \rtimes C_9) \rtimes C_2)$	$[144, 109]$
$(C_7 \times Q_8) \rtimes C_3$	$[168, 23]$
$((C_2 \times C_2 \times C_2) \rtimes C_7) \rtimes C_3$	$[168, 43]$
$SL(2, 5) \rtimes C_2$	$[240, 93]$
$((C_7 \times C_7) \rtimes C_3) \rtimes C_2$	$[294, 7]$
$((C_7 \times Q_8) \rtimes C_3) \rtimes C_2$	$[336, 134]$

Table 17: Groups with Strong Symmetric Genus 15

Structure Description	ID
$(C_3 \times C_3) \rtimes C_4$	$[36, 7]$
$C_3 \times Q_{16}$	$[48, 27]$
$C_2 \times (C_9 \rtimes C_3)$	$[54, 11]$
Q_{64}	$[64, 54]$
DC_{17}	$[68, 1]$
$C_9 \times D_8$	$[72, 10]$
$(C_3 \times C_3) \rtimes C_8$	$[72, 19]$
$(C_6 \times S_3) \rtimes C_2$	$[72, 22]$
$C_5 \times D_{16}$	$[80, 25]$
$C_3 \times D_{32}$	$[96, 61]$
$(C_5 \times C_5) \rtimes C_4$	$[100, 10]$
$D_{10} \times D_{10}$	$[100, 13]$
$C_{10} \times D_{10}$	$[100, 14]$
$S_3 \times D_{18}$	$[108, 16]$
$C_6 \times D_{18}$	$[108, 23]$
$C_2 \times ((C_9 \rtimes C_3) \rtimes C_2)$	$[108, 26]$
QD_{128}	$[128, 162]$
$(C_{34} \times C_2) \rtimes C_2$	$[136, 8]$
$((C_3 \times C_3) \rtimes C_8) \rtimes C_2$	$[144, 117]$
$(D_{10} \times D_{10}) \rtimes C_2$	$[200, 43]$
$C_5 \times A_5$	$[300, 22]$
$(C_5 \times A_5) \rtimes C_2$	$[600, 145]$
$A_6 \rtimes C_2$	$[720, 764]$

Table 18: Groups with Strong Symmetric Genus 16

Structure Description	ID
$C_4 \times Q_8$	$[32, 26]$
$(C_2 \times C_2).(C_2 \times C_2 \times C_2)$	$[32, 32]$
$C_4 \rtimes Q_8$	$[32, 35]$
$C_4 \times C_2 \times C_2 \times C_2$	$[32, 45]$
$C_2 \times C_2 \times C_2 \times C_2 \times C_2$	$[32, 51]$
$C_{10} \times C_2 \times C_2$	$[40, 14]$
$C_6 \times D_8$	$[48, 45]$
$(C_4 \times C_2) \rtimes C_8$	$[64, 5]$
$Q_8 \rtimes C_8$	$[64, 7]$
$(C_4 \times C_2).(C_4 \times C_2)$	$[64, 11]$
$(C_4 \times C_2).(C_4 \times C_2)$	$[64, 13]$
$(C_4 \times C_2).(C_4 \times C_2)$	$[64, 14]$
$(C_8 \times C_2) \rtimes C_4$	$[64, 17]$
$(C_8 \rtimes C_2) \rtimes C_4$	$[64, 24]$
$(C_8 \times C_2) \rtimes C_4$	$[64, 25]$
$(C_4 \times C_2).(C_4 \times C_2)$	$[64, 37]$
$(C_2 \times ((C_4 \times C_2) \rtimes C_2)) \rtimes C_2$	$[64, 60]$
$(C_4 \times C_2 \times C_2 \times C_2) \rtimes C_2$	$[64, 67]$
$(C_4 \times C_4 \times C_2) \rtimes C_2$	$[64, 71]$
$(C_2 \times ((C_4 \times C_2) \rtimes C_2)) \rtimes C_2$	$[64, 75]$
$C_2 \times (((C_4 \times C_2) \rtimes C_2) \rtimes C_2)$	$[64, 90]$
$(((C_4 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_2$	$[64, 91]$
$C_2 \times ((C_8 \times C_2) \rtimes C_2)$	$[64, 95]$
$(C_2 \times (C_8 \rtimes C_2)) \rtimes C_2$	$[64, 99]$
$C_2 \times ((C_4 \times C_4) \rtimes C_2)$	$[64, 101]$
$(C_2 \times (C_8 \rtimes C_2)) \rtimes C_2$	$[64, 102]$
$C_4 \times D_{16}$	$[64, 118]$
$(C_4 \times D_8) \rtimes C_2$	$[64, 123]$
$(C_2 \times QD_{16}) \rtimes C_2$	$[64, 131]$
$((C_4 \times C_4) \rtimes C_2) \rtimes C_2$	$[64, 136]$
$((C_4 \times C_4) \rtimes C_2) \rtimes C_2$	$[64, 137]$

