Wave Turbulence

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Abstract

In this article, we state and review the premises on which a successful asymptotic closure of the moment equations of wave turbulence is based, describe how and why this closure obtains, and examine the nature of solutions of the kinetic equation. We discuss obstacles that limit the theory's validity and suggest how the theory might then be modified. We also compare the experimental evidence with the theory's predictions in a range of applications. Finally, and most importantly, we suggest open challenges and encourage the reader to apply and explore wave turbulence with confidence. The narrative is terse but, we hope, delivered at a speed more akin to the crisp pace of a Hemingway story than the wordjumblingtumbling rate of a Joycean novel.

1. INTRODUCTION

Turbulence theory is about understanding the long-time statistical behavior of solutions of nonlinear field equations with additional external forcing and dissipation, e.g., the forced high–Reynolds number Navier-Stokes equations. Principal aims are to understand transport, such as the average flux of mass down a pipe as a function of the pressure head, and the spectral distributions that carry the energy or other conserved densities from the scales at which they are injected to the scales at which they are dissipated. Alas, despite some success, such as Kolmogorov's four-fifth's law (Frisch 1996) and predictions based on scaling arguments, quantitative results are hard to come by. The main obstacle is the lack of a consistent statistical closure of the infinite hierarchy of moment equations.

In contrast, the hierarchy of moment equations for wave turbulence, the turbulence of a sea of weakly interacting dispersive wave trains (the analogs of eddies), has a natural asymptotic closure (Benney & Newell 1967, 1969; Benney & Saffman 1966; Newell et al. 2001). All the long-time statistical quantities, the energy density, the nonlinear frequency renormalization, the long-time behaviors of the cumulants, and the structure functions can be calculated from a set of core particle densities $\{n^{(r)}(\mathbf{k}, t)\}$, which are proportional to the Fourier transforms of two-point averages. For simplicity, we use examples for which there is only one such density, $n_{\mathbf{k}} \equiv n(\mathbf{k}, t)$, such as the wave-action density in ocean gravity waves. Moreover, and central to the success of wave turbulence, this number density $n_{\mathbf{k}}$ satisfies a closed (Boltzmann-like, kinetic) equation (Hasselmann 1962, 1963a,b; Zakharov et al. 1992) with a form revealing that, to leading order, all transport is carried by N-wave resonances ($N = 3, 4, \ldots$). Furthermore, the kinetic equation admits stationary solutions that capture not only the entropy-maximizing, equipartition thermodynamic behavior of isolated systems, but also the finite-flux Kolmogorov behavior of nonisolated ones in which conserved densities such as energy and particle number flow from sources (in k space) to sinks (Zakharov & Filonenko 1967a,b). These Kolmogorov-Zakharov (KZ) solutions are the analogs of the familiar Kolmogorov energy spectrum prediction $E(k) = c\,P^{2/3}k^{-5/3}$ of high–Reynolds number hydrodynamics. In addition, the kinetic equation has time-dependent solutions of a self-similar type that describe how the stationary solutions are accessed.

Furthermore, nature and laboratories abound with applications in which wave turbulence theory should obtain. The most familiar example is that of ocean gravity waves on a wind-stirred sea, but, in principle, its signatures should also be found in magnetohydrodynamic waves in astrophysical contexts, in Rossby-like waves in the atmospheres of rotating planets, in the formation of condensates, in capillary waves, in acoustic waves, and in the music of vibrations on large, thin, elastic sheets. But does the hand of wave turbulence really guide the behavior of ocean waves, capillary waves, and all the examples above for which one might expect the theory to apply? Although there have been notable successes, the theory also has limitations. In short, both the good and bad news is that the wave turbulence story is far from over. One might compare its current standing, particularly with respect to experiments, to the situation regarding pattern formation in the late 1960s. By that time, there had been many theoretical advances, but the experimental confirmation of the predictions fell very much in the "looks like" category. It took the pioneering experimental works of Ahlers, Croquette, Fauve, Gollub, Libchaber, and Swinney in the mid- to late 1970s (which overcame some extraordinary challenges of managing long-time control of external parameters) to put some of the advances on a firm footing. For wave turbulence, we are only at the beginning of the experimental stage.

2. THE ASYMPTOTIC CLOSURE

2.1. The Set Up

We begin with equations governing the Fourier transforms $A^s_{\mathbf{k}} \equiv A^s(\mathbf{k}, t)$ of suitable combinations of the field variables $u^s(\mathbf{x}, t)$ chosen to diagonalize the linearized equations of the dynamical system under study,

$$\frac{dA^s_{\mathbf{k}}}{dt} - i\omega^s_{\mathbf{k}} A^s_{\mathbf{k}} = \sum_{r=2} \epsilon^{r-1} \sum_{s_1 \ldots s_r} \int L^{ss_1 \ldots s_r}_{\mathbf{k}\mathbf{k}_1 \ldots \mathbf{k}_r} A^{s_1}_{\mathbf{k}_1} \ldots A^{s_r}_{\mathbf{k}_r} \delta(\mathbf{k}_1 + \cdots + \mathbf{k}_r - \mathbf{k}) d\mathbf{k}_1 \ldots d\mathbf{k}_r, \qquad (1)$$

where $0 < \epsilon \ll 1$ is a small parameter (e.g., the wave slope); $\delta(x)$ is the Dirac delta function; and $\omega^s_{\mathbf{k}}$ is the linear dispersion relation, where s enumerates the set of cardinality $\{s\}$ of frequencies associated with wave vector $\mathbf{k}$. For gravity waves, $\omega^s_k = s\sqrt{gk}$, where $k = |\mathbf{k}|$, $s = \pm 1$, and $\{s\} = 2$, connoting waves with phases $\mathbf{kx} \pm \omega_{\mathbf{k}} t$. For magnetohydrodynamic waves, $\{s\} = 6$. The right-hand side of Equation 1 is obtained as convolutions of all the nonlinear terms and the coefficients $L^{ss_1 \ldots s_r}_{\mathbf{k}\mathbf{k}_1 \ldots \mathbf{k}_r}$ are symmetrized over $(1 \ldots r)$. The Fourier transforms $A^s_{\mathbf{k}}$ are generalized functions because the $u^s(\mathbf{x}, t)$ are bounded fields that do not decay at large $\mathbf{x}$. However, combinations of averages, called cumulants, will have ordinary Fourier transforms, at least initially, although they will develop weak (order ϵ^r, $r \geq 1$), but important, calculable generalized function behaviors over long times.

2.1.1. Equation 1 is easy to derive. Equation 1 does not require a priori knowledge of any Hamiltonian structure. All it requires is the diagonalization of the linear part of the equations and the ability to calculate convolutions. For example, atmospheric Rossby waves are described by the conservation of potential vorticity $(\partial_t + \psi_y \partial_x - \psi_x \partial_y)(\nabla^2 - \alpha^2)\psi + \beta \partial_x \psi = 0$, where ψ is the velocity stream function, α^{-1} is a length scale, and β measures the northward change of Earth's rotation. Here $\{s\} = 1$, $\omega(\mathbf{k}) = \beta k_x/(\alpha^2 + k^2)$, $k^2 = k_x^2 + k_y^2$, and $\epsilon L_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = (\mathbf{k}_1 \times \mathbf{k}_2)(\mathbf{k}_1^2 - \mathbf{k}_2^2)/(2(\alpha^2 + k^2))$. All higher-order coefficients are zero. For the nonlinear Schrödinger (NLS) equation, $u_t = -i\nabla^2 u - i\lambda u^2 u^*$, we take $u = u^+ = \int A^+_{\mathbf{k}} \exp(i\mathbf{kx}) d\mathbf{k}$, $u^* = u^- = \int A^-_{\mathbf{k}} \exp(i\mathbf{kx}) d\mathbf{k}$, so that $A^{+*}_{-\mathbf{k}} = A^-_{\mathbf{k}}$. Then $\omega^s_{\mathbf{k}} = sk^2$, $s = \pm 1$, and $\epsilon L^{ss_1s_2s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = -(i\lambda s/3) P_{123}(\delta_{s_1 s}\delta_{s_2 s}\delta_{s_3 -s})$, where P_{123} is the cyclic permutation over 1, 2, 3, and $\delta_{ss'}$ is the Kronecker delta. For gravity-capillary waves, we write the Fourier transforms of the surface elevation $\eta(\mathbf{x}, t)$ and velocity potential at the mean surface level as $\nu_{\mathbf{k}}^{-1}\sqrt{\omega_{\mathbf{k}}/2}(A^+_{\mathbf{k}} + A^-_{\mathbf{k}})$ and $(i\nu_{\mathbf{k}}/(\sqrt{2\omega_{\mathbf{k}}}))(A^+_{\mathbf{k}} - A^-_{\mathbf{k}})$, where $\omega^s_{\mathbf{k}} = s\omega_{\mathbf{k}}$; $\omega^2_{\mathbf{k}} = gk + (S/\rho)k^3 = k\nu^2_{\mathbf{k}}$; and g, S, and ρ are gravity, surface tension, and water density, respectively.

2.1.2. Symmetries. If there is only one physical process, such as in ocean gravity waves, $\omega_{\boldsymbol{k}}$ and $L^{ss_1 \ldots s_r}_{\mathbf{k}\mathbf{k}_1 \ldots \mathbf{k}_r}$ are homogeneous functions with degrees α and γ_r, respectively; i.e., $\omega(\lambda\mathbf{k}) = \lambda^\alpha \omega(\mathbf{k})$ and $L^{ss_1 \ldots s_r}_{\lambda\mathbf{k}\lambda\mathbf{k}_1 \ldots \lambda\mathbf{k}_r} = \lambda^{\gamma_r} L^{ss_1 \ldots s_r}_{\mathbf{k}\mathbf{k}_1 \ldots \mathbf{k}_r}$. For gravity waves, $\alpha = 1/2$, $\gamma_2 = 7/4$, and $\gamma_3 = 3$. For capillary waves, $\alpha = 3/2$, $\gamma_2 = 9/4$, and $\gamma_3 = 3$. In such situations, Equation 1 has a symmetry, $\mathbf{k} \to \mathbf{K} = \lambda\mathbf{k}$, $t \to T = \lambda^{-\alpha} t$, and $A^s_{\mathbf{k}} = \lambda^b B^s_{\mathbf{K}}$, under which it is invariant if $b = d + (\gamma_r - \alpha)/(r-1)$ for all $r, d = \dim(\mathbf{k})$. For what power law $n^s_{\mathbf{k}} \propto k^{-\alpha x}$ does the two-point average $\langle A^s_{\mathbf{k}} A^{-s}_{\mathbf{k}'} \rangle = \delta(\mathbf{k} + \mathbf{k}') n^s_{\mathbf{k}}$ inherit the above symmetry? By writing $A^s_{\mathbf{k}}$ in terms of $B^s_{\mathbf{K}}$ and $\mathbf{k}$ in terms of $\mathbf{K}$, one finds that $\alpha x = d + 2(\gamma_r - \alpha)/(r-1)$. For gravity waves, $\alpha x = 9/2$, which corresponds to a Phillips' (1985) spectrum of sharp, crested waves. We meet the generalized Phillips' spectrum again when we discuss the validity of the theory and the regularization of the KZ spectrum at high wave numbers.

