THE THEORY OF SEN & SEN'S OPERATOR

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1. Introduction

In this paper, we follow the method of Shankar Sen [3] by relating p-adic Galois representations with continuous cohomology groups and then developing a linear operator φ on such a representation that encodes the semilinear action of the Galois group. We are particularly interested in a $\mathbb{Z}_p$-extension of a local field K. That is, K is a field extension of $\mathbb{Q}_p$ with ascending extensions K_m such that the extension $K_\infty = \bigcup_m K_m$ has Galois group, which we will denote Γ, that is isomorphic to $\mathbb{Z}_p$. We will focus on the particular $\mathbb{Z}_p$-extension of adjoining p-power roots of unity to K, although it should be noted that all results follow in the more general setting.

We will provide more formal definitions and notation to be fixed throughout the paper in Section 2. In Section 3, we provide some preliminary results that take advantage of our $\mathbb{Z}_p$-extension.

Denoting by $\mathbb{C}$ the completion of the algebraic closure of K and $\mathcal{G}$ the Galois group of $\overline{K}/K$, a lemma of Ax states that $\mathbb{C}^{\mathcal{H}} = \widehat{K}_\infty$, the completion of K_∞, where $\mathcal{H}$ is the Galois group of $\overline{K}/K_\infty$. In Section 4 we will reduce the cohomology group $H^1_{\text{cont}}(\mathcal{G}, \mathrm{GL}_n(\mathbb{C}))$ first to $H^1_{\text{cont}}(\Gamma, \mathrm{GL}_n(\widehat{K}_\infty))$ by showing that $H^1_{\text{cont}}(\mathcal{H}, \mathrm{GL}_n(\mathbb{C}))$ is trivial, and then reduce $H^1_{\text{cont}}(\Gamma, \mathrm{GL}_n(\widehat{K}_\infty))$ to $H^1_{\text{cont}}(\Gamma, \mathrm{GL}_n(K_\infty))$ in a process known as decompletion.

We will then apply the results of Section 4 to $\mathbb{C}$-representations in Section 5 to show that for a $\mathbb{C}$-reprentation W there are two representations $\widehat{W}_\infty$ and W_∞ related to W. In particular, W_∞ is a K_∞-representation with an action of Γ. We then develop Sen's operator φ, showing that it is a K_∞-linear operator on W_∞ that captures the action of Γ. This operator can then be extended to W and $\mathcal{G}$. We finish the section by showing that φ determines the representation W up to isomorphism, at least in the appropriate sense.

2. Definitions & Notation

Let K be a local field whose residue field has characteristic p. The algebraic closure of K, $\overline{K}$, is not complete with respect to the p-adic valuation extended from K. We will denote the completion of $\overline{K}$ as $\mathbb{C}$. For two elements $a, b \in \mathbb{C}$, the congruence $a \equiv b \mod p^m$ will mean that $|a-b|_p \leq p^{-m}$ where $|\,|_p$ is the absolute p-adic valuation on $\mathbb{C}$ normalized so that $|p|_p = 1$.

Similarly, two matrices are congruent modulo p^m if each of their entries are congruent modulo p^m.

Definition 2.1. A **$\mathbb{Z}_p$-extension of** K is a set $\{K_n\}$ of ascending finite Galois extensions of K such that the Galois group $\mathrm{Gal}(K_n/K)$ is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. Denote their ascending union by K_∞. Then it follows that

$$\begin{aligned}\mathrm{Gal}(K_\infty/K) &\cong \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}\\ &= \mathbb{Z}_p.\end{aligned}$$

Equivalently, an extension K_∞/K is a $\mathbb{Z}_p$-extension if $\mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$ since one can take K_n to be the fixed field of the subbgroup $p^n\mathbb{Z}_p$ of the Galois group.

As an example, beginning with $m = 0$, let K_m be the finite field extension of K containing all of the p^{m+1} roots of unity. If m is the first integer where K_m/K_0 is not trivial, then $\mathrm{Gal}(K_{m+n}/K_n) \cong \mathbb{Z}/p^n\mathbb{Z}$ and K_∞/K_0 is a $\mathbb{Z}_p$-extension. We will therefore fix throughout the paper a field K with a finite extension K_0 and a $\mathbb{Z}_p$-extension K_∞/K_0.

Set $\mathcal{G} = \mathrm{Gal}(\overline{K}/K)$. The Galois group of K_∞/K is isomorphic to $\Delta\mathbb{Z}_p$ where Δ is a finite group. We will denote this Galois group by Γ. Then for each m set $\Gamma_m = \mathrm{Gal}(K_\infty/K_m)$. Define the subgroup $\mathcal{H}$ of $\mathcal{G}$ to be $\mathcal{H} = \mathrm{Gal}(\overline{K}/K_\infty)$ so that $\mathcal{G}/\mathcal{H} = \Gamma$. The Ax-Sen-Tate Lemma states, for our purposes, that the $\mathcal{H}$-invariants of $\mathbb{C}$, $\mathbb{C}^{\mathcal{H}}$, is the completion of the fixed field of $\mathcal{H}$, so

$$\mathbb{C}^{\mathcal{H}} = \widehat{K}_\infty.$$

See the original proof by Ax [1] and also remarks in [6].

That $\Gamma_0 \cong \mathbb{Z}_p$ is a key fact in the results that follow. To begin, it means that Γ_0 is topologically generated by an element $\gamma \in \Gamma$. In other words, the set of γ^m for m an integer is dense in Γ_0, where Γ_0 is given the profinite topology. To put it another way, every $\sigma \in \Gamma_0$ can be written as $\sigma = \gamma^t$ where $t \in \mathbb{Z}_p$. It then follows that Γ_m is the open subgroup of Γ topologically generated by γ^{p^m}. This also means that Γ itself is topologically generated by a finite number of elements.

Let χ be a cyclotomic character of $\mathcal{G}$. Namely, a map $\chi : \mathcal{G} \to \mathbb{Z}_p^\times$ with infinite image and where χ is trivial on $\mathcal{H}$. In the case where K_∞ is obtained by adjoining all p-power roots of unity, χ is the cyclotomic character defined by $\sigma(\zeta) = \zeta^{\chi(\sigma)}$ for any $\sigma \in \mathcal{G}$ and ζ a p-power root of unity.

The two main objects we will consider are continuous group cohomology and $\mathbb{C}$-representations. For the former, instead of focusing on the details of the cohomology, we will describe the properties pertinent to the paper.

Definition 2.2. Let G be a topological group and M a topological G-module written multiplicatively, meaning that M is a topological group, not necessarily abelian, such that G acts continuously on M with the expected properties: For any $\sigma, \tau \in G, x, y \in M$

(1): $\sigma(1) = 1$,
(2): $\sigma(xy) = \sigma(x)\sigma(y)$,
(3): $\sigma\tau(x) = \sigma(\tau(x))$.