Table 19: Groups with Strong Symmetric Genus 17

Structure Description	ID
$(((C_4 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_2$	$[64, 139]$
$(C_2 \times QD_{16}) \rtimes C_2$	$[64, 141]$
$(C_4 \times D_8) \rtimes C_2$	$[64, 144]$
$(C_8 \times C_4) \rtimes C_2$	$[64, 167]$
$((C_8 \times C_2) \rtimes C_2) \rtimes C_2$	$[64, 171]$
$(C_8 \times C_4) \rtimes C_2$	$[64, 173]$
$(C_8 \times C_4) \rtimes C_2$	$[64, 176]$
$C_2 \times QD_{32}$	$[64, 187]$
$C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2)$	$[64, 202]$
$C_2 \times ((C_4 \times C_4) \rtimes C_2)$	$[64, 211]$
$(C_2 \times C_2 \times D_8) \rtimes C_2$	$[64, 215]$
$(C_2 \times C_2 \times D_8) \rtimes C_2$	$[64, 216]$
$(C_2 \times ((C_4 \times C_2) \rtimes C_2)) \rtimes C_2$	$[64, 218]$
$D_8 \times D_8$	$[64, 226]$
$((C_4 \times C_2 \times C_2) \rtimes C_2) \rtimes C_2$	$[64, 241]$
$C_2 \times C_2 \times D_{16}$	$[64, 250]$
$C_2 \times ((C_2 \times D_8) \rtimes C_2)$	$[64, 254]$
$(C_2 \times (C_8 \rtimes C_2)) \rtimes C_2$	$[64, 256]$
$(C_2 \times D_{16}) \rtimes C_2$	$[64, 257]$
$C_2 \times (C_9 \rtimes C_4)$	$[72, 7]$
$(C_2 \times (C_5 \rtimes C_4)) \rtimes C_2$	$[80, 19]$
$C_5 \times ((C_4 \times C_2) \rtimes C_2)$	$[80, 21]$
$C_2 \times ((C_{10} \times C_2) \rtimes C_2)$	$[80, 44]$
$(C_{24} \times C_2) \rtimes C_2$	$[96, 27]$
$(C_3 \times (C_8 \rtimes C_2)) \rtimes C_2$	$[96, 30]$
$((C_2 \times (C_3 \rtimes C_4)) \rtimes C_2) \rtimes C_2$	$[96, 41]$
$C_3 \times (((C_4 \times C_2) \rtimes C_2) \rtimes C_2)$	$[96, 49]$
$C_4 \times SL(2, 3)$	$[96, 69]$
$D_{16} \times S_3$	$[96, 117]$
$C_2 \times ((C_3 \times D_8) \rtimes C_2)$	$[96, 138]$
$(C_6 \times D_8) \rtimes C_2$	$[96, 147]$
$C_2 \times (A_4 \rtimes C_4)$	$[96, 194]$
$C_2 \times C_4 \times A_4$	$[96, 196]$
$(SL(2, 3) \rtimes C_2) \rtimes C_2$	$[96, 201]$

Table 20: Groups with Strong Symmetric Genus 17 Continued

Structure Description	ID
$((C_2 \times D_8) \rtimes C_2) \rtimes C_3$	$[96, 204]$
$C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3)$	$[96, 229]$
$C_3 \times D_{34}$	$[102, 2]$
$C_2 \times ((C_2 \times C_2 \times C_2) \rtimes C_7)$	$[112, 41]$
$A_4 \times D_{10}$	$[120, 39]$
$((C_8 \times C_2) \rtimes C_4) \rtimes C_2$	$[128, 2]$
$(C_2 \times ((C_4 \times C_2) \rtimes C_2)) \rtimes C_4$	$[128, 36]$
$(((C_8 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_2$	$[128, 48]$
$((C_4 \times C_2) \rtimes C_8) \rtimes C_2$	$[128, 50]$
$((C_8 \rtimes C_2) \rtimes C_4) \rtimes C_2$	$[128, 77]$
$(C_4 \times C_4 \times C_2) \rtimes C_4$	$[128, 125]$
$((C_8 \rtimes C_4) \rtimes C_2) \rtimes C_2$	$[128, 135]$
$((C_4 \times C_2).(C_4 \times C_2)) \rtimes C_2$	$[128, 137]$
$(C_8 \times C_4) \rtimes C_4$	$[128, 141]$
$((C_4 \times C_2).(C_4 \times C_2)) \rtimes C_2$	$[128, 142]$
$(C_8 \rtimes C_4) \rtimes C_4$	$[128, 144]$
$((C_2 \times C_2 \times D_8) \rtimes C_2) \rtimes C_2$	$[128, 327]$
$((C_2 \times C_2 \times D_8) \rtimes C_2) \rtimes C_2$	$[128, 330]$
$((C_8 \times C_4) \rtimes C_2) \rtimes C_2$	$[128, 453]$
$(C_2 \times ((C_4 \times C_4) \rtimes C_2)) \rtimes C_2$	$[128, 738]$
$(C_2 \times ((C_2 \times D_8) \rtimes C_2)) \rtimes C_2$	$[128, 740]$
$(C_2 \times C_2 \times D_{16}) \rtimes C_2$	$[128, 743]$
$(C_2 \times (((C_4 \times C_2) \rtimes C_2) \rtimes C_2)) \rtimes C_2$	$[128, 753]$
$(C_2 \times C_2 \times D_{16}) \rtimes C_2$	$[128, 916]$
$((C_{16} \rtimes C_2) \rtimes C_2) \rtimes C_2$	$[128, 922]$
$(D_8 \times D_8) \rtimes C_2$	$[128, 928]$
$((C_4 \times C_2 \times C_2) \rtimes C_4) \rtimes C_2$	$[128, 932]$
$(C_2 \times D_{32}) \rtimes C_2$	$[128, 938]$
$(C_2 \times QD_{32}) \rtimes C_2$	$[128, 982]$
$(C_2 \times D_{32}) \rtimes C_2$	$[128, 995]$
$C_4 \times D_{34}$	$[136, 5]$
$(C_{36} \times C_2) \rtimes C_2$	$[144, 14]$
$(C_5 \times ((C_4 \times C_2) \rtimes C_2)) \rtimes C_2$	$[160, 13]$
$C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5)$	$[160, 235]$
$(C_3 \times (((C_4 \times C_2) \rtimes C_2) \rtimes C_2)) \rtimes C_2$	$[192, 33]$
$((C_4 \times C_4) \rtimes C_3) \rtimes C_4$	$[192, 185]$
$(((C_2 \times D_8) \rtimes C_2) \rtimes C_3) \rtimes C_2$	$[192, 201]$
$(((C_4 \times C_4) \rtimes C_3) \rtimes C_2) \rtimes C_2$	$[192, 956]$
$(C_2 \times C_4 \times A_4) \rtimes C_2$	$[192, 972]$