2.1.3. Properties. For simplicity of presentation, we take $\omega^s_{\mathbf{k}} = s\omega(|\mathbf{k}|)$, $s = \pm 1$. Then $L^{ss_1 \ldots s_r}_{\mathbf{k}\mathbf{k}_1 \ldots \mathbf{k}_r} = -L^{-s-s_1 \ldots -s_r}_{-\mathbf{k}-\mathbf{k}_1 \ldots -\mathbf{k}_r} = -(L^{ss_1 \ldots s_r}_{\mathbf{k}\mathbf{k}_1 \ldots \mathbf{k}_r})^*$ will be symmetric on $1 \ldots r$ and, on resonant manifolds, $\sum_{j=1}^r s_j \omega_j = s\omega$, $\sum_{j=1}^r \mathbf{k}_j = \mathbf{k}$, equal to $(s/s_1) L^{s_1 s -s_2 \ldots -s_r}_{\mathbf{k}_1 \mathbf{k} -\mathbf{k}_2 \ldots -\mathbf{k}_r}$. Also $L^{ss_1 \ldots s_r}_{\mathbf{0}\mathbf{k}_1 \ldots \mathbf{k}_r} = 0$, so that a zero initial mean will remain unchanged. We note that the NLS equation does not satisfy this. If H is the Hamiltonian

$$\int \omega_{\mathbf{k}} A^{+}_{\mathbf{k}} A^{-}_{-\mathbf{k}} d\mathbf{k} + \sum_{r=2}^{\infty} \epsilon^{r-1} \sum_{s_1 \ldots s_{r+1}} \int H^{s_1 \ldots s_{r+1}}_{\mathbf{k}_1 \ldots \mathbf{k}_{r+1}} A^{s_1}_{\mathbf{k}_1} \ldots A^{s_r}_{\mathbf{k}_r} \delta(\mathbf{k}_1 + \cdots + \mathbf{k}_{r+1}) d\mathbf{k}_1 \ldots d\mathbf{k}_{r+1}$$

and $A^{s}_{\mathbf{k}}$, $A^{-s}_{-\mathbf{k}} = A^{s^*}_{\mathbf{k}}$ are canonically conjugate, then Equation 1 is $dA^{s}_{\mathbf{k}}/dt = is\,\delta H/\delta A^{-s}_{-\mathbf{k}}$ and $L^{ss_1 \ldots s_r}_{\mathbf{k}\mathbf{k}_1 \ldots \mathbf{k}_r} = is(r+1) H^{s_1 \ldots s_r - s}_{\mathbf{k}_1 \ldots \mathbf{k}_r - \mathbf{k}}$.

2.2. The Cumulant Hierarchy

We seek to understand the long-time behavior of the statistical cumulants that are formed by suitable combinations of moments. Cumulants have the advantages of decaying sufficiently fast to zero (at least initially) as the separations in the configurations of N-space points become large so that they initially have ordinary Fourier transforms and the property that all cumulants of order three and higher are zero for joint Gaussian distributions. We now arrive at the first premise, which is assumed without much discussion in almost all turbulence theories but which turns out to be spontaneously violated in situations in which wave turbulence fails (see Section 5).

The first premise is that the field is spatially homogeneous. This means that statistical correlations such as $\langle u(\mathbf{x}) u(\mathbf{x}+\mathbf{r})\rangle$ (assume $\langle u(\mathbf{x})\rangle = 0$) only depend on the relative geometry, here $\mathbf{r}$, of the configuration. It also means that statistical (ensemble) averages are equivalent to averages over the base coordinate, here $\mathbf{x}$. As a direct consequence, the statistical averages of products of the generalized Fourier transforms $A^{s}_{\mathbf{k}}$ are Dirac delta correlated.

In particular, for zero mean fields, $\langle A^{s}_{\mathbf{k}} A^{s'}_{\mathbf{k}'}\rangle = \delta(\mathbf{k} + \mathbf{k}') Q^{ss'}(\mathbf{k}')$, $\langle A^{s}_{\mathbf{k}} A^{s'}_{\mathbf{k}'} A^{s''}_{\mathbf{k}''}\rangle = \delta(\mathbf{k} + \mathbf{k}' + \mathbf{k}'') Q^{ss's''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$, $\langle A^{s}_{\mathbf{k}} A^{s'}_{\mathbf{k}'} A^{s''}_{\mathbf{k}''} A^{s'''}_{\mathbf{k}'''}\rangle = \delta(\mathbf{k} + \mathbf{k}' + \mathbf{k}'' + \mathbf{k}''') Q^{ss's''s'''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'', \mathbf{k}''') + \delta(\mathbf{k} + \mathbf{k}') Q^{ss'}(\mathbf{k}')\delta(\mathbf{k}'' + \mathbf{k}''') Q^{s''s'''}(\mathbf{k}''') + \delta(\mathbf{k} + \mathbf{k}'') Q^{ss''}(\mathbf{k}'')\delta(\mathbf{k}' + \mathbf{k}''') Q^{s's'''}(\mathbf{k}''') + \delta(\mathbf{k}' + \mathbf{k}''') Q^{ss'''}(\mathbf{k}''')\delta(\mathbf{k}' + \mathbf{k}'') Q^{s's''}(\mathbf{k}'')$. The cumulant hierarchy is formed by multiplying Equation 1 by $A^{s'}_{\mathbf{k}'}$; replacing $s, \mathbf{k}$ by $s', \mathbf{k}'$; and adding and averaging. We use the symbols $P_{00'}$, $P_{00'0''}$ to connote the cyclical sums over $(s, \mathbf{k})$, $(s', \mathbf{k}')$, $(s'', \mathbf{k}'')$, and so on. Including only quadratic nonlinearities, the first two equations in the hierarchy are

$$\frac{dQ^{ss'}(\mathbf{k}')}{dt} - i(s\omega + s'\omega')Q^{ss'}(\mathbf{k}') = \epsilon P_{00'} \sum_{s_1 s_2} \int L^{ss_1 s_2}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} Q^{s's_1 s_2}(\mathbf{k}', \mathbf{k}_1, \mathbf{k}_2)\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d\mathbf{k}_1 d\mathbf{k}_2, \tag{2}$$

where $\mathbf{k} + \mathbf{k}' = 0$ and $\omega' = \omega(|\mathbf{k}'|)$; for $\mathbf{k} + \mathbf{k}' + \mathbf{k}'' = 0$,

$$\begin{aligned} &\frac{dQ^{ss's''}(\mathbf{k}, \mathbf{k}'\mathbf{k}'')}{dt} - i(s\omega + s'\omega' + s''\omega'')Q^{ss's''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \\ &= \epsilon P_{00'''} \sum_{s_1 s_2} \int L^{ss_1 s_2}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} Q^{s's''s_1 s_2}(\mathbf{k}', \mathbf{k}'', \mathbf{k}_1, \mathbf{k}_2)\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d\mathbf{k}_1 d\mathbf{k}_2 \\ &\quad + 2\epsilon P_{00'0''} \sum_{s_1 s_2} L^{ss_1 s_2}_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''} Q^{s_1 s'}(\mathbf{k}') Q^{s_2 s''}(\mathbf{k}''). \end{aligned} \tag{3}$$

2.3. The Solution Strategy and Asymptotic Closure

We solve the cumulant equations iteratively in power series in ϵ; i.e.,

$$Q^{(N)ss'\ldots s^{(N-1)}}(\mathbf{k}, \mathbf{k}', \ldots \mathbf{k}^{(N-1)}, t) = q_0^{(N)ss'\ldots s^{(N-1)}}(\mathbf{k}, \mathbf{k}', \ldots \mathbf{k}^{(N-1)}, 0) e^{i(s\omega + s'\omega' + \ldots)t} + \epsilon Q_1^{(N)} + \cdots. \tag{4}$$

We then calculate the long-time behavior, i.e., $t \to \infty$, $\tau_r = \epsilon^r t$ finite for $r = 2, 4, \ldots$ of the successive iterates. The iterates will contain some terms that will be bounded in t and others that will grow in time. By allowing the leading-order cumulants $q_0^{(N)}$ to vary slowly with time by writing $q_0^{(N)}(0)$ as a reverse Taylor series $q_0^{(N)}(\tau = \epsilon^r t) - \tau \partial q_0^{(N)}(\tau)/\partial \tau + \cdots$, we can remove these secular terms and render the asymptotic expansion given in Equation 4 uniformly valid in time. Closure occurs because the secular terms in $Q^{(N)}$ involve only cumulants less than or equal to N.