Define $C^1(G,M)$ to be the set of all continuous maps $U : G \to M$ such that $U(\sigma\tau) = U(\sigma)\sigma(U(\tau))$ for all $\sigma,\tau \in G$. This is the set of 1-cocycles. Define an equivalence relation on $C^1(G,M)$ where two cocycles U and U' are called cohomologous if there exists a $B \in M$ such that for every $\sigma \in G$, $U'(\sigma) = B^{-1}U(\sigma)\sigma(B)$. Then the continuous cohomology set $H^1_{\text{cont}}(G,M)$ is the set of equivalence classes of $C^1(G,M)$.

As an example (and the example we will be studying), let $G = \mathcal{G}$ with the Krull topology and $M = \mathrm{GL}_n(\mathbb{C})$ with the topology given by the valuation on $\mathbb{C}$. The last fact should be stressed. When the general linear group is given the discrete topology, then Hilbert's Theorem 90 says that $H^1_{\text{cont}}(\mathrm{Gal}(L/K), \mathrm{GL}_n(L))$ is trivial for any Galois extension of K, either finite or infinite. But as we will show, the very non-trivial set $H^1_{\text{cont}}(\mathcal{G}, \mathrm{GL}_n(\mathbb{C}))$ is in 1-1 correspondence with $H^1_{\text{cont}}(\Gamma, \mathrm{GL}_n(K_\infty))$ when given the valuation topology.

Definition 2.3. A **$\mathbb{C}$-representation** W is a finite-dimensional $\mathbb{C}$-vector space on which $\mathcal{G}$ acts semi-linearly and continuously with respect to the Krull topology on $\mathcal{G}$. In other words, the map $\mathcal{G} \times W \to W$ is continuous and

$$\sigma(cw) = \sigma(c)\sigma(w) \qquad \text{for } \sigma \in \mathcal{G}, c \in \mathbb{C}, \text{ and } w \in W$$
$$\sigma(w_1 + w_2) = \sigma(w_1) + \sigma(w_2) \qquad \text{for } \sigma \in \mathcal{G} \text{ and } w_1, w_2 \in W.$$

It turns out that n-dimensional $\mathbb{C}$-representations are related to the continuous cohomology group $H^1_{\text{cont}}(\mathcal{G}, \mathrm{GL}_n(\mathbb{C}))$. If we fix a basis for W, then there is a natural map $U : \mathcal{G} \to \mathrm{GL}_n(\mathbb{C})$ describing the action of $\mathcal{G}$ on W that is continuous by the continuity of the action of $\mathcal{G}$.

Claim 2.4. $U \in C^1(\mathcal{G}, \mathrm{GL}_n(\mathbb{C}))$

Let $\{e_1, ..., e_n\}$ be a basis for W. The semi-linearity of the action captures the cocycle condition for U. Suppose $\sigma, \tau \in \mathcal{G}$, $U(\sigma) = (a)_{i,j}$ and $U(\tau) = (b)_{i,j}$. Then

$$\begin{aligned}
\sigma(\tau(e_j)) &= \sigma(\sum_{k=1}^{n} b_{k,j}e_k) \\
&= \sum_{k=1}^{n} \sigma(b_{k,j})\sigma(e_k) \\
&= \sum_{k=1}^{n} \sigma(b_{k,j}) \sum_{i=1}^{n} a_{i,k}e_i \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{i,k}\sigma(b_{k,j})e_i,
\end{aligned}$$

and this shows that $U(\sigma\tau) = U(\sigma)\sigma(U(\tau))$. This shows the claim.

Similarly, changing the basis of W is equivalent to describing the action with another cocycle U' that is cohomologous to W via the change-of-basis matrix B. Hence, the cocycles derived from isomorphic $\mathbb{C}$-representations have the same cohomology class and there is a $1-1$ correspondence between $H^1_{\text{cont}}(\mathcal{G}, \mathrm{GL}_n(\mathbb{C}))$ and $\mathbb{C}$-representations of dimension n up to isomorphism.

3. Preliminary Results

We begin with some Propositions that will be used in the theory to follow. One should note that these results use the structure of K_∞ and Γ, which is important but can end up hidden in later proofs.

Proposition 3.1. *Let L be a finite separable extension of K_∞. Denote by $\mathcal{O}_L$ its ring of integers and $\mathfrak{m}_L$ its maximal ideal. Similarly define $\mathcal{O}_\infty$ and $\mathfrak{m}_\infty$ for K_∞. Then $\mathfrak{m}_\infty \subset Tr_{L/K_\infty}(\mathcal{O}_L)$.*

Proof. We follow the proof first shown by Tate [6]. First, we can assume that L/K_∞ is Galois. To see this, let L' be the Galois closure of L/K_∞ and then note that $\mathrm{Tr}_{L'/K_\infty} = \mathrm{Tr}_{L/K_\infty} \circ \mathrm{Tr}_{L'/L}$. There exists (see [4] p.89 Lemma 6) a finite extension L_0 of K (where it may be necessary to replace K by some K_n) such that $L = L_0K_\infty$, L_0 is linearly disjoint from K_∞, and it may be further assumed that L_0/K is Galois.

Let $L_m = L_0K_m$. The valuation of the different $\mathcal{D}_{L_m/K_m}$ is given by

$$v_p(\mathcal{D}_{L_m/K_m}) = \int_{-1}^{\infty} \frac{1}{|\mathrm{Gal}(K_m/K)^u|} - \frac{1}{|\mathrm{Gal}(L_m/K)^u|} du,$$

where G^u is the u-th ramification group of the Galois group G in the upper numbering. When u is large enough (say, $u \geq h$) so that $\mathrm{Gal}(L/K)^u \subset \mathrm{Gal}(L/L_0)$, then $\mathrm{Gal}(L_0/K)^u = 1$ and $\mathrm{Gal}(L_m/K)^u = \mathrm{Gal}(K_m/K)^u$ for any m. Thus, we can say that

$$v_p(\mathcal{D}_{L_m/K_m}) \leq \int_{-1}^{h} \frac{1}{|\mathrm{Gal}(K_m/K)^u|} du,$$

Set $\delta_m = v_p(\mathcal{D}_{L_m/K_m})$. By Herbrand's Theorem, $\mathrm{Gal}(K_m/K)^u = \Gamma_m^u = \Gamma_i\Gamma_m/\Gamma_m$ for some i dependent only on u (namely where $\Gamma^u = \Gamma_i$). Therefore for a fixed u, as $m \to \infty$, $|\mathrm{Gal}(K_m/K)| \to \infty$ and so $\delta_m \to 0$.