Table 21: Groups with Strong Symmetric Genus 17 Continued

Structure Description	ID
$(SL(2,3) \rtimes C_4) \rtimes C_2$	$[192, 988]$
$C_2 \times (((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2)$	$[192, 1000]$
$C_2 \times (((C_4 \times C_4) \rtimes C_3) \rtimes C_2)$	$[192, 1002]$
$(((C_4 \times C_4) \rtimes C_3) \rtimes C_2) \rtimes C_2$	$[192, 1008]$
$(((C_2 \times D_8) \rtimes C_2) \rtimes C_3).C_2$	$[192, 1491]$
$(((C_2 \times D_8) \rtimes C_2) \rtimes C_3) \rtimes C_2$	$[192, 1493]$
$(((C_2 \times D_8) \rtimes C_2) \rtimes C_3) \rtimes C_2$	$[192, 1494]$
$((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_4$	$[192, 1495]$
$C_2 \times (((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2)$	$[192, 1538]$
$SL(2,5) \rtimes C_2$	$[240, 90]$
$((((C_8 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_2) \rtimes C_2$	$[256, 382]$
$(((C_4 \times C_2) \rtimes C_8) \rtimes C_2) \rtimes C_2$	$[256, 390]$
$((C_4 \times C_4 \times C_2) \rtimes C_4) \rtimes C_2$	$[256, 396]$
$(((C_4 \times C_2).(C_4 \times C_2)) \rtimes C_2) \rtimes C_2$	$[256, 515]$
$C_2 \times PSL(3,2)$	$[336, 209]$
$((C_2 \times ((C_4 \times C_2) \rtimes C_2)) \rtimes C_4) \rtimes C_3$	$[384, 4]$
$((((C_2 \times D_8) \rtimes C_2) \rtimes C_3) \rtimes C_2) \rtimes C_2$	$[384, 5602]$
$((((C_2 \times D_8) \rtimes C_2) \rtimes C_3) \rtimes C_2) \rtimes C_2$	$[384, 5604]$
$(C_2 \times (((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2)) \rtimes C_2$	$[384, 5657]$
$((((C_4 \times C_4) \rtimes C_3) \rtimes C_2) \rtimes C_2) \rtimes C_2$	$[384, 5677]$
$(((C_2 \times ((C_4 \times C_2) \rtimes C_2)) \rtimes C_4) \rtimes C_3) \rtimes C_2$	$[768, 1085341]$
$(C_2 \times C_2 \times C_2).PSL(3,2)$	$[1344, 814]$

Table 22: Groups with Strong Symmetric Genus 17 Continued

Structure Description	ID
$C_7 \rtimes C_8$	$[56, 1]$
$DC_3 \times C_5$	$[60, 1]$
DC_{18}	$[72, 4]$
DC_{19}	$[76, 1]$
$C_{13} \times S_3$	$[78, 3]$
$C_5 \times QD_{16}$	$[80, 26]$
$C_7 \times A_4$	$[84, 10]$
$C_{14} \times S_3$	$[84, 13]$
$C_5 \times D_{18}$	$[90, 1]$
$(C_3 \times Q_{16}) \rtimes C_2$	$[96, 35]$
$(C_7 \times D_8) \rtimes C_2$	$[112, 14]$
$(C_6 \times D_{10}) \rtimes C_2$	$[120, 12]$
$C_{17} \rtimes C_8$	$[136, 12]$
$C_{72} \rtimes C_2$	$[144, 7]$
$(C_{38} \times C_2) \rtimes C_2$	$[152, 7]$
$(C_7 \times A_4) \rtimes C_2$	$[168, 46]$

Table 23: Groups with Strong Symmetric Genus 18

Structure Description	ID
$Q_8 \times S_3$	$[48, 40]$
$C_3 \times ((C_4 \times C_2) \rtimes C_2)$	$[48, 47]$
$(C_3 \times C_3 \times C_3) \rtimes C_2$	$[54, 14]$
$C_3 \times (C_5 \rtimes C_4)$	$[60, 2]$
$C_8.(C_4 \times C_2)$	$[64, 43]$
$C_{16} \rtimes C_4$	$[64, 46]$
$C_3 \times (C_3 \rtimes C_8)$	$[72, 12]$
$(C_3 \rtimes C_4) \times S_3$	$[72, 20]$
$DC_3 \times C_6$	$[72, 29]$
$C_4 \times ((C_3 \times C_3) \rtimes C_2)$	$[72, 32]$
$(C_6 \times C_6) \rtimes C_2$	$[72, 35]$
$C_4 \times (C_5 \rtimes C_4)$	$[80, 11]$
$(C_5 \rtimes C_4) \rtimes C_4$	$[80, 12]$
$C_{20} \rtimes C_4$	$[80, 13]$
$QA_4(2) \times C_5$	$[80, 24]$
$(C_{20} \times C_2) \rtimes C_2$	$[80, 38]$
$(C_9 \times C_3) \rtimes C_3$	$[81, 3]$
$(C_9 \times C_3) \rtimes C_3$	$[81, 8]$
$C_{15} \times S_3$	$[90, 6]$
$C_3 \times ((C_8 \times C_2) \rtimes C_2)$	$[96, 52]$
$C_3 \times ((C_4 \times C_4) \rtimes C_2)$	$[96, 54]$
$(D_8 \times S_3) \rtimes C_2$	$[96, 121]$
$(C_8 \times S_3) \rtimes C_2$	$[96, 126]$
$((C_3 \times C_3) \rtimes C_4) \rtimes C_3$	$[108, 8]$
$C_2 \times (((C_3 \times C_3) \rtimes C_3) \rtimes C_2)$	$[108, 28]$
$C_3 \times ((C_3 \times C_3) \rtimes C_4)$	$[108, 36]$
$((C_3 \times C_3) \rtimes C_2) \times S_3$	$[108, 39]$
$C_3 \times D_{38}$	$[114, 4]$
$C_3 \times D_{42}$	$[126, 13]$
$D_8 \times D_{18}$	$[144, 41]$
$(C_4 \times D_{18}) \rtimes C_2$	$[144, 44]$
$((C_3 \times C_3) \rtimes C_4) \rtimes C_4$	$[144, 120]$
$S_3 \times SL(2, 3)$	$[144, 128]$
$((C_3 \times C_3) \rtimes C_8) \rtimes C_2$	$[144, 130]$