Carrying out this procedure involves studying the long-time asymptotics of integrals such as

$$Q_1^{ss'}(\mathbf{k}, \mathbf{k}') = P_{00'} \sum_{s_1 s_2} \int L_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{ss_1s_2} q_0^{(3)s's_1s_2}(\mathbf{k}', \mathbf{k}_1, \mathbf{k}_2, 0) \Delta(s_1\omega_1 + s_2\omega_2 - s\omega, t) \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d\mathbf{k}_1 d\mathbf{k}_2, \tag{5}$$

where $\Delta(x, t) = \int_0^t \exp(ixt)dt = (\exp(ixt) - 1)/(ix)$. As we progress up the series of iterates, we find we need to evaluate integrals of the form $\int_{-\infty}^{\infty} f(x)\Delta(x, t)dx$, $\int_{-\infty}^{\infty} f(x)\Delta(x, t)\Delta(-x, t)dx$, $\int_{-\infty}^{\infty} f(x, y)\mathrm{E}(x, y, t)dxdy$, and $\int_{-\infty}^{\infty} f(x)\mathrm{E}(x, 0, t)dx$, where $\mathrm{E}(x, y, t) = \int_0^t \Delta(x - y, t')\exp(iyt')dt'$. Here we list those necessary for the first closure. Under very weak assumptions on the smoothness of $f(x)$ [we will assume $f(x)$ and its first derivative exist and are absolutely integrable], we find (Benney & Newell 1969) $\exp(ixt) \sim 0$, $\Delta(x, t) \sim \tilde{\Delta}(x) = \pi\delta(x)\mathrm{sgn}(t) + i\mathrm{P}(1/x)$, $\Delta(x, t)\Delta(-x, t) \sim 2\pi\delta(x)\mathrm{sgn}(t) + 2\mathrm{P}(1/x)\frac{\partial}{\partial x}$, $\mathrm{E}(x, 0, t) \sim \tilde{\Delta}(x)(t - i\frac{\partial}{\partial x})$, and $\mathrm{E}(x, y, t) \sim \tilde{\Delta}(x)\tilde{\Delta}(y)$. The first asymptotic result is the Riemann-Lebesgue (RL) lemma, $\lim_{t\to\infty} \int f(x)\exp(ixt)dx = 0$; the second means $\lim_{t\to\infty} \int_{\infty}^{\infty} f(x)\Delta(x, t)dx = \pi f(0)\mathrm{sgn}(t) + i\mathrm{P}\int_{-\infty}^{\infty} x^{-1} f(x)dx$, where P denotes the Cauchy principal value. The x's in these expressions are combinations $h(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$ of frequencies such as $\pm\omega(\mathbf{k}_1) \pm \omega(\mathbf{k}_2) \mp \omega(\mathbf{k})$, $\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k} = 0$. We check that the coordinate x crosses the resonant manifold $h \equiv \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) - \omega(\mathbf{k}) = 0$, $\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k} = 0$, transversely, namely that $\nabla_{\mathbf{k}_1} h \neq \mathbf{0}$ on $h = 0$. In some cases (e.g., acoustic waves), the zero of h is double and the asymptotic analysis needs to be done more carefully (Newell & Aucoin 1971).

We now introduce the second and third premises. In the second premise, at some time $t = 0$, e.g., at the beginning of the storm action, the statistical cumulants such as $\{u(\mathbf{x})u(\mathbf{k} + \mathbf{r})\}$ [equal to $\langle u(\mathbf{x})u(\mathbf{k} + \mathbf{r})\rangle - \langle u(\mathbf{x})\rangle^2$ if the mean is nonzero] decay sufficiently rapidly as the separations $\mathbf{r}, \mathbf{r}', \ldots$ become large independently so as to admit ordinary Fourier transforms. This is a weak assumption and perfectly reasonable as one fully expects that, at least initially, distant points are statistically uncorrelated.

In the third premise, in carrying out the asymptotic analysis, we treat all leading-order cumulants $q_0^{(N)}$ as being constant in time or at worst slowly varying compared with the fast time t measured in units of inverse frequency. In other words, we require timescale separation. As $\epsilon \to 0$, the linear time $t_L(\mathbf{k})$ must be much less than the nonlinear time $t_{NL}(\mathbf{k})$ on which the leading-order cumulants evolve.

Because the triple correlation $\langle u^s(\mathbf{x})u^{s'}(\mathbf{x} + \mathbf{r})u^{s''}(\mathbf{x} + \mathbf{r}')\rangle$ tends to zero very rapidly at $t = 0$, its Fourier transform $q_0^{(3)ss's''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'', \mathbf{0})$, $\mathbf{k} + \mathbf{k}' + \mathbf{k}'' = \mathbf{0}$, is sufficiently smooth so that $Q_1^{(2)ss'}(\mathbf{k}, \mathbf{k}')$, $\mathbf{k} + \mathbf{k}' = \mathbf{0}$, is bounded as $t \to 0$. The leading-order contribution of the third-order cumulant does not induce any secular behavior in the second-order moment. As far as the long-time statistics is concerned, it could have very well been taken to be zero.

Next, when we compute the first correction $Q_1^{ss's''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$ to the third-order cumulant, we find that it is given by an integral containing the leading-order cumulant of fourth-order $q_0^{s's''s_1s_2}(\mathbf{k}', \mathbf{k}'', \mathbf{k}_1, \mathbf{k}_2, \mathbf{0})$ multiplied by $\Delta(s_1\omega_1 + s_2\omega_2 - s\omega)$ and a term that integrates out to $2P_{00'0''}\{\sum_{s_1s_2} L_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''}^{ss_1s_2} q_0^{s_1s'}(\mathbf{k}')q_0^{s_2s''}(\mathbf{k}'')\Delta(s_1\omega' + s_2\omega'' - s\omega)\exp(i(s\omega + s'\omega' + s''\omega'')t)\}$. The first integral is bounded. To take the long-time limit of the second, one must return to physical space $R_1^{ss's''}(\mathbf{r}, \mathbf{r}') = \int Q_1^{ss's''}\exp(i(\mathbf{k}'\mathbf{r} + \mathbf{k}'\mathbf{r}'))d\mathbf{k}'d\mathbf{k}''$, $\mathbf{k} + \mathbf{k}' + \mathbf{k}'' = \mathbf{0}$. Only when $s_1 = -s'$, $s_2 = -s''$, do we obtain a nonvanishing contribution (the other combinations vanish by the RL lemma), which we call the asymptotic survivor,

$$Q_1^{ss's''} \sim 2P_{00'0''}L^{s-s'-s''}_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''}q_0^{-s's'}(\mathbf{k}')q_0^{-s''s''}(\mathbf{k}'')\tilde{\Delta}(s\omega+s'\omega'+s''\omega''). \tag{6}$$

Thus the third- (fourth-) order physical space cumulant $R_1^{ss's''}(\mathbf{r},\mathbf{r}') = \langle u^s(\mathbf{x})u^{s'}(\mathbf{x}+\mathbf{r})u^{s''}(\mathbf{x}+\mathbf{r}')\rangle$ will see a long-time non-Gaussian behavior only at order ϵ (ϵ^2) and, more important for closure, this asymptotic survivor only depends on the leading-order behavior $q_0^{-ss}(\mathbf{k})$ (which for simplicity we assume to be s independent and call $n_{\mathbf{k}}$) of the number density $n_{\mathbf{k}}$.

Likewise, $Q_2^{s's}(\mathbf{k})$, the second correction to the two-point cumulant, will be given by the integration over $(0, t)$ of an integral involving $Q_1^{(3)s's_1s_2}(\mathbf{k},\mathbf{k}',\mathbf{k}_1,\mathbf{k}_2)$. In the long-time limit, the secular terms that arise depend only on integrals containing quadratic products of $n_{\mathbf{k}}$ if $s'=-s$ (directly related to the asymptotic survivor given in Equation 6) or, if $s'=s$, as a product $iQ_0^{s's}(\mathbf{k})(\Omega_2^s[n_{\mathbf{k}}]+\Omega_2^{s'}[n_{\mathbf{k}'}])$, where $\Omega_2^s[n_{\mathbf{k}}]$ is an integral over $n_{\mathbf{k}}$ given by

$$\Omega_2^s[n_{\mathbf{k}}] = \sum_{s_2}\int\Bigg(-3iL^{sss_2-s_2}_{\mathbf{k}\mathbf{k}\mathbf{k}_2-\mathbf{k}_2} - 4\sum_{s_1}\int L^{ss_1s_2}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}L^{s_1s-s_2}_{\mathbf{k}_1\mathbf{k}-\mathbf{k}_2}$$
$$\left(P\frac{1}{s_1\omega_1+s_2\omega_2-s\omega} + i\pi\,\mathrm{sgn}(t)\delta(s_1\omega_1+s_2\omega_2-s\omega)\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\right)\Bigg)n_{\mathbf{k}_2}d\mathbf{k}_1d\mathbf{k}_2, \tag{7}$$

where the cubic coefficient $L^{ss_1s_2s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ has been restored. In a similar fashion, the only unbounded (grows as t) contribution to $Q_2^{(N)ss'\ldots s^{(N-1)}}[\mathbf{k},\cdots\mathbf{k}^{(N-1)}]$ has the form $iQ_0^{(N)ss'\cdots}(\Omega_2^s[n_{\mathbf{k}}]+\Omega_2^{s'}[n_{\mathbf{k}'}]+\ldots+\Omega_2^{s^{(N-1)}}[n_{\mathbf{k}^{(N-1)}}])$.

We remove the secular terms by allowing the hitherto time-independent leading-order cumulants $n_{\mathbf{k}}(q_0^{-ss}(\mathbf{k}))$, $q_0^{(N)ss'\ldots s^{(N-1)}}(\mathbf{k},\ldots)$ to vary slowly in time. The resulting equation for $dn_{\mathbf{k}}/dt$, which contains only $n_{\mathbf{k}}$, is the kinetic equation. The equations for $q_0^{(N)}$ can be (miraculously) simultaneously solved for all N by renormalizing the frequency $s\omega_{\mathbf{k}} \to s\omega_{\mathbf{k}} + \epsilon^2\Omega_2^s[n_{\mathbf{k}}]$. A natural asymptotic closure is thereby achieved. In physical terms, the reasons for the natural asymptotic closure are the following: For times much greater than $t_L(\mathbf{k})$ but shorter than $t_{NL} = n_{\mathbf{k}}(dn_{\mathbf{k}}/dt)^{-1}$, the field dynamics is dominated by linear, dispersive wave propagation. Even acoustic waves in dimensions greater than one are dispersive, as are oppositely traveling Alfvén waves! The statistics of the field at any cluster of points is governed approximately by a linear superposition of independent contributions that have traveled from afar, and, by central limit theorem arguments, the field relaxes close to [within $\mathcal{O}(\epsilon)$] joint Gaussian. The second premise in Section 2.3 and the RL lemma ensure that this result is also manifested by the mathematics. But the dynamics is nonlinear. The third- and higher-order cumulants are regenerated on the longer timescale t_{NL} by a combination of higher-order cumulants and products of equal and lower-order cumulants. The asymptotic closure occurs because, in the long-time limit, the latter dominate the former.