Now, $\mathrm{Tr}_{L_m/K_m}(\mathcal{O}_{L_m}) \supset \mathfrak{m}_{K_m}^j$ if and only if $\mathcal{O}_{L_m} \subset \mathfrak{m}_{K_m}^j \mathcal{D}_{L_m/K_m}^{-1}$ (c.f. [4]) and therefore if $\mathcal{D}_{L_m/K_m} = \mathfrak{m}_{L_m}^{d_m}$, then $\mathrm{Tr}_{L_m/K_m}(\mathcal{O}_{L_m}) = \mathfrak{m}_{K_m}^j$ where $j = \left\lfloor \frac{d_m}{e_{L_m/K_m}} \right\rfloor$ amd where e_{L_m/K_m} is the ramification index of L_m/K_m. In fact, we have

$$\begin{aligned} d_m &= \frac{\delta_m}{v_p(\pi_{L_m})} \\ &= \frac{\delta_m e_{L_m/K_m}}{v_p(\pi_{K_m})} \end{aligned}$$

where π_{L_m} and π_{K_m} are uniformizers for $\mathcal{O}_{L_m}$ and $\mathcal{O}_{K_m}$, respectively. But then we have that $\pi_{L_m}^j \in \mathrm{Tr}_{L_m/K_m}(\mathcal{O}_{L_m}) \subset \mathrm{Tr}_{L/K_\infty}(\mathcal{O}_L)$ and

$$\begin{aligned} v_p(\pi_{K_m}^j) &= jv_p(\pi_{K_m}) \\ &\leq \frac{d_m}{e_{L_m/K_m}} v_p(K_m) \\ &= \delta_m. \end{aligned}$$

From this and $\delta_m \to 0$, it can be seen that $\mathrm{Tr}_{L/K_\infty}(\mathcal{O}_L)$ contains elements of arbitrarily small positive valuation and thus contains all of $\mathfrak{m}_\infty$. $\square$

Remarks 3.2. Note the following:

(1) As seen at the start of the previous proof, given a finite Galois extension L/K_∞, one can find an L_0 such that $L = L_0 K_\infty$ and L_0 is linearly disjoint from K_∞. Thus, the $L_n = L_0 K_n$ form a $\mathbb{Z}_p$-extension of L/L_0.
(2) The following generalization holds by the last remark: If M/K_∞ is a finite subextension of L/K_∞, then $\mathrm{Tr}_{L/M}(\mathcal{O}_L) \supset \mathfrak{m}_M$.
(3) Proposition 3.1 is actually true for K_∞ an infinite APF-extenstion of K, as shown in [5].

Corollary 3.3. *Let L be a finite separable extension of K_∞ and M/K_∞ a finite subextension. Then there exists some $c \in L$ such that $v_p(c) \geq -1$ and $Tr_{L/M}(c) = 1$.*

Proof. By Proposition 3.1 and the remark that followed, there is an $a \in \mathcal{O}_L$ such that $\mathrm{Tr}_{L/M}(a) = p$. Then set $c = \frac{a}{p}$. Since $v_p(a) \geq 0$, it follows that $v_p(c) \geq -1$ and $\mathrm{Tr}_{L/M}(c) = \frac{1}{p}\mathrm{Tr}_{L/M}(a) = 1$. $\square$

Define the map $t : K_\infty \to K$ by

$$t(x) = \frac{1}{p^n}\mathrm{Tr}_{K_n/K}(x), \qquad \text{for } x \in K_n.$$

By composition of trace maps and since K_{n+1}/K_n is totally ramified of degree p, it can be shown that $t(x)$ does not depend on the choice of n where $x \in K_n$.

Proposition 3.4. *Let γ be a generator for Γ_0. There exists a constant $a > 0$ such that for all $x \in K_\infty$,*

$$\left|x - t(x)\right|_p \leq a \left|\gamma(x) - x\right|_p.$$

Proof. This is Proposition 6 of [6]. $\square$

The above proposition shows that t is continuous on K_∞ and so extends by continuity to the completion, $\widehat{K}_\infty$.

Proposition 3.5. *For a nonnegative integer m, let γ_m be a topological generator of Γ_m. Let $Y_m = (\gamma_m - 1)\widehat{K}_\infty$. Then*

- **(a):** $\widehat{K}_\infty = K_m \oplus Y_m$.
- **(b):** *There is an integer $d > 0$, independent of m, such that the congruence $c + (\gamma_m - 1)y \equiv 0 \mod p^{md}$ implies $y \equiv 0 \mod p^{(m-1)d}$ for $c \in K_m$ and $y \in Y_m$.*
- **(c):** *$\gamma_m - 1$, which is 0 on K_m, is bijective with a continuous inverse on Y_m.*
- **(d):** *If $\lambda \in K_m$ is a principal unit, meaning $\left|\lambda - 1\right|_p < 1$ but λ is not a root of unity, then $\gamma_m - \lambda$ is bijective with a continuous inverse on $\widehat{K}_\infty$.*

Proof. Except for (b), this is Proposition 7 of [6] once one notes that the constant a in Proposition 3.4 remains the same when replacing the ground field K with K_m and that $\ker t = Y_m$. For part (b), suppose we have a $c \in K_m$ and a $y \in Y_m$ with $|c + (\gamma_m - 1)y|_p \leq p^{-md}$. We can assume that $y \in K_\infty$ since if y_k is a sequence in K_∞ approaching y with $y_k \equiv 0 \mod p^{(m-1)d}$, then $y \equiv 0 \mod p^{(m-1)d}$.

Using Proposition 3.4 twice, since $t = 0$ on Y_m,

$$\begin{aligned} |y|_p &\leq a\,|(\gamma_m - 1)y|_p \\ &\leq a^2 \left|(\gamma_m - 1)^2 y\right|_p. \end{aligned}$$

Let d be an integer with $a^2 \leq p^d$. Then $(\gamma_m - 1)(c + (\gamma_m - 1)y) = (\gamma_m - 1)^2 y$ as $\gamma_m(c) = c$, so

$$\begin{aligned} |y|_p &\leq a^2\,|(\gamma_m - 1)(c + (\gamma_m - 1)y)|_p \\ &\leq a^2\,|c + (\gamma_m - 1)y|_p \\ &\leq p^d p^{-md} \\ &= p^{-(m-1)d}, \end{aligned}$$

which completes the proof. □

Proposition 3.6. *If $V \subset \widehat{K}_\infty$ is a finite dimensional K_m-vector space and V is Γ_m-stable, then $V \subset K_\infty$.*

Proof. Let γ_m be a generator of Γ_m. Since V is Γ_m stable, $\gamma_m(V) \subset V$, and γ_m is K_m-linear, so γ_m can be viewed as a K_m-linear transformation of V. Let E be the field obtained by adjoining to K_m all eigenvalues of γ_m. Setting $E_\infty = EK_\infty$, the extension E_∞/E has the same properties hypothesized as K_∞/K with Galois group Γ'. Moreover, restriction to K_∞ gives an injective map $\Gamma' \hookrightarrow \Gamma_m \subset \Gamma_0$, so that $\gamma_m^{p^r}$ generates the image of Γ' for some r. Let γ' be the generator of Γ' whose image is $\gamma_m^{p^r}$.