Table 24: Groups with Strong Symmetric Genus 19

Structure Description	ID
$((C_3 \times C_3) \rtimes C_8) \rtimes C_2$	$[144, 131]$
$C_4 \times ((C_3 \times C_3) \rtimes C_4)$	$[144, 132]$
$(C_{12} \times C_3) \rtimes C_4$	$[144, 133]$
$D_8 \times ((C_3 \times C_3) \rtimes C_2)$	$[144, 172]$
$(C_3 \times C_3 \times Q_8) \rtimes C_2$	$[144, 175]$
$C_2 \times ((C_3 \times C_3) \rtimes C_8)$	$[144, 185]$
$C_2 \times ((S_3 \times S_3) \rtimes C_2)$	$[144, 186]$
$C_6 \times S_4$	$[144, 188]$
$C_2 \times ((C_3 \times A_4) \rtimes C_2)$	$[144, 189]$
$C_4 \times D_{38}$	$[152, 4]$
$(C_{40} \times C_2) \rtimes C_2$	$[160, 28]$
$(C_5 \times (C_8 \rtimes C_2)) \rtimes C_2$	$[160, 32]$
$((C_9 \times C_3) \rtimes C_3) \rtimes C_2$	$[162, 5]$
$A_3 \wr S_3$	$[162, 10]$
$((C_3 \times C_3 \times C_3) \rtimes C_3) \rtimes C_2$	$[162, 11]$
$((C_9 \times C_3) \rtimes C_3) \rtimes C_2$	$[162, 13]$
$(C_3 \times C_3 \times Q_8) \rtimes C_3$	$[216, 42]$
$C_2 \times (((C_3 \times C_3) \rtimes C_3) \rtimes C_4)$	$[216, 100]$
$((C_3 \times C_3) \rtimes C_4) \times S_3$	$[216, 156]$
$C_3 \times (((C_4 \times C_4) \rtimes C_3) \rtimes C_2)$	$[288, 397]$
$((C_{12} \times C_3) \rtimes C_4) \rtimes C_2$	$[288, 430]$
$(C_4 \times ((C_3 \times C_3) \rtimes C_4)) \rtimes C_2$	$[288, 433]$
$(C_2 \times ((C_3 \times C_3) \rtimes C_8)) \rtimes C_2$	$[288, 841]$
$((C_3 \times C_3 \times Q_8) \rtimes C_3) \rtimes C_2$	$[432, 269]$
$C_2 \times A_6$	$[720, 766]$

Table 25: Groups with Strong Symmetric Genus 19 Continued

Structure Description	ID
$C_7 \times Q_8$	$[56, 10]$
$Q_8 \rtimes C_9$	$[72, 3]$
DC_{20}	$[80, 8]$
DC_{21}	$[84, 5]$
$C_{11} \times D_8$	$[88, 9]$
$C_9 \times D_{10}$	$[90, 2]$
$C_3 \times QD_{32}$	$[96, 62]$
$(C_7 \times Q_8) \rtimes C_2$	$[112, 16]$
$C_2 \times (C_{19} \rtimes C_3)$	$[114, 2]$
$(C_{10} \times S_3) \rtimes C_2$	$[120, 13]$
$C_3 \times D_{40}$	$[120, 18]$
$S_3 \times D_{22}$	$[132, 5]$
$(Q_8 \rtimes C_9) \rtimes C_2$	$[144, 32]$
$C_{80} \rtimes C_2$	$[160, 7]$
$(C_{42} \times C_2) \rtimes C_2$	$[168, 38]$
$C_2 \times ((C_{19} \rtimes C_3) \rtimes C_2)$	$[228, 7]$

Table 26: Groups with Strong Symmetric Genus 20

Structure Description	ID
$C_2 \times C_2 \times Q_8$	$[32, 47]$
$C_2 \times C_2 \times (C_3 \rtimes C_4)$	$[48, 42]$
$C_{12} \times C_2 \times C_2$	$[48, 44]$
$C_7 \rtimes C_9$	$[63, 1]$
$C_8 \rtimes C_8$	$[64, 3]$
$C_8 \rtimes C_8$	$[64, 15]$
$C_8 \rtimes C_8$	$[64, 16]$
$C_4.(C_4 \times C_4)$	$[64, 19]$
$C_4.(C_4 \times C_4)$	$[64, 22]$
$C_{16} \rtimes C_4$	$[64, 27]$
$C_{16} \rtimes C_4$	$[64, 28]$
$C_4 \rtimes C_{16}$	$[64, 44]$
$C_8.D_8$	$[64, 45]$
$(C_8 \times C_2 \times C_2) \rtimes C_2$	$[64, 97]$
$(C_2 \times (C_8 \rtimes C_2)) \rtimes C_2$	$[64, 98]$
$(C_8 \times C_4) \rtimes C_2$	$[64, 124]$
$((C_4 \times C_4) \rtimes C_2) \rtimes C_2$	$[64, 125]$
$(C_2 \times C_2 \times Q_8) \rtimes C_2$	$[64, 129]$
$(C_2 \times Q_{16}) \rtimes C_2$	$[64, 133]$
$(C_8 \times C_2 \times C_2) \rtimes C_2$	$[64, 146]$
$(C_2 \times (C_8 \rtimes C_2)) \rtimes C_2$	$[64, 149]$
$(C_2 \times QD_{16}) \rtimes C_2$	$[64, 152]$
$(C_2 \times (C_4 \rtimes C_4)) \rtimes C_2$	$[64, 161]$
$(C_2 \times (C_4 \rtimes C_4)) \rtimes C_2$	$[64, 162]$
$((C_8 \times C_2) \rtimes C_2) \rtimes C_2$	$[64, 163]$
$C_2 \times (C_5 \rtimes C_8)$	$[80, 32]$
$(C_5 \rtimes C_8) \rtimes C_2$	$[80, 33]$
$C_2 \times C_4 \times D_{10}$	$[80, 36]$
$C_2 \times C_2 \times (C_5 \rtimes C_4)$	$[80, 50]$
$C_2 \times C_2 \times C_2 \times D_{10}$	$[80, 51]$
$C_2 \times (C_{11} \rtimes C_4)$	$[88, 6]$
$C_{16} \times S_3$	$[96, 4]$
$C_{48} \rtimes C_2$	$[96, 5]$
$(C_{12} \times C_2) \rtimes C_4$	$[96, 38]$
$C_3 \times ((C_8 \times C_2) \rtimes C_2)$	$[96, 48]$
$C_3 \times ((C_8 \rtimes C_2) \rtimes C_2)$	$[96, 50]$