Now we are in a position to discuss the connection with the so-called random phase approximation, which certain authors employ unnecessarily in the derivation of the kinetic equation. In calculating $Q^{-ss}(\mathbf{k})$, we found that the secular terms in the various iterates depend only on $n_{\mathbf{k}}$. All terms involving q_0^N, $n \geq 3$, were bounded. Although we did not, we could have ignored them for this part of the calculations. They have no long-time cumulative effect. This means that, had we initially expanded the Fourier amplitudes $A^s_{\mathbf{k}}$ as $A^s_{\mathbf{k}0} + \epsilon A^s_{\mathbf{k}1} + \cdots$, then all product averages $\langle A^s_{\mathbf{k}0}A^{s'}_{\mathbf{k}'0}A^{s''}_{\mathbf{k}''0}\ldots\rangle$ could be decomposed as if the zeroth-order amplitudes had random phases or as if they were joint Gaussian so that only products of two-point functions survive (Wick's theorem). But let us be clear. Averages of products of the complex amplitudes $A^s_{\mathbf{k}}$ cannot be expanded as if they had random phases or as if they were joint Gaussian. We have seen that moments such as $P_{00'0''}\langle A^s_{\mathbf{k}1}A^{s'}_{\mathbf{k}'0}A^{s''}_{\mathbf{k}''0}\rangle$ have asymptotic survivors that play central roles in both producing a nontrivial closure and inducing over long times weakly decaying long-distance correlations. Only the zeroth-order products of the amplitudes can be expanded as if they had random phases. In the language of cumulant discard schemes, what we have shown using only very weak assumptions (the three premises listed in Sections 2.2 and 2.3) is that one could have found the correct kinetic equations to order ϵ^2 by discarding all cumulants of order four and higher, and to order ϵ^4 by discarding all cumulants of order six and higher, and so on.

The asymptotic closure can be carried out systematically, as long as the asymptotics are done correctly, to all even orders in ϵ. The result is a kinetic equation

$$\frac{dn_{\mathbf{k}}}{dt} = T\,[n_{\mathbf{k}}] = \epsilon^2 T_2[n_{\mathbf{k}}] + \epsilon^4 T_4[n_{\mathbf{k}}] + \cdots, \tag{8}$$

$T_{2r} = \partial n_{\mathbf{k}}/\partial \tau_r$, and a frequency renormalization

$$s\,\omega_{\mathbf{k}} \to s\,\omega_{\mathbf{k}} + \epsilon^2 \Omega_2^s[n_{\mathbf{k}}] + \epsilon^4 \Omega_4^s[n_{\mathbf{k}}] + \cdots, \tag{9}$$

$\mathrm{sgn}(t)\mathrm{Im}\Omega_{2N}^s[n_{\mathbf{k}}] > 0$, together with expressions for regenerated behaviors of the higher-order cumulants that only depend on $n_{\mathbf{k}}$ and from which all the Fourier and physical space statistics can be calculated. The expression for $T_2[n_{\mathbf{k}}]$ is

$$T_2[n_{\mathbf{k}}] = 4\pi\ \mathrm{sgn}(t) \sum_{s_1 s_2} \int L^{s s_1 s_2}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} \left(\frac{L^{-s-s_1-s_2}_{-\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}}{n_{\mathbf{k}}} + \frac{L^{s_1 s-s_2}_{\mathbf{k}_1\mathbf{k}-\mathbf{k}_2}}{n_{\mathbf{k}_1}} + \frac{L^{s_2 s-s_1}_{\mathbf{k}_2\mathbf{k}-\mathbf{k}_1}}{n_{\mathbf{k}_2}} \right)$$
$$\delta(s_1\omega_1 + s_2\omega_2 - s\,\omega)\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})d\mathbf{k}_1 d\mathbf{k}_2. \tag{10}$$

When three-wave resonances are forbidden, $T_2[n_{\mathbf{k}}] \equiv 0$, and

$$T_4[n_{\mathbf{k}}] = 12\pi\ \mathrm{sgn}(t) \sum_{s_1 s_2 s_3} \int G^{s s_1 s_2 s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} \left(\frac{G^{-s-s_1-s_2-s_3}_{-\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3}}{n_{\mathbf{k}}} + P_{123} \frac{G^{s_1 s-s_2-s_3}_{\mathbf{k}_1\mathbf{k}-\mathbf{k}_2-\mathbf{k}_3}}{n_{\mathbf{k}_1}} \right)$$
$$\delta(s_1\omega_1 + s_2\omega_2 + s_3\omega_3 - s\,\omega)\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k})d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}, \tag{11}$$

where

$$G^{s s_1 s_2 s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = L^{s s_1 s_2 s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} - \frac{2i}{3}\mathrm{P}_{123} \sum_{s_4} L^{s-s_4 s_1}_{\mathbf{k}\mathbf{k}_2+\mathbf{k}_3\mathbf{k}_1} L^{-s_4 s_2 s_3}_{\mathbf{k}_2+\mathbf{k}_3\mathbf{k}_2\mathbf{k}_3} / (s_2\omega_2 + s_3\omega_3 + s_4\omega(\mathbf{k}_2 + \mathbf{k}_3)).$$

For surface gravity waves, there are no resonances for which all the sign parameters are equal. As a consequence, to this order, wave number is conserved and Equation 11, after appropriate summations, becomes, for $t > 0$,

$$T_4[n_{\mathbf{k}}] = 12\pi \int |G_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|^2 n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} \left(\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} - \frac{1}{n_{\mathbf{k}_3}} \right)$$
$$\delta(\omega_1 + \omega_2 - \omega_3 - \omega)\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k})d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \tag{12}$$

where $G_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ is closely related to the $G^{s s_1 s_2 s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ given above. Assuming isotropy and angle averaging so that $N_\omega = \Omega_0 k^{d-1} \frac{dk}{d\omega} n_k(k(\omega))$ (where Ω_0 is the solid angle in d dimensions), the kinetic equation for four-wave resonances can be further simplified as

$$\frac{dN_\omega}{dt} = S(\omega)$$
$$= \int S_{\omega\omega_1\omega_2\omega_3} n_k n_{k_1} n_{k_2} n_{k_3} \left(\frac{1}{n_k} + \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} - \frac{1}{n_{k_3}} \right) \delta(\omega_1 + \omega_2 - \omega_3 - \omega) d\omega_1 d\omega_2 d\omega_3, \tag{13}$$

where Δ is the region $\omega_2, \omega_3, \omega_2 + \omega_3 > 0$ in the ω_2, ω_3 plane. If $G^{s s_1 s_2 s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ has homogeneity degree γ_3, $S_{\omega\omega_1\omega_2\omega_3}$ has homogeneity degree $\sigma = (2\gamma_3 + 3d)/\alpha - 4$. Before we discuss solutions, we make several further remarks.

First, the kinetic equation is solved for times $t = \mathcal{O}(1/\epsilon^{2r})$, $r = 1, 2, \ldots$, by successive truncation. For the theory to remain valid, it is important to check that solutions of the truncated equations keep Equations 8 and 9 uniformly asymptotic in $\mathbf{k}$, and that the ratio t_L/t_{NL} is uniformly small in $\mathbf{k}$. Second, with forcing and damping, the first truncation leads to a universal, statistically steady state involving all wave vectors, except in cases such as acoustic waves or Alfvén waves in which the resonant manifolds foliate wave-vector space (see Section 5.2.1 and the shape of the KZ spectrum in Galtier et al. 2000). Third, the kinetic equation $dn_{\mathbf{k}}/dt = \epsilon^4 T_4[n_{\mathbf{k}}]$ is independent of the sign of the coefficient $G_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$, but the first frequency correction is not. This has important ramifications for the equations of NLS type. Fourth, the energy density $e_{\mathbf{k}}$ is related to $n_{\mathbf{k}}$ to order ϵ^2 by $e_{\mathbf{k}} = (\omega_{\mathbf{k}} + \epsilon^2 s^{-1}\Omega_2^s[n_{\mathbf{k}}])n_{\mathbf{k}}$. Formal energy conservation follows from the properties of $L^{s_1 s -s_1 \ldots -s_r}_{\mathbf{k}_1 \mathbf{k} -\mathbf{k}_1 \ldots -k_r} = (s_1/s)L^{s s_1 \ldots s_r}_{\mathbf{k}\mathbf{k}_1\ldots\mathbf{k}_r}$ given earlier. Fifth, if one redoes the initial value problem from $t = t_1 = \mathcal{O}(\epsilon^{-2}) > 0$, the coefficients in Equations 10 and 11 are still sgn(t) not sgn($t - t_1$) because, in addition to Equation 10 with the sign factor sgn($t - t_1$), there is an extra term with sign factor (sgn(t) − sgn($t - t_1$)) arising from the fact that the third-order cumulant now has a nonsmooth, order ϵ, initial value (see Equation 6). Thus the isolated system will always relax to its thermodynamic state for large positive or negative time measured from $t = 0$.

Finally, let us emphasize what we mean by a valid wave turbulence theory. Above we only use the conservative part of the underlying dynamical system. We append, phenomenologically, to the kinetic equation (Equation 13) two terms S_{IN} and S_{OUT} representing forcing and dissipation. Only if input and output can be represented as $\gamma(\omega)A^s_{\mathbf{k}}$ in Equation 1 is Equation 13 a natural closure. In that special case, $S_{IN} = 2\gamma_{IN}(\omega)N(\omega)$ and $S_{OUT} = 2\gamma_{OUT}(\omega)N(\omega)$. But the input and output are often too complicated for this to be the case. Therefore, in general, we look at Equation 13 with S_{IN} and S_{OUT} added as being a valid approximation because each is much smaller than the first term (which feeds $n_{\mathbf{k}}$) and second term (proportional to $n_{\mathbf{k}}$) in Equation 11. We can also take account of weak nonspatial homogeneity by writing $\partial n_{\mathbf{k}}/\partial t$ as $\partial n_{\mathbf{k}}/\partial t + \nabla_{\mathbf{k}}\omega\nabla_{\mathbf{x}}n_{\mathbf{k}} - \nabla_{\mathbf{x}}\omega\nabla_{\mathbf{k}}n_{\mathbf{k}}$, where ω is the renormalized frequency. We say that wave turbulence theory is valid if solutions of the original or extended equations match what is observed.

3. SOLUTIONS OF THE KINETIC EQUATION

In systems where a UV cutoff avoids energy leakage to an infinite wave number, zero-flux thermodynamic solutions (Rumpf 2008) are the statistically steady states. But most wave turbulent systems of interest have sources and/or sinks that oftentimes are widely separated in wave-number space. In the windows of transparency between sources and sinks, one expects the statistically steady state to be of Kolmogorov finite-flux type. It is quite remarkable that, of all the pioneers in establishing the kinetic equation as the centerpiece of wave turbulence theory, only Zakharov saw that the finite-flux KZ solutions were more important for nonisolated systems. For over 20 years, few of his Western colleagues took much notice of them, and his was a lone voice crying in the wilderness.