Let $V' = EV$ with an action by Γ' inherited from Γ. Then $\gamma'(ev) = e\gamma_m^{p^r}(v)$ for any $v \in V$ and $e \in E$ and hence V' is stable as an E-vector space under γ'. But this time, any eigenvalue of γ' is a p^r-power of an eigenvalue of γ_m and must therefore lie in E. Let $\lambda \in E$ be an eigenvalue of γ' and $v' \in V'$ a corresponding eigenvector. Then $\gamma'^{p^s} v' = \lambda^{p^s} v'$ for any integer s. But $\gamma'^{p^s} \to 1$ as $s \to \infty$. As the action of Γ' is continuous on E_∞, it follows that $\lambda^{p^s} \to 1$ as $s \to \infty$. But the only way that is possible is if $|\lambda - 1|_p < 1$, or in other words λ is a principal unit. Applying Proposition 3.5(d) to E_∞/E, as $(\gamma' - \lambda)$ is not bijective on $\widehat{E}_\infty$, since $(\gamma' - \lambda)v' = 0$, it must be that λ is in fact a root of unity and whose order is a power of p as $\lambda^{p^s} \to 1$. Because this is true for all eigenvalues of γ', γ'^{p^s} is unipotent for some s. This means that $(\gamma'^{p^s} - 1)$ is nilpotent on V' and so could not have an inverse on V'. But using Proposition 3.5(c) with γ'^{p^s}, $(\gamma'^{p^s} - 1)$ should be bijective on $(\gamma'^{p^s} - 1)\widehat{E}_\infty$. This must mean that $(\gamma'^{p^s} - 1)$ is actually 0 on V'. But then $(\gamma_m^{p^i} - 1)$ is 0 on V for some i, which implies that $V \subset K_\infty$. □

4. Continuous Cohomology

The goal of this section is to reduce the problem of understanding $H^1_{\text{cont}}(\mathcal{G}, \text{GL}_n(\mathbb{C}))$ to uderstanding $H^1_{\text{cont}}(\Gamma, \text{GL}_n(K_\infty))$ by showing that the two sets are in 1-1 correspondence.

Proposition 4.1. *$H^1_{cont}(\mathcal{H}, \text{GL}_n(\mathbb{C}))$ is trivial.*

Proof. Suppose $U : \mathcal{H} \to \text{GL}_n(\mathbb{C})$ is a cocycle. By continuity, there exists an open subgroup $\mathcal{H}_0$ of $\mathcal{H}$ such that $U(\sigma) \equiv 1 \mod p^2$ for all $\sigma \in \mathcal{H}_0$. Consider the exact inflation-restriction sequence

$$1 \to H^1_{\text{cont}}(\mathcal{H}/\mathcal{H}_0, \text{GL}_n(\mathbb{C}^{\mathcal{H}_0})) \to H^1_{\text{cont}}(\mathcal{H}, \text{GL}_n(\mathbb{C})) \to H^1_{\text{cont}}(\mathcal{H}_0, \text{GL}_n(\mathbb{C})).$$

The group $\mathcal{H}/\mathcal{H}_0$ is finite and so the corresponding field extension $\mathbb{C}^{\mathcal{H}_0}/\widehat{K_\infty}$ is finite. Any cocyle of $\mathcal{H}/\mathcal{H}_0$ is actually continuous when GL_n is given the dicrete topology, but then as remarked above (also p.151 Proposition 3 of [4]), all cocycles are cohomologous to the trivial cocycle. So $H^1_{\text{cont}}(\mathcal{H}/\mathcal{H}_0, GL_n(\mathbb{C}^{\mathcal{H}_0}))$ is trivial and hence it suffices to show that U restricted to $\mathcal{H}_0$ is trivial. In other words, we want to find $B \in GL_n(\mathbb{C})$ such that $B^{-1}U_\sigma\sigma(B) = 1$ for all $\sigma \in \mathcal{H}_0$.

We'll do so by constructing a sequence of matrices $\{B_m\}$ and a sequence of cocycles $\{U_m\}$ such that $U_1 = U$, $B_m \equiv 1 \mod p^m$, $U_{m+1}(\sigma) = B_m^{-1}U_m(\sigma)\sigma(B_m)$ and $U_m(\sigma) \equiv 1 \mod p^{m+1}$ for all $\sigma \in \mathcal{H}_0$. As we already have U_1, suppose we have U_m. We will show that there exists $B_m \in \text{GL}_n(\mathbb{C})$ such that $B_m \equiv 1 \mod p^m$ and $B_m^{-1}U_m(\sigma)\sigma(B_m) \equiv 1 \mod p^{m+2}$ for any $\sigma \in \mathcal{H}_0$.

By continuity of U_m, there exists an open, normal subgroup $\mathcal{H}_1$ of $\mathcal{H}_0$ such that $U_m(\sigma) \equiv 1 \mod p^{m+3}$ for all $\sigma \in \mathcal{H}_1$. Let T be a set of coset representatives of $\mathcal{H}_0/\mathcal{H}_1$. By Corollary 3.3 there exists $c \in \mathbb{C}^{\mathcal{H}_1}$ such that $v_p(c) \geq -1$ is integral and $\sum_{\tau \in T} \tau(c) = 1$.

Define B_m by

$$B_m = \sum_{\tau \in T} U_m(\tau)\tau(c).$$

Since $U_m(\tau) \equiv 1 \mod p^{m+1}$ and $v_p(c) \geq -1$, it follows that $B_m \equiv \sum_{\tau \in T} \tau(c) \mod p^m \equiv 1 \mod p^m$. Now fix $\sigma \in \mathcal{H}_0$. For any $\tau \in T$, there exists a unique $\tau' \in T$ and $\sigma' \in \mathcal{H}_1$ such that $\sigma\tau = \tau'\sigma'$. Then

$$\begin{aligned} U_m(\sigma\tau) &= U_m(\tau'\sigma') \\ &= U_m(\tau')\tau'(U_m(\sigma')) \\ &\equiv U_m(\tau') \mod p^{m+3} \end{aligned}$$

since $\sigma' \in \mathcal{H}_1$. Now,

$$\begin{aligned}\sigma(B_m) &= \sum_{\tau\in T} \sigma(U_m(\tau))\sigma\tau(c)\\ &= \sum_{\tau\in T} U_m(\sigma)^{-1}U_m(\sigma\tau)\sigma\tau(c)\\ &= \sum_{\tau\in T} U_m(\sigma)^{-1}U_m(\tau'\sigma')\tau'\sigma'(c)\\ &\equiv \sum_{\tau\in T} U_m(\sigma)^{-1}U_m(\tau')\tau'(c) \mod p^{m+2}\\ &\equiv U_m(\sigma)^{-1}B_m \mod p^{m+2}.\end{aligned}$$

For the desired congruence, $B_m^{-1}U_m(\sigma)\sigma(B_m) \equiv 1 \mod p^{m+2}$, we must use that fact that B_m^{-1} exists as it is formally the geometric series $1 + (1 - B_m) + (1 - B_m)^2 + ...$ which clearly converges as it is Cauchy with respect to the p-adic valuation. Specifically, if S_k represents the sum of the first k terms, then for any $l \geq k$, $S_l - S_k \equiv 0 \mod p^{km}$. Then the sequence of entries $\{(S_k)_{i,j}\}_n$ for any $1 \leq i, j \leq n$ is Cauchy with respect to $||_p$ and so converges in $\mathbb{C}$.