Table 27: Groups with Strong Symmetric Genus 21

Structure Description	ID
$A_4 \rtimes C_8$	[96, 65]
$C_8 \times A_4$	[96, 73]
$((C_8 \times C_2) \rtimes C_2) \rtimes C_3$	[96, 74]
$(C_2 \times C_4 \times S_3) \rtimes C_2$	[96, 91]
$(C_{12} \times C_2 \times C_2) \rtimes C_2$	[96, 137]
$(C_2 \times C_2 \times C_2 \times S_3) \rtimes C_2$	[96, 144]
$(C_2 \times D_{24}) \rtimes C_2$	[96, 156]
$A_4 \rtimes Q_8$	[96, 185]
$C_2 \times C_2 \times SL(2,3)$	[96, 198]
$C_5 \times (C_5 \rtimes C_4)$	[100, 9]
$C_8 \times D_{14}$	[112, 3]
$C_{56} \rtimes C_2$	[112, 4]
$C_6 \times (C_5 \rtimes C_4)$	[120, 40]
$C_2 \times (C_{15} \rtimes C_4)$	[120, 41]
$C_2 \times S_3 \times D_{10}$	[120, 42]
$(C_{16} \times C_4) \rtimes C_2$	[128, 63]
$(C_{16} \rtimes C_4) \rtimes C_2$	[128, 65]
$(C_8 \times C_8) \rtimes C_2$	[128, 67]
$(C_8 \rtimes C_8) \rtimes C_2$	[128, 68]
$(C_4.(C_4 \times C_4)) \rtimes C_2$	[128, 73]
$(C_4.(C_4 \times C_4)) \rtimes C_2$	[128, 81]
$(C_4 \rtimes C_{16}) \rtimes C_2$	[128, 92]
$(C_4 \rtimes C_{16}) \rtimes C_2$	[128, 93]
$(C_2 \times (C_5 \rtimes C_8)) \rtimes C_2$	[160, 78]
$(C_{20} \times C_2) \rtimes C_4$	[160, 81]
$((C_2 \times (C_5 \rtimes C_4)) \rtimes C_2) \rtimes C_2$	[160, 86]
$((C_5 \rtimes C_8) \rtimes C_2) \rtimes C_2$	[160, 88]
$(C_2 \times C_2 \times C_2 \times D_{10}) \rtimes C_2$	[160, 103]
$(C_2 \times C_4 \times D_{10}) \rtimes C_2$	[160, 116]
$(C_2 \times D_{40}) \rtimes C_2$	[160, 129]
$C_4 \times D_{42}$	[168, 35]
$(C_{44} \times C_2) \rtimes C_2$	[176, 13]
$(C_4.(C_4 \times C_4)) \rtimes C_3$	[192, 4]
$(C_3 \times ((C_8 \times C_2) \rtimes C_2)) \rtimes C_2$	[192, 29]
$(C_3 \times ((C_8 \rtimes C_2) \rtimes C_2)) \rtimes C_2$	[192, 34]
$C_2 \times (((C_4 \times C_4) \rtimes C_3) \rtimes C_2)$	[192, 944]