We begin with the special pure power-law solutions of Equation 13, $n_{\mathbf{k}}(\omega) = c_1,\ c_2\omega^{-1}$, $c_3 P^{1/3}\omega^{-2\gamma_3/3\alpha - d/\alpha}$, and $c_4 Q^{1/3}\omega^{-2\gamma_3/3\alpha - d/\alpha + 1/6}$, corresponding to equipartition of number density, energy density, finite energy flux P (with zero number flux Q), and finite number flux Q (with zero energy flux P). These are obtained when the coupling coefficient $S_{\omega\omega_1\omega_2\omega_3}$ is homogeneous of degree $\sigma = (2\gamma_3 + 3d)/\alpha - 4$. Using the properties of the coupling coefficients, the first two solutions follow by inspection. The last two are obtained by dividing the region of integration Δ into four subregions (Δ_1, $0 < \omega_2, \omega_3 < \omega$, $\omega_2 + \omega_3 + > \omega$; Δ_2, $0 < \omega_2 < \omega$, $\omega_3 > \omega$; Δ_3, ω_2, $\omega_3 > \omega$; and Δ_4, $0 < \omega_3 < \omega$, $\omega_2 > \omega$) and mapping each of Δ_2, Δ_3, and Δ_4 conformally onto Δ_1. Setting $\omega_j = \omega\zeta_j$, $j = 1, 2, 3$, and $n = c\omega^{-x}$ allows us to write $S(\omega)$ as $c^3\omega^{-y-1}I(x, y(x))$, where

$$I(x, y) = \int_{\Delta_1'} S_{1\zeta_1\zeta_2\zeta_3}(\zeta_1\zeta_2\zeta_3)^{-x}\delta(1+\zeta_1-\zeta_2-\zeta_3)(1+\zeta_1^x-\zeta_2^x-\zeta_3^x)(1+\zeta_1^y-\zeta_2^y-\zeta_3^y)d\zeta_1 d\zeta_2 d\zeta_3, \tag{14}$$

Δ'_1 is $0 < \zeta_2, \zeta_3 < 1, \zeta_2+\zeta_3 > 1$, and $y = 3x+1-2\gamma_3/\alpha-3d/\alpha$. The choices $x = 0, 1$ and $y = 0, 1$ are the pure equipartition solutions and finite-flux solutions, respectively. Depending on the behavior of $S_{\omega\omega_1\omega_2\omega_3}$ near $\omega_1 = \omega_2+\omega_3-\omega = 0$, $I(x, y)$ converges in the ranges including the finite-flux solutions. For gravity waves, the KZ finite-flux solutions are $x = 23/3$ and $x = 8$, respectively, and $I(x,y(x))$ converges for $5 < x < 19/2$ (see the second remark below). The Kolmogorov constant can be found by setting $S(\omega) = \partial Q/\partial\omega$ and $\omega S(\omega) = -\partial P/\partial\omega$, respectively, where $Q = -\lim_{y\to 0} c_4^3 Q y^{-1}\omega^{-y} I(x, y) = c_4^3 Q(dI/dy)_{y=0}$ and $P = +\lim_{y\to 0} c_3^3 P(y-1)^{-1} I(x, y) = c_3^3 P(dI/dy)_{y=1}$, respectively. The slopes of $I(x, y)$ at $y = 0, 1$ are negative and positive, respectively. A similar analysis for the kinetic equation $dn_{\mathbf{k}}/dt = \epsilon^2 T_2[n_{\mathbf{k}}]$ dominated by three-wave resonances gives two special stationary solutions $n_{\mathbf{k}} = T/\omega_{\mathbf{k}}$ and $n_{\mathbf{k}} = c P^{1/2}\omega^{-(\gamma_2/\alpha+d/\alpha)}$ corresponding to energy equipartition and finite energy flux P.

For reasons of pedagogy, it is helpful to rederive the stationary solutions $T_4[n_{\mathbf{k}}] = 0$ another way. If the coupling coefficient $S_{\omega\omega_1\omega_2\omega_3}$ is localized and supported only near $\omega = \omega_1 = \omega_2 = \omega_3$, one can replace $S(\omega)$ by a differential representation $\partial^2 K/\partial\omega^2$, where $K = S_0\omega^{3x_0+2} n^4 d^2 n^{-1}/d\omega^2$, S_0 is a well-defined integral, and $x_0 = 2\gamma_3/(3\alpha) + d/\alpha$. We can identify the particle flux as $Q = \partial K/\partial\omega$ (Q is positive when particles flow from high to low wave numbers), and P, the direct energy flux, is $K-\omega\partial K/\partial\omega$. The stationary solutions are clearly $K = A\omega+B$, where we can identify the constants A and B with Q and P. In general, the solutions of $K = S_0\omega^{3x_0+2} n^4 (d^2(1/n))/(d\omega^2) = Q\omega + P$ will be a four-parameter family with two additional constants T and μ (temperature and chemical potential) arising from the double integration of $K = Q\omega + P$. If $P = Q = 0$, then $n = T/(\omega-\mu)$, the Rayleigh-Jeans solution for classical waves. (Similar solutions for bosons and fermions generalizing the Bose-Einstein and Fermi-Dirac distributions by the addition of finite fluxes can also easily be found.) The pure KZ solutions occur when we take in turn $Q = 0$ and $P = 0$ and look for power-law solutions $n = c_3\omega^{-x}$, $n = c_4\omega^{-x}$, for $K = \omega Q + P$. It is easy to show that these two solutions are $n = c_4 Q^{1/3}\omega^{-(2\gamma_3)/(3\alpha)-d/\alpha+1/6}$ and $n = c_3 P^{1/3}\omega^{-2\gamma_3/(3\alpha)-d/\alpha}$, respectively. In many practical applications, however, the actual steady-state solution may be a complicated combination of these special solutions. We now make several further remarks.

First, the pure finite energy (zero particle) flux solution is strictly relevant only when the energy is inserted at the boundary $k = 0$ and removed at $k = \infty$, equivalent to solving Equation 13 with boundary conditions $K = P$, $\partial K/\partial\omega = 0$, at $\omega = 0$ and $\omega = \infty$. Likewise, the pure particle flux KZ inverse cascade (zero energy flux) obtains when particles are added to the system at $k = \infty$ and removed at $k = 0$.

Second, $T_4[n_{\mathbf{k}}]$ in Equation 8 [and $S(\omega)$ in Equation 13] can be written as two integrals, a feed $F_{\mathbf{k}}$ involving integrals over the product $n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3}$ and a term $-n_{\mathbf{k}}\Gamma_{\mathbf{k}}$ proportional to $n_{\mathbf{k}}$. Indeed $\Gamma_{\mathbf{k}}$ is exactly $\mathrm{Im}\Omega_4[n_{\mathbf{k}}]$. $F_{\mathbf{k}}$ and $\Gamma_{\mathbf{k}}$ are often divergent at low wave numbers on the KZ solution, but the singularity cancels because of the combination. The first consequence is that when compared with any input or dissipation terms, the contributions $F_{\mathbf{k}}$ and $n_{\mathbf{k}}\Gamma_{\mathbf{k}}$ dominate the phenomenologically added S_{IN} and S_{OUT}. The second consequence is the breakdown criteria developed in the fifth remark below can curtail the range of validity of the KZ spectrum even further.

Third, both energy and particle number conservation may break down after a finite time. The idea goes back to Onsager. If we compute the amount of energy, $\int_{\omega_0}^{\infty}\omega N_\omega d\omega$, in the range (ω_0, ∞), for example, under the direct energy flux KZ spectrum $n_{\mathbf{k}} = c_3 P^{1/3}\omega^{-(2\gamma_3)/(3\alpha)-d/\alpha}$, or $\omega N_\omega = (\Omega_0/\alpha) c_3 P^{1/3}\omega^{-(2\gamma_3)/(3\alpha)}$, the integral converges (the spectrum supports finite energy) if $\gamma_3 > 3\alpha/2$ and diverges otherwise (infinite capacity). In the finite-capacity case, if energy is added at $\omega = \omega_0$ at a constant rate and is not confined to finite frequencies, it must escape to the dissipation sink at $\omega = \infty$ in a finite time t^* after which energy conservation no longer holds. Likewise, there is an equivalent notion of finite capacity for the particle flux. Ocean gravity waves have finite energy capacity at large wave numbers and infinite wave-action capacity at low wave numbers so that the inverse cascade does not build condensates (in infinite wave tanks). Alternatively, in optical turbulence for which $\alpha = 2$, $\gamma_3 = 0$, the direct cascade of the Hamiltonian density has infinite capacity, whereas the power density has finite capacity, and, as a result, condensates and collapses can form (see Dyachenko et al. 1992). For three-wave resonances, the direct energy flux cascade $n_{\mathbf{k}} = c\,P^{1/2}\omega^{-(\gamma_3+d)/\alpha}$ has finite capacity if $\gamma_2 > \alpha$ (e.g., capillary waves).

Fourth, pure KZ spectra are generally realized by self-similar solutions $n_{\mathbf{k}} = t^{-b}n(kt^{-a})$ of Equation 13 (Falkovich & Shafarenko 1991). In the infinite-capacity case, both exponents a and b are found by scaling arguments, and the spectrum consists of a front that joins an exponentially small precursor to a wake through a front at $k_f \propto t^a$, $a > 0$. The spectrum behind the front relaxes to the pure KZ spectrum. In the finite-capacity case, an anomaly occurs. The front now travels as $k_f \propto (t^* - t)^a$, $a < 0$, and leaves in its wake a spectrum k^{-x} for which the exponent $x = b/|a|$ is greater than the KZ value and can only be determined by solving a nonlinear eigenvalue problem. Only after the front hits the dissipation range does the KZ spectrum build up backward from $k = \infty$. It is not known whether such anomalies are present for fully turbulent systems (Connaughton et al. 2003, Connaughton & Newell 2010).