To complete the proof, let $B = B_1B_2...$ be the product which again converges using the same argument as above. Then for any positive integer m and any $\sigma \in \mathcal{H}_0$,

$$\begin{aligned}B^{-1}U(\sigma)\sigma(B) &\equiv B_{m-1}^{-1}...B_1^{-1}U(\sigma)\sigma(B_1)...\sigma(B_{m-1}) \mod p^m\\ &\equiv U_{m-1}\\ &\equiv 1 \mod p^m.\end{aligned}$$

And therefore $B^{-1}U(\sigma)\sigma(B) = 1$ for all $\sigma \in \mathcal{H}_0$ and U is trivial on $\mathcal{H}_0$. □

Corollary 4.2. *There is a bijection* $H^1_{cont}(\Gamma, \mathrm{GL}_n(\widehat{K}_\infty)) \to H^1_{cont}(\mathcal{G}, \mathrm{GL}_n(\mathbb{C}))$.

Proof. The exact inflation-restriction sequence for $\mathcal{H} \subset \mathcal{G}$ gives

$$1 \to H^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_n(\widehat{K}_\infty)) \to H^1_{\mathrm{cont}}(\mathcal{G}, \mathrm{GL}_n(\mathbb{C})) \to H^1_{\mathrm{cont}}(\mathcal{H}, \mathrm{GL}_n(\mathbb{C}))$$

since $\mathcal{G}/\mathcal{H} = \Gamma$ and $\mathbb{C}^{\mathcal{H}} = \widehat{K}_\infty$. Then by Proposition 4.1, $H^1_{\mathrm{cont}}(\mathcal{H}, \mathrm{GL}_n(\mathbb{C})) = 1$ and $H^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_n(\widehat{K}_\infty)) \to H^1_{\mathrm{cont}}(\mathcal{G}, \mathrm{GL}_n(\mathbb{C}))$ is bijective. □

We now want to establish a bijection from $H^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_n(K_\infty))$ to $H^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_n(\widehat{K}_\infty))$ in a process called decompletion.

Proposition 4.3. *The inclusion* $\mathrm{GL}_n(K_\infty) \hookrightarrow \mathrm{GL}_n(\widehat{K}_\infty)$ *induces a bijection on cohomology:*

$$H^1_{cont}(\Gamma, \mathrm{GL}_n(K_\infty)) \to H^1_{cont}(\Gamma, \mathrm{GL}_n(\widehat{K}_\infty)).$$

Proof. We will prove injectivity and surjectivity separately. For each, we need a lemma.

Lemma 4.4. *If U is a cocycle of Γ in $\mathrm{GL}_n(K_\infty)$, then there exists an m such that $U(\sigma) \in \mathrm{GL}_n(K_m)$ for all $\sigma \in \Gamma$.*

Proof. Let m be large enough so that $U(\gamma), U(\sigma_i) \in \mathrm{GL}_n(K_m)$, where $\{\gamma, \sigma_1, ..., \sigma_k\}$ generate Γ topologically. Then as K_m is stable under γ and each σ_i, $U(\sigma) \in \mathrm{GL}_n(K_m)$ for any σ that is a product of the generators by the cocyle condition. Finally, since these products are dense in Γ and U is continuous, $U(\sigma)$ is in the completion of $\mathrm{GL}_n(K_m)$ for any $\sigma \in \Gamma$. But $\mathrm{GL}_n(K_m)$ is complete (as K_m/K is a finite extension). So the image of U lies in $\mathrm{GL}_n(K_m)$. □

Now suppose that U and U' are cocyles of Γ in $\mathrm{GL}_n(K_\infty)$ that are cohomologous in $\mathrm{GL}_n(\widehat{K}_\infty)$. That is, there exists an $M \in \mathrm{GL}_n(\widehat{K}_\infty)$ such that $U'(\sigma) = M^{-1}U(\sigma)\sigma(M)$for every $\sigma \in \Gamma$. Or, equivalently, that $\sigma(M) = U(\sigma)^{-1}MU'(\sigma)$. Using the lemma, let m be large enough such that the images of both U and U' lie in $\mathrm{GL}_n(K_m)$. Let V be the K_m-vector space spanned by the entries of M. It follows from this coboundary condition that V is stable under Γ. Then by Proposition 3.6, V, and hence the entries of M, lie in K_∞. So $M \in \mathrm{GL}_n(K_\infty)$, U and U' are cohomologous in $\mathrm{GL}_n(K_\infty)$, and we have shown injectivity.

For surjectivity, suppose that U is a cocyle of Γ in $\mathrm{GL}_n(\widehat{K}_\infty)$. We want to find a cocyle U' of Γ in $\mathrm{GL}_n(K_\infty)$ that is cohomologous (in $\mathrm{GL}_n(\widehat{K}_\infty)$) to U. To do so, we repeat the strategy of Proposition 4.1 by building up sequences of matrices.

Lemma 4.5. *Let γ_r be a generator of Γ_r for a fixed r, d the constant of Proposition* 3.5*, and $A \in \mathrm{GL}_n(\widehat{K}_\infty)$, $R \in \mathrm{GL}_n(K_r)$ such that $A \equiv 1 \mod p^{2d}$ and $A \equiv R \mod p^{md}$ for some $m \geq 3$. Then there exists a $B \in \mathrm{GL}_n(\widehat{K}_\infty)$ and an $R' \in \mathrm{GL}_n(K_r)$ such that $B \equiv 1 \mod p^{(m-1)d}$, $B^{-1}A\gamma_r(B) \equiv 1 \mod p^{2d}$, and $B^{-1}A\gamma_r(B) \equiv R' \mod p^{(m+1)d}$.*

Proof. Parts (a) and (c) of Proposition 3.5 state that any element of $\widehat{K}_\infty$ is a sum of an element in K_r and an element of $Y_r = (\gamma_r - 1)\widehat{K}_\infty$. So $A = R' + (\gamma_r - 1)S$ for some $R' \in M_n(K_r)$ and $S \in M_n(\widehat{K}_\infty)$ with entries in Y_r. But then

$$\begin{aligned} A - R &= (R' - R) + (\gamma_r - 1)S \\ &\equiv 0 \mod p^{md}. \end{aligned}$$