Table 28: Groups with Strong Symmetric Genus 21 Continued

Structure Description	ID
$(C_8 \times A_4) \rtimes C_2$	$[192, 960]$
$(C_8 \times A_4) \rtimes C_2$	$[192, 961]$
$(((C_8 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2$	$[192, 963]$
$(((C_8 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2$	$[192, 964]$
$(D_8 \times A_4) \rtimes C_2$	$[192, 974]$
$(C_2 \times C_2 \times SL(2,3)) \rtimes C_2$	$[192, 980]$
$D_{10} \times (C_5 \rtimes C_4)$	$[200, 41]$
$A_5 \rtimes C_4$	$[240, 91]$
$(C_6 \times (C_5 \rtimes C_4)) \rtimes C_2$	$[240, 96]$
$C_2 \times C_2 \times A_5$	$[240, 190]$
$(((C_5 \rtimes C_8) \rtimes C_2) \rtimes C_2) \rtimes C_2$	$[320, 202]$
$((C_{20} \times C_2) \rtimes C_4) \rtimes C_2$	$[320, 206]$
$((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_4$	$[320, 1635]$
$((C_8 \times C_8) \rtimes C_3) \rtimes C_2$	$[384, 568]$
$((C_4.(C_4 \times C_4)) \rtimes C_3) \rtimes C_2$	$[384, 570]$
$(C_2 \times C_2 \times A_5) \rtimes C_2$	$[480, 951]$
$C_4 \times D_{42}$	$[168, 35]$
$(C_{44} \times C_2) \rtimes C_2$	$[176, 13]$
$(C_4.(C_4 \times C_4)) \rtimes C_3$	$[192, 4]$
$(C_3 \times ((C_8 \times C_2) \rtimes C_2)) \rtimes C_2$	$[192, 29]$
$(C_3 \times ((C_8 \rtimes C_2) \rtimes C_2)) \rtimes C_2$	$[192, 34]$
$C_2 \times (((C_4 \times C_4) \rtimes C_3) \rtimes C_2)$	$[192, 944]$
$(C_8 \times A_4) \rtimes C_2$	$[192, 960]$
$(C_8 \times A_4) \rtimes C_2$	$[192, 961]$
$(((C_8 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2$	$[192, 963]$
$(((C_8 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2$	$[192, 964]$
$(D_8 \times A_4) \rtimes C_2$	$[192, 974]$
$(C_2 \times C_2 \times SL(2,3)) \rtimes C_2$	$[192, 980]$
$D_{10} \times (C_5 \rtimes C_4)$	$[200, 41]$
$A_5 \rtimes C_4$	$[240, 91]$
$(C_6 \times (C_5 \rtimes C_4)) \rtimes C_2$	$[240, 96]$
$C_2 \times C_2 \times A_5$	$[240, 190]$
$(((C_5 \rtimes C_8) \rtimes C_2) \rtimes C_2) \rtimes C_2$	$[320, 202]$
$((C_{20} \times C_2) \rtimes C_4) \rtimes C_2$	$[320, 206]$
$((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_4$	$[320, 1635]$
$((C_8 \times C_8) \rtimes C_3) \rtimes C_2$	$[384, 568]$
$((C_4.(C_4 \times C_4)) \rtimes C_3) \rtimes C_2$	$[384, 570]$
$(C_2 \times C_2 \times A_5) \rtimes C_2$	$[480, 951]$

Table 29: Groups with Strong Symmetric Genus 21 Continued

Structure Description	ID
$(C_3 \times C_3) \rtimes Q_8$	[72, 24]
$C_3 \times (C_3 \rtimes Q_8)$	[72, 26]
$C_3 \times C_3 \times D_8$	[72, 37]
$(C_5 \rtimes C_8) \rtimes C_2$	[80, 16]
$C_4 \times (C_7 \rtimes C_3)$	[84, 2]
$C_2 \times C_2 \times (C_7 \rtimes C_3)$	[84, 9]
DC_{22}	[88, 3]
DC_{23}	[92, 1]
$C_{18} \times S_3$	[108, 24]
$C_5 \times D_{22}$	[110, 4]
$(C_7 \rtimes C_9) \rtimes C_2$	[126, 1]
$C_3 \times ((C_7 \rtimes C_3) \rtimes C_2)$	[126, 7]
$S_3 \times (C_7 \rtimes C_3)$	[126, 8]
$C_6 \times (C_7 \rtimes C_3)$	[126, 10]
$C_6 \times D_{22}$	[132, 7]
$(C_3 \times (C_3 \rtimes C_8)) \rtimes C_2$	[144, 57]
$C_3 \times D_{48}$	[144, 72]
$C_3 \times ((C_3 \times D_8) \rtimes C_2)$	[144, 80]
$((C_3 \times C_3) \rtimes Q_8) \rtimes C_2$	[144, 118]
$(C_2 \times ((C_7 \rtimes C_3) \rtimes C_2)) \rtimes C_2$	[168, 9]
$(C_2 \times C_2 \times (C_7 \rtimes C_3)) \rtimes C_2$	[168, 11]
$C_{88} \rtimes C_2$	[176, 5]
$(C_{46} \times C_2) \rtimes C_2$	[184, 7]
$S_3 \times ((C_7 \rtimes C_3) \rtimes C_2)$	[252, 26]
$A_4 \times (C_7 \rtimes C_3)$	[252, 27]
$C_2 \times ((C_3 \times (C_7 \rtimes C_3)) \rtimes C_2)$	[252, 30]
$C_3 \times PSL(3, 2)$	[504, 157]
$((C_7 \rtimes C_3) \times A_4) \rtimes C_2$	[504, 160]
$(C_3 \times PSL(3, 2)) \rtimes C_2$	[1008, 881]

Table 30: Groups with Strong Symmetric Genus 22

Structure Description	ID
$C_2 \times (C_3 \rtimes Q_8)$	[48, 34]
$C_4.D_{16}$	[64, 49]
$(C_2 \times Q_{16}) \rtimes C_2$	[64, 191]
$(C_3 \rtimes Q_8) \rtimes C_4$	[96, 23]
$C_{24} \rtimes C_4$	[96, 24]
$C_{24} \rtimes C_4$	[96, 25]
$(C_2 \times (C_3 \rtimes C_8)) \rtimes C_2$	[96, 39]
$(C_4 \times (C_3 \rtimes C_4)) \rtimes C_2$	[96, 44]
$QA_5(2) \times C_3$	[96, 60]
$(C_{24} \times C_2) \rtimes C_2$	[96, 111]
$(D_8 \times S_3) \rtimes C_2$	[96, 118]
$(C_8 \times S_3) \rtimes C_2$	[96, 123]
$C_2 \times (C_2.S_4)$	[96, 188]
$(C_5 \times (C_3 \rtimes C_4)) \rtimes C_2$	[120, 10]
$(C_{32} \times C_2) \rtimes C_2$	[128, 149]
$(C_{32} \rtimes C_2) \rtimes C_2$	[128, 151]
$C_3 \times D_{46}$	[138, 2]
$D_8 \times D_{22}$	[176, 31]
$(C_4 \times D_{22}) \rtimes C_2$	[176, 34]
$C_4 \times D_{46}$	[184, 4]
$(C_{48} \times C_2) \rtimes C_2$	[192, 68]
$(C_3 \times (C_{16} \rtimes C_2)) \rtimes C_2$	[192, 77]