Fifth, to test the validity of the third premise in Section 2.3 (i.e., the breakdown of KZ spectra), let us compute $t_L/t_{NL} = (n_{\mathbf{k}}\omega_k)^{-1}dn_{\mathbf{k}}/dt$ on the spectrum $n_k = ck^{-\alpha x}$ for the case in which $T_2[n_{\mathbf{k}}] \equiv 0$. Setting each k_j in T_4 to $k\zeta_j$, we obtain

$$\frac{t_L}{t_{NL}} = \frac{k^{2\gamma_3}c^3k^{-3\alpha x}k^{-\alpha}k^{2d}}{ck^{-\alpha x}k^{\alpha}}I, \tag{15}$$

where I is some k-independent integral that we assume (for the moment) converges. We find

$$\frac{t_L}{t_{NL}} \propto c^2k^{2(\gamma_3+d-\alpha-\alpha x)}. \tag{16}$$

Only on the generalized Phillips spectrum is $\mathbf{k}$ independent (i.e., has a critical balance between linear and nonlinear terms). The ratio then depends on the nondimensional size of c. On the direct energy cascade KZ spectrum, $\alpha x = 2\gamma_3/3 + d$, $c = CP^{1/3}$, we find $t_L/t_{NL} = C^2P^{2/3}k^{2(\gamma_3/3-\alpha)}$. On the inverse number density (wave action, power) cascade, $\alpha x = 2\gamma_3/3+d-\alpha/3$, $c = CQ^{1/3}$, we find $t_L/t_{NL} = C^2Q^{2/3}k^{2(\gamma_3-2\alpha)/3}$. We have absorbed the small parameter ϵ into c and thereby both into P and Q. For $\gamma_3 > 3\alpha$, as is the case for ocean gravity waves, the ratio is of order unity or greater, and the third premise in Section 2.3 is violated for wave numbers $k > k_U$ when $P^{2/3}k_U^{2(\gamma_3/3-\alpha)} \sim 1$. For gravity waves, $P^{2/3}k_U/g \sim 1$. For $\gamma_3 < 2\alpha$, as is the case for the NLS equation, the ratio becomes unity at small wave numbers $k < k_I$ when $Q^{2/3}k_I^{2(\gamma_3-2\alpha)/3} \sim 1$. On the direct energy cascade for three-wave resonances, breakdown occurs for $k > k_U$, $P^{1/2}k_U^{\gamma_2-2\alpha} \sim 1$, when $\gamma_2 > 2\alpha$. For capillary waves, $\gamma_2 = 9/4 < 2\alpha = 3$, so the wave turbulence requirement that $t_L/t_{NL} \ll 1$ gets better the larger k is. As long as the relevant integral I in Equation 15 converges, breakdown occurs at the same point for all measures of wave turbulence validity, t_L/t_{NL}, T_4/T_2, Ω_2/ω, Ω_4/ω, and $(S_4 - 3S_2^2)/S_2^2$ (with k, ω replaced by r^{-1}, τ^{-1}). But as indicated in an earlier remark, it turns out that, for gravity waves, the ratio Ω_4/ω does not converge, and this adds a factor $(k_U/k_p)^{1/2}$ to $P^{2/3}k_U/g$, with k_p the peak wave number in n_k. The breakdown range becomes even larger in this case. For wave numbers larger than the breakdown wave number k_U, the KZ spectrum must be modified. We suggest how in Section 5. For the NLS or MMT equations (see Section 5), the ratio Ω_2/ω contains an additional logarithmic factor $c_3P^{1/3}k_0^{-1/2}\ln(k_d/k_0)$ necessitated by the logarithmic divergence of the KZ energy flux spectrum $c_3P^{1/3}k^{-d}$ and the introduction of an infrared cutoff at k_0. Again, the range of validity of the KZ spectrum is further diminished.

Sixth, given that $n_{\mathbf{k}}$ has power-law behavior over an inertial range, one can ask what is the corresponding universal behavior of the structure functions $S_N(\mathbf{r}, \tau) = \langle(\eta(\mathbf{x}+\mathbf{r}, \mathbf{t}+\tau) - \eta(\mathbf{x}, \mathbf{t}))^{\mathbf{N}}\rangle$, where $\eta(\mathbf{x}, \mathbf{t})$ is the surface elevation, for example (Biven et al. 2003). As measurements usually involve the time signal at a given point, we have to settle for $S_N(0, \tau)$. For surface waves, the second-order structure function $S_2(\tau) = 2\int_0^\infty I(\omega)(1-\cos(\omega\tau))d\omega$, where $I(\omega) = (\pi\omega/2v^2)(dk^2/d\omega)n_k(\omega)$, $\omega^2 = kv^2 = gk + (S/\rho)k^3$, is the Fourier transform of the two-point correlation $\langle\eta(\mathbf{x}, \mathbf{t})\eta(\mathbf{x}, \mathbf{t}+\tau)\rangle$. For capillary waves, $I(\omega) = cP^{1/2}(S/\rho)^{1/6}\omega^{-17/6}$ is the direct energy flux spectrum. If this shape was valid for the entire range, then $S_2(\tau) = 2cP^{1/2}(S/\rho)^{1/6}\tau^{11/6}\int_0^\infty \xi^{-17/6}(1-\cos\xi)d\xi$. But, of course, in practice, the universal power law holds at best over a finite range (ω_1, ω_2), where ω_1 (ω_2) denotes the upper (lower) boundary of the forcing (dissipation) range. We can generally take $\omega_2 = \infty$ so that $S_2(\tau) = 2\int_0^{\omega_1} I(\omega)(1-\cos(\omega\tau))d\omega + 2cP^{1/2}(S/\rho)^{1/6}\tau^{11/6}J(\omega_1\tau)$, where $J(\omega_1\tau) = \int_{\omega_1\tau}^\infty \xi^{17/6}(1-\cos\xi)d\xi$. The nonuniversal part of $S_2(\tau)$ behaves, for small τ, as τ^2. The universal part behaves as $\tau^{11/6}$. It is difficult to distinguish the two. To gain separation, it is better to work with averages of higher time differences such as $S'_N(\tau) = \langle(\eta(\mathbf{x}, \mathbf{t}+\tau) - 2\eta(\mathbf{x}, \mathbf{t}) + \eta(\mathbf{x}, \mathbf{t}-\tau))^{\mathbf{N}}\rangle$. In this case, the nonuniversal behavior of S_2 now is τ^4 for small τ, whereas the universal behavior remains at $\tau^{11/6}$.

Finally, with finite box effects, size matters. If the spectrum is not continuous but quantized, then it is more difficult to satisfy the resonance condition. Some help, however, is gained by the fact that the nonlinear correction to the frequency effectively replaces the Dirac delta function $\delta(\omega_1 + \omega_2 - \omega)$ with a Lorentzian, $\mathrm{Im}(1/(\omega'_1 + \omega'_2 - \omega'))$, $s\omega'_1 = s\omega + \epsilon^2\Omega_2^s$. This means that the resonance manifold is broadened by an amount proportional to ϵ^2 (ϵ^4 in the four-wave interaction case). In order for the quantized spectrum to allow lots of resonances within the band, we require that the ratio of the box size l to the typical wavelength λ participating in triad resonances be greater than ϵ^{-2} multiplied by a calculable factor. For gravity waves when $\epsilon \sim 0.1$, resolving resonances involving waves of 60 m would require a tank of approximately 60 km. Finally, for finite boxes, it is important to make sure that repeated boundary reflections lead to the same almost-joint Gaussian behavior in the linear limit. In large boxes, the near-Gaussian behavior is guaranteed dynamically by the addition of influences from statistically uncorrelated far-away sources. It may be necessary to use corrigated boundaries to achieve the same conditions.

4. EXPERIMENTAL EVIDENCE

In this section, we examine the available experimental evidence. Although there is some evidence of consistency between theory and experiment, much remains to be done. In particular, it would be valuable to measure the joint space-time power spectrum $\Phi(\mathbf{k}, \omega)$, the Fourier transform of $\langle\eta(\mathbf{x}, \mathbf{t})\eta(\mathbf{x}+\mathbf{r}, \mathbf{t}+\tau)\rangle$. A necessary condition for a valid theory is that its support be concentrated on the modified dispersion relation given in Equation 9.

4.1. Capillary Wave Turbulence

Weakly nonlinear capillary waves support three-wave resonances (Pushkarev & Zakharov 1996). Two groups, based in Paris (Falcon et al. 2007a,b, 2008, 2009; Falcon 2010) and Chernogolovka (Kolmakov et al. 2004, 2006), have carried out series of experiments on the surface response to broad- and narrowband forcing at wave numbers $k_f < k_0 = \sqrt{\rho g/S}$. To increase the range in which pure capillary influences are dominant, both groups have sought to decrease viscosity and the effective gravity. The Paris group used shallow layers of mercury ($\lambda_0 = 2\pi/k_0 \sim 1$ cm) and layers of ethanol and water in very-low-gravity situations (kudos to their courage in flying loop the loops) where $\lambda_0 \sim 10$ cm. The forcing in both cases was sinusoidal (via subharmonic generation) and low pass filtered, broadband and random in the 0–6-Hz range. There is clear evidence of the theoretically predicted pure KZ energy flux frequency spectrum $I(\omega) = (2\pi)^{-1}\int\langle\eta(x,t)\eta(x,t+\tau)\rangle\exp(-i\omega\tau)d\tau = cP^{1/2}(S/\rho)^{1/6}\omega^{-17/6}$ over at least a decade of

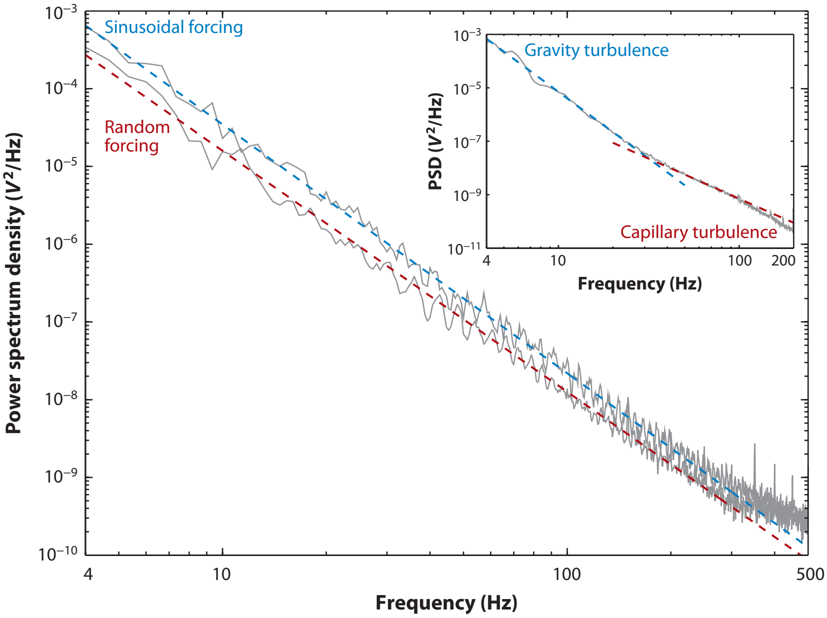

Figure 1

Power spectrum density of surface wave height on the surface of a fluid layer in low gravity, showing capillary wave turbulence. The lower curve is random forcing at 0–6 Hz, and the upper curve is sinusoidal forcing at 3 Hz. Dashed lines have slopes of −3.1 (*lower curve*) and −3.2 (*upper curve*). (*Inset*) Same as in main panel, but with gravity. Here the slopes of the dashed lines are −5 (*upper curve*) and −3 (*lower curve*), corresponding to gravity and capillary wave turbulence regimes, respectively. Figure courtesy of Eric Falcon, taken from Falcon et al. (2009).

frequencies in the first experiment and over two decades in the second. The observed power law $I(\omega) \sim \omega^{-s}$ found $s = 3.1$ for broadband and $s = 3.2$ for narrowband input and is shown in **Figure 1**. The probability density function (PDF) for the surface elevation η (Falcon et al. 2007b) is almost Gaussian with the usual Tayfun correction expected from second harmonics excited by quadratic interactions. In the normal gravity experiments, the spectrum in the gravity wave range is much steeper than the KZ spectrum predictions, consistent with what is seen for gravity waves in finite tanks (Nazarenko et al. 2010). However, the dependence of $I(\omega)$ on P, the energy flux, is neither $P^{1/2}$ nor $P^{1/3}$ as predicted but seems to be proportional to P. The reason for this is unclear, but one might argue that P, the constant flux in the inertial range and dissipation rate, is not measured simply by the mean of very widely distributed input flux (as measured by forces on the driving paddles), which has fluctuations much larger than the mean itself and can take on both positive and negative values. In addition, the structure-function measurements do not corroborate the theory, but that may be attributable to the sixth remark in Section 3.