Now Proposition 3.5(b) states that every entry of S is congruent to 0 modulo $p^{(m-1)d}$. Let $B = 1 - S$ and write $A = 1 + N$ where N is a matrix with $N \equiv 0 \mod p^{2d}$. Then $B^{-1} = 1 + S + S^2 + ...$, $S^2 \equiv 0 \mod p^{2(m-1)d}$, and $NS \equiv 0 \mod p^{(m+1)d}$. We can therefore say that

$$\begin{aligned} B^{-1}A\gamma_r(B) &\equiv (1+S)(1+N)(1-\gamma_r(S)) \mod p^{2(m-1)d} \\ &\equiv 1 + N + S - \gamma_r(S) \mod p^{(m+1)d} \\ &\equiv A - (\gamma_r - 1)S \mod p^{(m+1)d} \\ &\equiv R' \mod p^{(m+1)d}. \end{aligned}$$

Furthermore, modulo p^{2d}, $B^{-1}A\gamma_r(B)$ is congruent to A which is congruent to 1 and this compeltes the proof. □

Now, for our cocycle U, let r be such that $U(\sigma) \equiv 1 \mod p^{3d}$ for any $\sigma \in \Gamma_r$, which we can do by continuity of the action of Γ. Let γ_r be a generator for Γ_r. Beginning with $m = 3$, let $A_3 = U(\gamma_r)$ and $R_3 = 1$. By the lemma, we have sequences of matrices A_m, B_m, R_m with $A_{m+1} = B_m^{-1} A_m \gamma_r(B_m)$, $A_m \equiv R_m \mod p^{md}$, and $B_m \equiv 1 \mod p^{(m-1)d}$ for any m.

Let $B = B_3 B_4 ... = \prod_{m=3}^{\infty} B_m$. For a fixed m_0, $A_m \equiv A_{m_0} \equiv R_{m_0} \mod p^{(m_0-1)d}$ for any $m \geq m_0$. Hence, A_m and R_m both converge to $R \in \mathrm{GL}_n(K_r)$, and furthermore $B^{-1}U(\gamma_r)\gamma_r(B) = R$.

Define the cocycle U' as $U'(\sigma) = B^{-1}U(\sigma)\sigma(B)$. Fix a $\sigma \in \Gamma$. Using the cocycle condition and the fact that $\gamma_r \sigma = \sigma \gamma_r$ for any $\sigma \in \Gamma$, it follows that $\gamma_r(U'(\sigma)) = (U'(\gamma_r))^{-1} U'(\sigma) \sigma(U'(\gamma_r))$. But then the entries of $U'(\sigma)$ span a K_r-vector space that is stable under γ_r and hence all of Γ_r. Then by Proposition 3.6, $U'(\sigma)$ has entries in $\mathrm{GL}_n(K_\infty)$. Thus U' is a cocyle in $\mathrm{GL}_n(K_\infty)$ that is cohomologous to U via the matrix B, which proves surjectivity. □

5. Representations and Sen's Operator

We now apply the results of the previous section to study $\mathbb{C}$-representations. To begin, define, using suggestive notation, the $\mathcal{H}$-invariants of a $\mathbb{C}$-representation W by

$$\widehat{W}_\infty = \{w \in W : \sigma(w) = w \text{ for all } \sigma \in \mathcal{H}\}.$$

$\widehat{W}_\infty$ can clearly be viewed as a $\widehat{K}_\infty$-vector space.

Theorem 5.1. *The natural map $\widehat{W}_\infty \otimes_{\widehat{K}_\infty} \mathbb{C} \to W$ is an isomorphism.*

Proof. Suppose W corresponds to a cocycle $\mathcal{G} \to \mathrm{GL}_n(\mathbb{C})$ given by some basis of W. By Proposition 4.1, this cocycle restricted to $\mathcal{H}$ is cohomologous to the trivial cocycle. In other words, there is a basis $\{e_1, ..., e_n\}$ for W on which $\mathcal{H}$ acts trivially. Therefore, these basis elements are contained in $\widehat{W}_\infty$ and the map $\widehat{W}_\infty \otimes_{\widehat{K}_\infty} \mathbb{C} \to W$ is surjective.

It now suffices to show that the elements $\{e_1, ..., e_n\}$ span $\widehat{W}_\infty$ as a $\widehat{K}_\infty$-vector space. So suppose $w \in W$ is fixed by $\mathcal{H}$ and $w = \sum c_i e_i$ with $c_i \in \mathbb{C}$. Then for $\sigma \in \mathcal{H}$, $\sigma(w) = \sum \sigma(c_i) e_i = w$ and so each c_i must also be fixed by $\mathcal{H}$, hence in $\widehat{K}_\infty$. So w is in the $\widehat{K}_\infty$-span of $\{e_1, ..., e_n\}$. □

Since $\widehat{W}_\infty$ is fixed by $\mathcal{H}$, $\widehat{W}_\infty$ has a continuous and $\widehat{K}_\infty$-semilinear action by Γ. As before, we want to decomplete to K_∞. Define, again suggestively, W_∞ as the set of all vectors in $\widehat{W}_\infty$ whose orbits under Γ generate a finite dimensional K-vector space. Equivalently, W_∞ is the set of vectors whose stabilizers have finite index in Γ. Any such vector is called K-finite. As any single element of K_∞ lies in a finite extension of K, W_∞ is a K_∞-vector space.

Theorem 5.2. *The natural map $W_\infty \otimes_{K_\infty} \widehat{K}_\infty \to \widehat{W}_\infty$ is an isomorphism.*

Proof. The surjectivity of Proposition 4.3 shows that there is a basis $\{e_1, ..., e_n\}$ of $\widehat{W}_\infty$ such that $\sigma(e_i)$ lies in the K_∞-span of $\{e_1, ..., e_n\}$ for any i and any $\sigma \in \Gamma$. Let $U : \Gamma \to \mathrm{GL}_n(K_\infty)$ be the corresponding cocycle. Furthermore, as in Lemma 4.4, there is a finite extension K_m such that $U(\sigma) \in \mathrm{GL}_n(K_m)$ for all $\sigma \in \Gamma$. As K_m is stable under Γ, we deduce that $K_m e_1 \oplus ... \oplus K_m e_n$ is stable under Γ. This means that the Γ-orbit of each e_i generates a finite dimensional K-vector space and so each $e_i \in W_\infty$.