Table 31: Groups with Strong Symmetric Genus 23

Structure Description	ID
$C_9 \rtimes C_8$	$[72, 1]$
$C_5 \rtimes Q_{16}$	$[80, 18]$
DC_{24}	$[96, 8]$
$C_{25} \rtimes C_4$	$[100, 1]$
$C_{17} \times S_3$	$[102, 1]$
$C_{13} \times D_8$	$[104, 10]$
$(C_2 \times C_2) \rtimes C_{27}$	$[108, 3]$
$C_7 \times D_{16}$	$[112, 24]$
$C_3 \times ((C_{10} \times C_2) \rtimes C_2)$	$[120, 20]$
$C_5 \times D_{24}$	$[120, 23]$
$C_5 \times S_4$	$[120, 37]$
$D_{10} \times D_{14}$	$[140, 7]$
$(C_9 \times D_8) \rtimes C_2$	$[144, 16]$
$S_3 \times D_{26}$	$[156, 11]$
$C_{96} \rtimes C_2$	$[192, 8]$
$(C_{50} \times C_2) \rtimes C_2$	$[200, 8]$
$((C_2 \times C_2) \rtimes C_{27}) \rtimes C_2$	$[216, 21]$

Table 32: Groups with Strong Symmetric Genus 24

5 Comments and Open Problems

Our code relies heavily on the GAP Small Groups Library and the limitations of our code become clear as we compute for higher strong symmetric genera. Problems of this sort first surface for groups with strong symmetric genus twenty-five when we encounter groups of order 2016, which are not fully classified in the GAP Small Groups Library. Our code is not able to process the last valid signature $(0; 2, 3, 7)$ which gives rise to these groups of order 2016. We can resolve this problem by observing that there is no perfect group of order 2016. We know that if G is isomorphic to the image of a surface kernel epimorphism from a group given by a signature of the form $(0; m_1, m_2, m_3)$ where the m_i's are pairwise coprime, then G is a perfect group. We can check using the GAP Perfect Groups Library that there are no perfect groups of order 2016. Therefore, we can eliminate the signature $(0; 2, 3, 7)$ from consideration and the classification of the groups with strong symmetric genus twenty-five is complete.

Table 2 gives the number of groups for each strong symmetric genus up to twenty-five. Notice that the number of groups for strong symmetric genera of the form $2^n + 1$ for $n \in \mathbb{N}$ is higher than that for all preceding strong symmetric genera. For example, consider the 55 groups of strong symmetric genus nine. The number of groups for all previous strong symmetric genera is significantly less than 55. We believe this trend will continue for higher values of $2^n + 1$.

Strong Symmetric Genus	Number of Groups	Strong Symmetric Genus	Number of Groups
2	6	14	15
3	10	15	31
4	10	16	23
5	22	17	126
6	11	18	16
7	21	19	59
8	13	20	16
9	55	21	88
10	36	22	29
11	28	23	22
12	14	24	17
13	49	25	216

Table 33: Number of Groups for Each Strong Symmetric Genus

We found that the primary obstacle in our computation is the processing time of our functions. In particular, we noticed substantial increases in processing time for strong symmetric genera five, nine, and seventeen. We suspect that processing time is the highest for strong symmetric genera of the form $2^n + 1$ for $n \in \mathbb{N}$. Recall that the number of groups for these particular strong symmetric genera are also the highest out of all strong symmetric genera. We believe this is not a coincidence.

Furthermore, the processing time for strong symmetric genera thirteen and twenty-one are also relatively high. This leads us to believe that the compositeness of the number $g-1$ in the Riemann-Hurwitz formula may play a part in the overall computation process. Notice that the number of groups for strong symmetric genera thirteen and twenty-one are also relatively high in comparison to less composite numbers. Further investigation into this phenomenon may yield a trend that can determine when more resources are needed to run the code.

We conjecture that we can collect data for groups with strong symmetric genus up to sixty using the GAP perfect groups library. However, further modifications to our present code are needed in order to proceed.

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7 Appendix

7.1 Proof of Theorem 6

Theorem 5. *If Γ is a finitely-generated, abelian group and G is a finite abelian group, then $\Gamma \cong \mathbb{Z}^r \oplus H$ for some $r \geq 0$ and a finite group H, and there is an epimorphism $\Phi : \Gamma \to G$ if and only if, for each prime divisor p of $|G|$, there is an epimorphism $\Phi_p : \mathbb{Z}^r \oplus H_p \to G_p$, where H_p and G_p are the unique Sylow p-subgroups of H and G, respectively.*

Proof. Let $\Phi : \Gamma \to G$ be an epimorphism where p is a prime that divides $|G|$. Clearly $\Phi(H_p) \subseteq G_p$. Let $\Pi_p : G \to G_p$ be the canonical projection onto G_p and let $\Phi_p = \Pi_p \circ (\Phi|_{\mathbb{Z}^r \oplus H_p})$. Then Φ_p is a homomorphism. It is surjective because for each $x \in G$, we can take a pre-image y under Φ with $y = y_0 + \sum_{p \,|\, |H|}(y_p)$ where $y_0 \in \mathbb{Z}^r$. Then we have $\Phi_p(y_0 + y_p) = x$.

Now let $\Phi_p : \mathbb{Z}^r \oplus H_p \to G_p$ be epimorphisms for all primes p that divides $|G|$. Define $\Phi : \mathbb{Z}^r \oplus H \to G$ by mapping the generators of $\mathbb{Z}^r$ via $x \mapsto \sum_p \Phi_p(x)$ and mapping the generators of H_p via $x \mapsto \Phi_p(x)$.

Then Φ is a homomorphism. It is also surjective because of the following: it suffices to show that generators of all G_p are in the image of Φ. So take $x \in G_p$ and choose a pre-image y under Φ_p. Then $y = y_0 + y_p$ with $y_0 \in \mathbb{Z}^r$, $y_p \in H_p$ and $\Phi(y) = \Phi_p(y_p) + \sum_q \Phi_q(y_0) = \Phi_p(y) + \sum_{q \neq p} \Phi_q(y_0)$. The order of $\Phi_p(y)$ is a power of p, say p^k. Then the order of $\sum_{q \neq p} \Phi_q(y_0)$, say N, is coprime to p.