The Russian group studied capillary turbulence on quantum fluids, liquid hydrogen, and helium in both its normal and superfluid states. For broadband forcing, they found $I(\omega) \sim \omega^{-s}$, where $s = 2.8 \pm 0.2$. The result for narrowband forcing is steeper, $s = 3.7$, and the group studied the transition between this and the KZ regime. We do not understand why the difference between narrowband forcing (not expected to follow wave turbulence theory) and broadband forcing in the Russian experiments should be so much greater than that observed by the Paris group.

4.2. Gravity Wave Turbulence

The views expressed in this section have been informed by the experimental results of Donelan et al. (1992, 2005, 2006) and Toba (1972, 1973a,b, 1997), by observations by Long & Resio (2007), by the books of Young (1999) and Phillips (1977), by the numerical simulations of both the forced and damped kinetic and water wave equations by Badulin et al. (2005, 2007) and Korotkevich (2008), and by the recent review articles of Zakharov (2005). Although most measurements have involved time signals at a fixed location, Hwang and colleagues (Hwang et al. 2000, Hwang & Wang 2004, Hwang 2006) (see **Figure 2**) measured spatial correlations by flying precision parallel courses over the ocean surface.

Despite the fact that Hasselmann derived the kinetic equation in 1962 and Zakharov and Filonenko found the finite-flux solutions in 1968, it took a long time for the oceanographic community to accept the fact that, over the largest range, the observed spectra $E(k) = 2\pi k\omega_{\mathbf{k}} n_{\mathbf{k}}$ and $I(\omega)$ had more connections with the pure KZ spectra $n_k = c_3 P^{1/3} k^{-4}$ $[E(k) = 2\pi c_3 g^{1/2} P^{1/3} k^{-5/2}$, $I(\omega) = d_3 P^{1/3} g\omega^{-4}]$ and $n_k = c_4 Q^{1/3} k^{-23/6}$ $[E(k) = 2\pi c_4 g^{1/2} Q^{1/3} k^{-7/3}$, $I(\omega) = d_4 Q^{1/3} g\omega^{-11/3}]$ than with the flux-independent spectrum $I(\omega) \sim g^2\omega^{-5}$ that Phillips had proposed. Eventually in the 1980s, Donelan et al. (1985) argued (Young 1999, p. 119) that the data did not support the earlier JONSWAP spectrum proportional to ω^{-5} but rather supported the one given by an

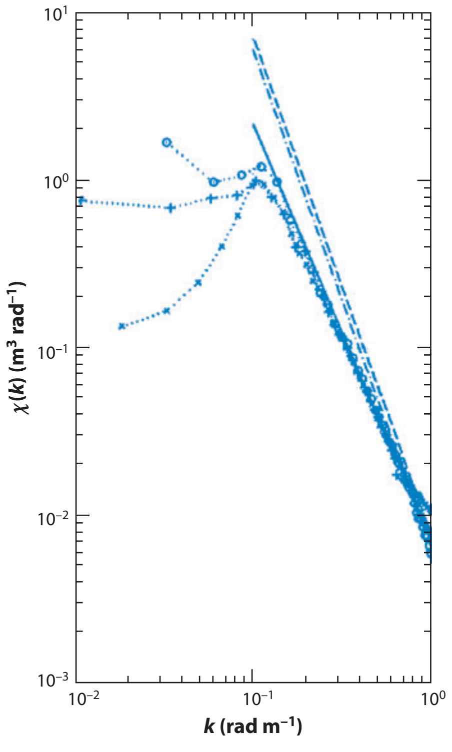

Figure 2

Surface elevation spectra proportional to $E(k)/g$ with three comparison slopes: solid line is $k^{-5/2}$, and the dashed and dashed-dotted curves are k^{-3} (Phillips' spectrum) with different normalizations. Near the peak, the spectrum is slightly less steep (the wave-action flux has slope 7/3), whereas for meter length scales, the spectrum is steeper and closer to Phillips'. Figure courtesy of Paul A. Hwang, adapted from Hwang et al. (2000).

$I(\omega)$ with a tail frequency of ω^{-4}. Furthermore, the spectrum $I(\omega) = d_3 P^{1/3}\omega^{-4}$ is consistent with the observationally deduced Toba's law. The total energy per square meter of ocean surface is proportional to $I = \langle\eta^2\rangle \approx \int_{\omega_p}^{\infty} I(\omega)d\omega = (1/3)d_3 P^{1/3} g\omega_p^{-3}$, which translates into the law that the mean square height $\sqrt{\langle\eta^2\rangle}$ is proportional to $T_p^{3/2}$, the 3/2 power of the period of the peak wave. This is analogous to the widely accepted result that $\epsilon = \langle\eta^2\rangle$ satisfies $\epsilon\omega_p^4/g^2 = \alpha((d\epsilon/dt)\omega_p^3/g^2)^{1/3}$, where the flux P is replaced by $g d\epsilon/dt$ and $d\epsilon/dt$ is interpreted as a total dissipation rate. Note that neither this nor Toba's law is much changed if we were to use the pure inverse wave-action flux KZ spectrum instead of the pure direct energy flux KZ spectrum.

Although the evidence seems to favor the wave turbulence prediction, there is a rub. Although little is known about the precise form of input and dissipation, it is generally agreed that the main input occurs at small (<1 m) scales by a variant of the Miles (1967) instability. [The Kelvin-Helmholz instability, which amplifies perturbations of wave number k on the surface between two layers of fluid (e.g., air, water) with constant tangential velocities (the ultra-simple model) U and U_w with densities ρ_a and ρ ($\rho_a U^2 \simeq \rho U_w^2$), has a growth rate $(\rho + \rho_a)\sigma = -ik(\rho_a U + \rho U_w) + \nu$, $\nu^2 = -g(\rho^2 - \rho_a^2)k(1 - k\rho\rho_a(U - U_w)^2/(g(\rho^2 - \rho_a^2)) + Sk^2/(g(\rho - \rho_a))$. It would require a U of at least 5 m s^{-1} [$\rho/\rho_a \sim 10^{-3}$, $S/(\rho g) \sim 7 \times 10^{-2}$ cm^2] to excite waves of wavelengths more than centimeters. The fact that wave generation occurs at these centimeter scales for much lower wind speeds would seem to rule out the Kelvin-Helmholz instability as the primary mechanism. However, as we argue in Section 5, it may play a role in initiating whitecap events.] The generation of longer waves is primarily thought to be the result of an inverse cascade in which both wave action and energy are carried to long waves. But the pure KZ ω^{-4} spectrum is predicated on an energy flux from long to short waves. Numerical simulations (Badulin et al. 2007) suggest that the observed spectrum—an evolving spectral shape $I(\omega)$ with a front at ω_f moving from high toward low frequencies, rising quickly to a peak at ω_p, and then decaying algebraically as ω^{-s}, $11/3 \le s \le 4$ in the wake $\omega > \omega_p$—is a combination of an almost constant wave-action flux (especially near ω_p) and a nonconstant energy flux that changes sign so as to provide a net direct energy flux. The change of sign can be attributed to the deposition of energy by the dual inverse fluxes that conserve both energy and wave action. The conclusion we draw is that the observed spectrum is consistent with a wave turbulence solution but one that is more complicated than that of pure wave action or energy fluxes. An additional observation is that at small scales, in strongly driven seas, the spectrum seems to be steeper and more aligned with Phillips' prediction. The appearance of the Phillips' spectrum in strongly driven situations is consistent with the findings of Korotkevich (2008) in direct numerical simulations.

4.3. Vibrating Plate Turbulence: Can One Hear the Kolmogorov Spectrum?

During et al. (2006a) derived and analyzed the wave turbulence of vibrations $\omega_{\mathbf{k}} \propto |\mathbf{k}|^2$ on large, thin, elastic plates with normal deformations $\eta(\mathbf{x}, t)$ governed by the von Kármán–Donnell equations. They found a kinetic equation similar to Equation 13, but with the distinct feature that, unlike gravity waves and optical waves of diffraction, the restriction that the four-wave interaction preserves wave numbers is no longer required. As a result, there is only one equipartition solution $n_{\mathbf{k}} = T/\omega_{\mathbf{k}} \propto k^{-2}$ and the KZ direct energy flux solution $n_{\mathbf{k}} \propto k^{-2}$. Because of the degeneracy, the KZ solution requires a log correction, and one can show $\langle|\eta_k|^2\rangle = d(P^{1/3}/k^4)\ln^{1/3}(k^*/k)$, where k^* is some cutoff wave number. Numerical simulations appear to corroborate these findings to within a surprising accuracy (even for the log power fit!). However, subsequent experiments by Mordant (2008) and Boudaoud et al. (2008) have failed to observe these spectra. In frequency space, the corresponding KZ spectrum is $\ln^{1/2}(\omega^*/\omega)$, but the experiments show a definite power-law decay ω^{-s}, where $s \approx 0.6$. Space-time spectra confirm that, except for very large waves, the joint power spectrum $\Phi(\mathbf{k}, \omega)$ is supported on a very thin curve that closely follows the linear dispersion relation. The real (nonlinear) frequency correction (Ω_2 in Equation 7) is found, however, to be proportional to P^s, where $s \geq 1/2$ rather than $s = 1/3$.