Again, it remains to prove that each $w \in W_\infty$ can be written as a K_∞-linear combination of the e_i's. Suppose $w = \sum c_i e_i$ with $c_i \in \widehat{K}_\infty$. As each of $w, e_1, ..., e_n$ are K-finite, they have stabilizers with finite index in Γ. Therefore, their intersection, H, also has finite index in Γ. But then for any $\sigma \in H$,

$$\begin{aligned}\sigma(w) &= \sum_{i=1}^{n} \sigma(c_i)e_i \\ &= w,\end{aligned}$$

and so it must be that $\sigma(c_i) = c_i$ for each i. Because c_i is fixed by H and H has finite index in Γ, it follows that $c_i \in K_r$ for some r. Hence, we can conclude that $c_i \in K_\infty$ for all i. □

While we have simplified the study of $\mathbb{C}$-representations W to the space W_∞, the action of Γ is still semilinear. But as it turns out, there is a linear operator on W_∞, and which can be extended to all of W, that encapsulates the action of Γ. This is known as Sen's operator φ.

Let γ be a generator for Γ_0 so that any element of Γ_0 is γ^t for some $t \in \mathbb{Z}_p$. Informally, imagine defining an operator ψ on W_∞ by

$$\psi(w) = \lim_{t\to 0} \frac{\gamma^t(w) - w}{t}$$

In other words, ψ is in essence the derivative (with respect to t) of $t \mapsto \gamma^t$ at $t = 0$, i.e. $\psi = \log \gamma$. In addition, notice that for any $c \in K_\infty$, c is fixed by some open subset of Γ and hence, for t small enough, $\gamma^t(cw) = c\gamma^t(w)$. Then it follows that $\psi(cw) = c\psi(w)$. The end result is a K_∞-linear operator on W_∞ that encodes the action of γ (and hence Γ) on the space.

Of course, the key word is 'Informally', so we need to show that this limit exists and that the result is the logarithm of γ. Moreover, we would prefer to have an operator that does not depend on the choice of generator γ. Suppose that $\gamma' = \gamma^s$ is another generator of Γ_0. Then in the above, we would have $\log(\gamma^s) = s\log(\gamma)$. The correction is to consider the character χ of $\mathcal{G}$, and divide ψ by $\log(\chi(\gamma))$. Then $\log(\chi(\gamma^s)) = s\log(\chi(\gamma))$, which removes the dependence on the choice of generator.

Theorem 5.3. *There exists a K_∞-linear operator φ on W_∞ such that, for every $w \in W_\infty$, there exists an open subgroup Γ_w of Γ such that*

$$\sigma(w) = exp(\varphi \log \chi(\sigma))w$$

for all $\sigma \in \Gamma_w$

Proof. As in the proof of Theorem 5.2, there exists a basis $e_1, ..., e_n$ of W_∞ such that $W_m = K_m e_1 \oplus ... \oplus K_m e_n$ is stable under Γ. Then the action of Γ_m on W_m is linear, and so can be given by a linear homomorphism $\rho : \Gamma_m \to \mathrm{GL}_n(K_m)$.

Let $r \in \mathbb{Z}_p$ be small enough so that $\gamma^r \in \Gamma_m$ and $\rho(\gamma^r) \equiv 1 \mod p$ and replace Γ_m by an open subgroup if necessary so that γ^r generates Γ_m. Define the logarithm on $M_n(K_m)$ by

$$\log(A) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(A-1)^k}{k},$$

which will converge when $A \equiv 1 \mod p$. In particular, $\log \rho(\gamma^r)$ makes sense. Note also that for $B \equiv 0 \mod p$, the exponential function defined as

$$\exp(B) = \sum_{k=1}^{\infty} \frac{B^k}{k!}$$

also makes sense. Finally, $\log(A) \equiv 0 \mod p$ and $\exp(\log(A)) = A$ for $A \equiv 1 \mod p$. For more details, see [2].

Now define the operator φ on W_m by

$$\varphi = \frac{\log \rho(\gamma^r)}{\log \chi(\gamma^r)},$$

which is K_m-linear since $\gamma^r \in \Gamma_m$. It is then clear that the formula of the theorem holds on W_m using $\Gamma_w = \Gamma_m$.

Extend φ by linearity to all of W_∞ and suppose $w \in W_\infty$. Write w as $\sum \lambda_i e_i$. Then there is an open subgroup Γ' of Γ that fixes all of the λ_i. Let $\Gamma_w = \Gamma_m \cap \Gamma'$. Then for any $\sigma \in \Gamma_w$, $\sigma = (\gamma^r)^t$ for some $t \in \mathbb{Z}_p$. Then

$$\begin{aligned}
\exp(\varphi \log \chi(\sigma)) &= \exp(\frac{\log \rho(\gamma^r)}{\log \chi(\gamma^r)} \log \chi(\gamma^r)^t) \\
&= \exp(t \log \rho(\gamma^r)) \\
&= \rho(\gamma^r)^t \\
&= \rho((\gamma^r)^t) \\
&= \rho(\sigma).
\end{aligned}$$

For uniqueness, suppose φ' also satisfies the formula. Then there exists a t such that $(\gamma^r)^t \in \Gamma_{e_i}$ for each basis element e_i. But then it follows that

$$\rho((\gamma^r)^t) = \exp(\varphi' \log \chi((\gamma^r)^t)),$$

or that

$$\begin{aligned}
\varphi' &= \frac{\log \rho(\gamma^r)^t}{\log \chi((\gamma^r)^t)} \\
&= \frac{\log \rho(\gamma^r)}{\log \chi(\gamma^r)} \\
&= \varphi.
\end{aligned}$$

□

As a remark, notice that

$$\begin{aligned}\frac{1}{\log \chi(\gamma)} \lim_{t\to 0} \frac{\gamma^t(w) - w}{t} &= \frac{1}{\log \chi(\gamma)} \lim_{t\to 0} \frac{(\exp(t\varphi \log \chi(\gamma)) - 1)(w)}{t} \\ &= \frac{1}{\log \chi(\gamma)} \log \chi(\gamma)\varphi(w) \\ &= \varphi(w)\end{aligned}$$

Finally, one can continue to extend φ by linearity to all of W, via Theorem 5.1, though the formula of the theorem does not hold for all of W.

To end the section, we want to show that two $\mathbb{C}$-representations are isomorphic if and only if they have Sen Operators that are similar to each other. We will need a couple of facts about the operator φ to continue.

First, as the action of Γ on W_∞ is continuous, it is clear from the definition of φ (and the fact that Γ is abelian) that the actions of φ and Γ commute. Moreover, by the extension of φ to W, it follows that the actions of φ and $\mathcal{G}$ commute.

Proposition 5.4. *Suppose that W_1 and W_2 are $\mathbb{C}$-representations with respective Sen operators φ_1 and φ_2. The $\mathbb{C}$-vector space $W = \mathrm{Hom}_{\mathbb{C}}(W_1, W_2)$ also has an action of $\mathcal{G}$:*

$$(\sigma \cdot f)(w) = \sigma(f(\sigma^{-1}(w))).$$

Then the Sen Operator for W is given by $\varphi : f \mapsto \varphi_2 \circ f - f \circ \varphi_1$.