Now choose $\tilde{N}$ with the properties $\tilde{N} \cong 0$ (mod N) and $\tilde{N} \cong 1$ (mod p^k). Then $\Phi(\tilde{N} \cdot y) = \Phi_p(y) = x$.

□

7.2 Function Schematic

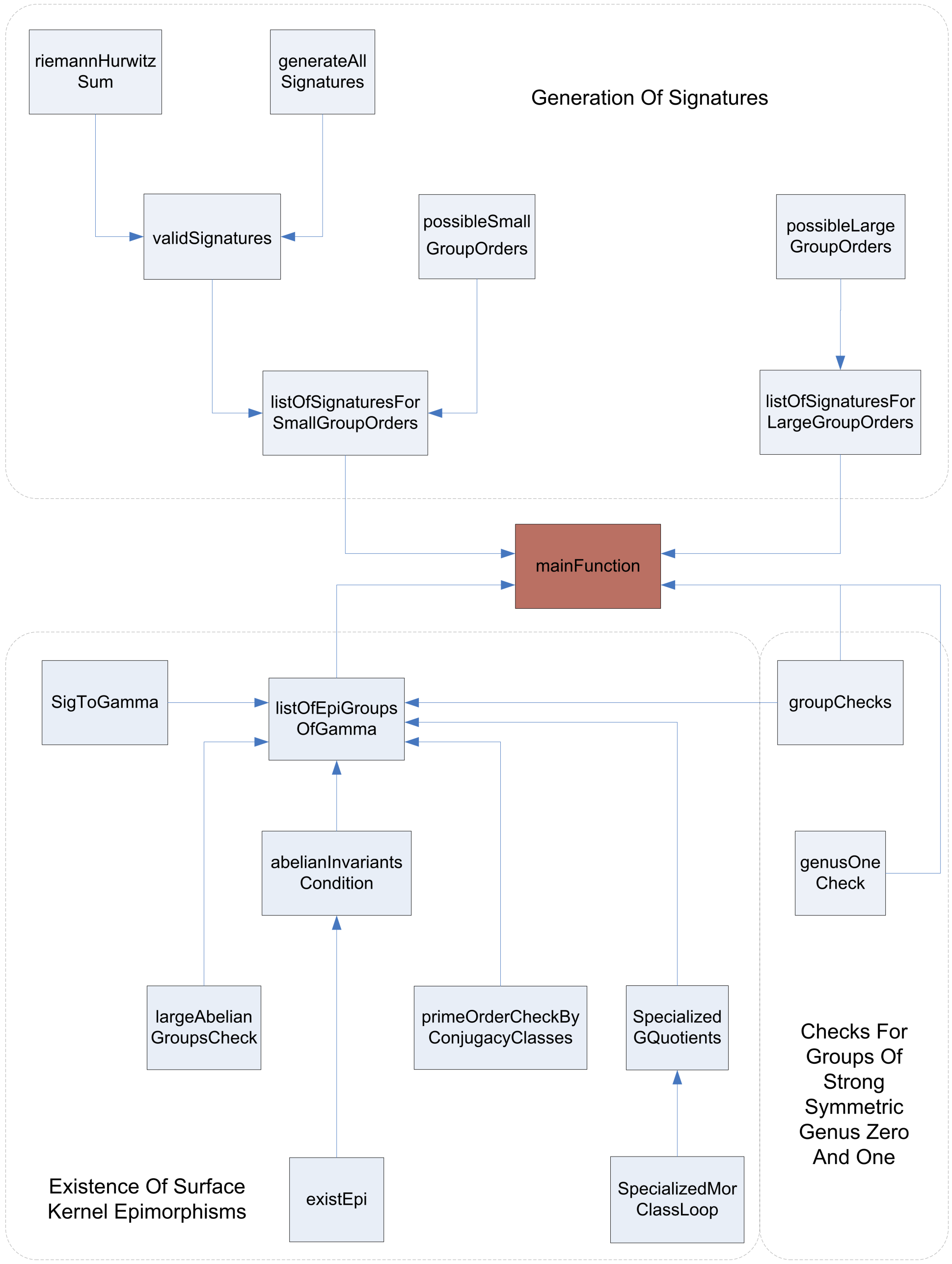

Figure 1: Function Schematic

References

[Bea01] A. F. Beardon. The geometry of Riemann surfaces. In *Topics on Riemann surfaces and Fuchsian groups (Madrid, 1998)*, volume 287 of *London Math. Soc. Lecture Note Ser.*, pages 1–28. Cambridge Univ. Press, Cambridge, 2001.

[Bre00] Thomas Breuer. *Characters and automorphism groups of compact Riemann surfaces*, volume 280 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2000.

[JS87] Gareth A. Jones and David Singerman. *Complex functions*. Cambridge University Press, Cambridge, 1987. An algebraic and geometric viewpoint.

[Kat92] Svetlana Katok. *Fuchsian groups*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.

[Leh64] Joseph Lehner. *Discontinuous groups and automorphic functions*. Mathematical Surveys, No. VIII. American Mathematical Society, Providence, R.I., 1964.

[Mas71] Bernard Maskit. On Poincaré's theorem for fundamental polygons. *Advances in Math.*, 7:219–230, 1971.

[MZ00] Coy L. May and Jay Zimmerman. Groups of small strong symmetric genus. *J. Group Theory*, 3(3):233–245, 2000.

[MZ05] Coy L. May and Jay Zimmerman. The groups of strong symmetric genus 4. *Houston J. Math.*, 31(1):21–35 (electronic), 2005.

[Sah69] Chih-han Sah. Groups related to compact Riemann surfaces. *Acta Math.*, 123:13–42, 1969.