Two reasons can be given to explain the steeper spectrum. The first has been suggested by C. Josserand (private communication), who included broadband damping and found that the spectrum indeed did decay as ω^{-s}, $s > 0$, where s could be tuned by the choice of damping. The second is that the spectrum is steepened by finite box size effects.

5. QUESTIONS AND OPEN CHALLENGES

5.1. Questions

Two questions naturally emerge from the discussion to date. First, we have learned that the finite-flux spectra are almost never valid over all wave numbers in the sense that, either for very short or for very long waves, the premises on which wave turbulence theory rests are violated by the KZ solutions. Is there any way to modify the theory for those ranges so that its basic validity, namely closure and spectral energy transfer via resonances, still obtains? Second, are there situations for which all the ingredients for a successful wave turbulence appear to be in place, namely a dispersive wave system with weak nonlinearity for early times, but which over long times realize statistically stationary states that are completely inconsistent with the predictions of wave turbulence theory? Is there any way in which one can recognize these situations a priori?

We provide some answers to the first question by considering two examples. Given, over some range, a direct flux KZ spectrum for gravity waves, we know that the KZ spectrum breaks the premises of theory for wave numbers $k > k_U$, where $P^{2/3}k_U/g$ is of order unity. It is then natural to ask if there is another physical process, in this case capillary wave action, that may come into play for wave numbers (k_0, ∞) with $k_0 < k_U$, $k_0 = (\rho g/S)^{1/2}$, which can regularize the breakdown or, if $k_0 > k_U$, if there is a new spectral shape that we can find for $n_{\mathbf{k}}$ in (k_U, ∞) (we take $k_0 = \infty$ in this case) that can be legitimately attached to the KZ direct energy spectrum cascade at k_U. For ocean waves, $k_0 < k_U$ if $P < (gS/\rho)^{3/4}$ or [because $P \sim (\rho_a/\rho)^{3/2} U^3$] at wind speeds U of less than 5 m s^{-1}. Then the direct three-wave resonance energy transfer carries the energy flux from the four-wave KZ spectrum to the millimeter scale at which viscosity acts to absorb the energy. For $P > (gS/\rho)^{3/4}$, for $U > (\rho/\rho_a)^{1/2}(gS/\rho)^{1/4}$, or for wind speeds much greater than 5 m s^{-1}, a new spectrum must be appended for $k > k_U$. We note that this is precisely the criterion for which there is a range of wave numbers for which the Kelvin-Helmholz instability is active. Such an instability leads to wave breaking and may be responsible for whitecapping events. It has been our suggestion (Newell & Zakharov 1992, 2008) that the new spectrum is the Phillips' spectrum $n_{\mathbf{k}} = c g^{1/2} k^{-9/2}$. It has exactly the right properties to satisfy the amended kinetic equations $S(\omega) + S_{OUT} = 0$ in the integral sense so that, ignoring surface tension, all energy crossing k_U with flux P is absorbed in (k_U, ∞) by whitecapping events. Moreover, because the constant c is very small (the estimate is 0.2), the wave turbulence approximation (see Equation 15) is still valid.

The second example concerns the NLS equation with the Hamiltonian $(1/2)\int(|\nabla u|^2 - \lambda|u|^4)d\mathbf{x})$ and the Majda-McLaughlin-Tabak (MMT) equation (Majda et al. 1997). In the MMT equation, the Fourier space representation of the dispersion is $\omega = k^{1/2}$ rather than $\omega = k^2$. In the case of the focusing NLS equation ($\lambda = +1$) in two dimensions, number density inserted at intermediate wave numbers at rate Q_0 flows to longer and longer waves and will form an unstable condensate. The condensate breaks into large-amplitude collapse events that carry number density to small scales at which it is dissipated. The imperfect burnout of fraction f (see Dyachenko et al. 1992, Newell & Zakharov 1995) leads to a source of number density and energy at large wave numbers. By the joint action of waves and collapses, a statistically steady state with flux Q_0/f is achieved. Such a scenario is seen for the discrete NLS equation (Rumpf & Newell 2004) and the Benjamin-Feir stable ($\lambda = 1$) MMT equation (Cai et al. 1999, 2001; Rumpf & Biven 2005). Driving feeds wave action to the system at low amplitudes, and dissipation removes it at high amplitudes, a flux from low to high amplitudes. The nonlinearity of energy offsets the loss of energy by the dissipation of low-amplitude waves at high wave numbers. For sufficiently weak damping, the collapse events are rare enough so as not to affect the spectrum of lower moments. The challenge is to devise a two-species gas (waves and collapses) description that takes advantage of both the weak interaction of waves and the special structures of the collapse events.

With regard to the second question, we know from the work of Fermi, Pasta, and Ulam how an initially weakly nonlinear system can avoid thermalization for a very long time by the creation of coherent structures, a result that led to soliton theory. In their seminal paper, Majda et al. (1997) challenged the notion that a weakly nonlinear system of wave turbulence type would always reach the KZ attractor as its statistically steady state. Rumpf et al. (2009) have verified and explained that conclusion. For $\lambda = -1$, monochromatic waves of the MMT equation are Benjamin-Feir unstable and lead to the creation of solitary pulses, which in turn excite radiating tails. The ensemble of such pulses leads both to the observed MMT spectrum and to the correct prediction of a wave-action inverse cascade. But how might we have known a priori that wave turbulence theory must fail? We have recently discovered (with Zakharov) that, for $\lambda = -1$, the KZ solution is unstable to perturbations in $n(\mathbf{k}, \mathbf{x})$ that are not spatially homogeneous, namely weakly dependent on $\mathbf{x}$ (B. Rumpf, V.E. Zakharov & A.C. Newell, unpublished manuscript). The assumption of spatial homogeneity is temporarily violated although it will again be recovered when the system moves away from a wave turbulent state and reaches its new, nonlinear MMT statistically steady spectrum.

5.2. Open Challenges

To conclude, we list some additional challenges.

5.2.1. Acoustic turbulence, isotropy or shocks? The resonant manifolds for the dispersion relation $\omega = c|\mathbf{k}|$ are rays in wave-vector space. The first closure transfers spectral energy along but not between the rays. Given an initial anisotropic energy distribution, do the nonlinear interactions of the next closure lead to an isotropic distribution or to condensation along particular rays, which would likely become fully nonlinear shocks (L'vov et al. 1997)?

5.2.2. Energy exchange times. For a discrete set of interacting triads, the nonlinear energy exchange time is ϵ^{-1}. For a continuum set of such triads, cancellations cause this time to extend to ϵ^{-2}. Why?

5.2.3. Condensate formation. Condensate formation modeled by the defocusing ($\lambda = -1$) NLS equation is an open and hot topic (Connaughton et al. 2005; During et al. 2006b; L'vov et al. 1998, 2003). We list three questions here. (*a*) Given an input Q_0, P_0 of number and energy flux at finite wave number k_0, can one follow, using wave turbulence theory, the creation of a condensate by inverse flux action and the subsequent relaxation of waves and vortices on the condensate (Dyachenko et al. 1992, Lacaze et al. 2001, Nazarenko & Onorato 2006)? (*b*) Given a finite total energy and number of particles (and an UV cutoff k_c), can one find in the subcritical temperature range $T < T_c$ [at $T = T_c$, μ, the chemical potential is zero; for $T < T_c$, μ would be positive and the Rayleigh-Jeans equilibrium $n_k = T/(\omega_k - \mu)$ singular] a wave turbulence description with a gradual transition from free waves ($\omega_k = k^2$) to Bogoliubov waves ($\omega = \pm\sqrt{2|\lambda|n_0 k^2 + k^4}$), where n_0 is the number of particles in the condensate (Zakharov & Nazarenko 2005)? (For $T \approx T_c$, Equation 9 is no longer uniformly asymptotic.) (*c*) How is the second-order phase transition in question *b* affected if one initiates the dynamics with nonzero fluxes?

5.2.4. Wave turbulence in astrophysics. Magnetized plasmas, found in the solar corona, solar wind, and Earth's magnetosphere, support waves, and, like ocean waves, they have a continuum of scales (up to 18 decades!) and are a natural playground for wave turbulence (Boldyrev & Perez 2009; Galtier 2003, 2006, 2009; Galtier et al. 2000; Goldreich & Sridhar 1995, 1997; Goldstein & Roberts 1999; Kuznetsov 1972, 2001; Ng & Bhattacharjee 1996; Ng et al. 2003; Sahraoui et al. 2003, 2007; Sridhar & Goldreich 1994). To date, however, only the signatures of strong turbulence (Kolmogorov, rather than Iroshnikov-Kraichnan) have been found experimentally although wave turbulence behavior has been clearly seen in direct numerical simulations by Bigot et al. (2008). Given present satellite capabilities, what are the best hopes for observing wave turbulence spectra such as the $E(\mathbf{k}_\perp, k_\parallel) \sim f(k_\parallel)\mathbf{k}_\perp^{-2}$, $k_\parallel = \mathbf{kb}$, $\mathbf{k}_\perp = \mathbf{k} - \mathbf{k}_\parallel \mathbf{b}$, $f(k_\parallel)$ nonuniversal, $\mathbf{b}$ unit vector in the magnetic field direction, behavior for a sea of oppositely traveling Alfven waves? For a review, we refer the reader to Galtier (2009).

5.2.5. Continuum limit of finite-dimensional wave turbulence. In a box L^d, one can define a natural probability measure on Fourier amplitudes avoiding the difficulties of such measures in infinite dimensions. The resulting Liouville hierarchy for the Fourier amplitude PDF leads, via the Brout-Prigogine equation for its vacuum component, to a kinetic equation if one assumes that the vacuum PDF can be factored into a product of its marginals (a closure assumption!). Can one show that a natural closure occurs in L^d or, if not, how the natural closure arises in taking the $L \to \infty$ limit (Jakobsen & Newell 2004)?

5.2.6. A priori conditions for wave turbulence. Can one find mathematically rigorous a priori conditions on the governing equation given in Equation 1 or its statistical hierarchy that guarantees that wave turbulence theory will obtain? (We also refer the reader to earlier comments in this section.)

5.2.7. Homogeneity. Is broken spatial homogeneity (the first premise listed above) a potential problem for all turbulence theories?

5.2.8. Anomalous exponents. Are all finite-capacity Kolmogorov solutions reached with anomalous exponents? Do they have anything to do with positive entropy production?

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