Proof. For a fixed $w \in (W_1)_\infty$, consider $\gamma^t(f(\gamma^{-t}w)) - f(w)$. For simplicity, assume (as we will be taking $t \to 0$) that $\gamma = \gamma^r$, where γ^r can define both φ_1 and φ_2 as in the proof of Theorem 5.3. The matrix representation of γ under both W_1 and W_2 is within the radius of convergence for the logarithm. Again for simplicity denote these by $\log_1 \gamma$ and $\log_2 \gamma$, respectively. Then by Taylor expansion around $t = 0$,

$$\begin{aligned}\gamma^t(f(\gamma^{-t}w)) - f(w) &= (1 + t\log_2 \gamma)(f(\gamma^{-t}w)) + O(t^2)f(\gamma^{-t}w) - f(w) \\ &= (1 + t\log_2 \gamma)f((1 - t\log_1 \gamma)w + O(t^2)w) + O(t^2)f(w + O(t)w) - f(w) \\ &= f(w) - tf(\log_1 \gamma\, w) + t\log_2 \gamma\, f(w) + O(t^2)f(w) - f(w) \\ &= t\log_2 \gamma\, f(w) - tf(\log_1 \gamma\, w) + O(t^2)f(w).\end{aligned}$$

Now, if φ is a Sen Operator for W, then

$$\begin{aligned}\varphi(f)(w) &= \frac{1}{\log \chi(\gamma)} \lim_{t\to 0} \frac{\gamma^t \cdot f(w) - f(w)}{t} \\ &= \frac{1}{\log \chi(\gamma)} \left(\lim_{t\to 0} \frac{t\log_2 \gamma\, f(w)}{t} - \lim_{t\to 0} \frac{tf(\log_1 \gamma\, w)}{t} \right) \\ &= \frac{1}{\log \chi(\gamma)} \left(\lim_{t\to 0} \frac{\gamma^t f(w) - f(w)}{t} - \lim_{t\to 0} \frac{f(\gamma^t(w)) - f(w)}{t} \right) \\ &= (\varphi_2 \circ f - f \circ \varphi_1)(w).\end{aligned}$$

□

Finally, one more proposition about the kernel of φ:

Proposition 5.5. *The kernel of φ on W is the $\mathbb{C}$-subspace generated by the $\mathcal{G}$ invariants of W.*

Proof. First, any element of W fixed by $\mathcal{G}$ is fixed by $\mathcal{H}$ and hence in $\widehat{W}_\infty$, but is also fixed by Γ and so is in W_∞. But from the limit definition of φ, or via properties of the exponential, it follows that φ of any such element is zero.

Now consider $\ker\varphi$. Since φ and $\mathcal{G}$ commute, if $w \in \ker\varphi$, then $\varphi(\sigma(w)) = \sigma(\varphi(w)) = 0$ for any $\sigma \in \mathcal{G}$ and therefore $\ker\varphi$ is stable under $\mathcal{G}$. As the Sen Operator of a subspace is just the restriction to that subspace, we can assume that $W = \ker\varphi$ with Sen Operator $\varphi = 0$. Suppose $w \in W_\infty$. Then since $\varphi = 0$, w is fixed by Γ_w, where Γ_w is as in Theorem 5.3. Then the orbit of Γ on w is the same as the orbit of Γ/Γ_w on w, which is finite as Γ/Γ_w is finite. But then the action of Γ on W_∞ is continuous for the discrete topology of W_∞. Hilbert's Theorem 90 states that the group cohomology in this case is trivial. Then similar to the proof of Theorem 5.1, there exists a basis $\{e_1, ..., e_n\}$ of W_∞, such that each of the e_i's are fixed by Γ. This implies that the e_i's are $\mathcal{G}$-invariant and these generate the $\mathbb{C}$-space $W = \ker\varphi$. □

We are now ready to prove our desired result.

Theorem 5.6. *Let W_1 and W_2 be two $\mathbb{C}$-representations, and φ_1 and φ_2 their respective operators. Then W_1 and W_2 are isomorphic as $\mathbb{C}$-representations if and only if φ_1 and φ_2 are similar.*

Proof. First, suppose that $F : W_1 \to W_2$ is an isomorphism of $\mathbb{C}$-representations. This means that $\sigma \circ F = F \circ \sigma$ for every $\sigma \in \mathcal{G}$. Put another way $\sigma \circ F \circ \sigma^{-1} = F$, which is to say that F is $\mathcal{G}$-invariant. Let $W = \mathrm{Hom}_{\mathbb{C}}(W_1, W_2)$. By Proposition 5.4, its operator is given by $\varphi : g \to \varphi_2 \circ g - g \circ \varphi_1$. Since F is also a morphism of $\mathbb{C}$-vector spaces, $F \in W$. But F is also $\mathcal{G}$-invariant, so by Proposition 5.5 $\varphi(F) = 0$. That is, $\varphi_2 \circ F - F \circ \varphi_1 = 0$, so $\varphi_1 = F^{-1}\varphi_2 F$.

The other direction nearly follows the same idea, except that φ_1 being similar to φ_2 implies that there is a $\mathbb{C}$-vector space isomorphism $f : W_1 \to W_2$ such that $\varphi_2 \circ f = f \circ \varphi_1$. The issue is that f may not be a morphism of $\mathbb{C}$-representations. That is, it may not be the case that $\sigma \circ f = f \circ \sigma$ for every $\sigma \in \mathcal{G}$. However, as we will see, f gives rise to another morphism F that does commute with $\mathcal{G}$.

Using the same W as above, $f \in W$, and the similarity condition means f is killed by φ. Now by Proposition 5.5, f is the $\mathbb{C}$-linear combination of $\mathcal{G}$-invariant elements $f_1, ..., f_m$ of W. Let $f = c_1 f_1 + ... + c_m f_m$. Viewing the f_i as matrices via fixed bases of W_1 and W_2, because f is an isomorphism,

$$\det(c_1 f_1 + ... + c_m f_m) \neq 0.$$

Moreover, this means that the polynomial in m variables, $t_1, ..., t_m$, given by $\det(t_1 f_1 + ... + t_m f_m)$ is not zero. As K is an infinite field, there exist elements $\lambda_1, ..., \lambda_m \in K$ such that

$$\det(\lambda_1 f_1 + ... + \lambda_m f_m) \neq 0.$$

Let $F = \lambda_1 f_1 + ... + \lambda_m f_m$. Each f_i is $\mathcal{G}$-invariant, i.e. $\sigma \circ f_i \circ \sigma^{-1} = f_i$, meaning that each f_i is a morphism of $\mathbb{C}$-representations. Since the $\lambda_i \in K$ are also fixed by $\mathcal{G}$, F is a $\mathbb{C}$-representation. Finally, by the non-zero determinant condition, F is an isomorphism between W_1 and W_2. □

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