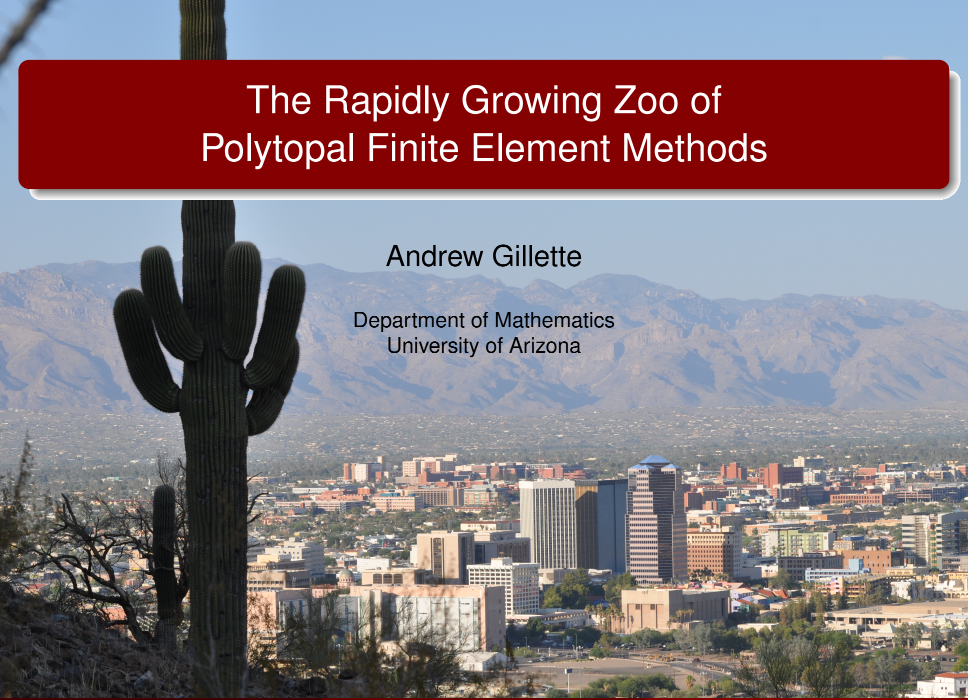


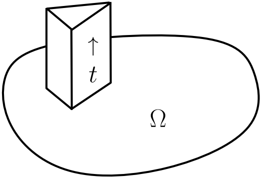
The Rapidly Growing Zoo of Polytopal Finite Element Methods

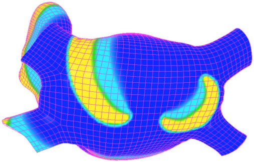
Andrew Gillette

Department of Mathematics
University of Arizona

What are finite element methods?

The **finite element method** is a way to numerically approximate the solution to PDEs.





$$\begin{bmatrix} \quad \mathbb{A} \quad \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \end{bmatrix}$$

CHARACTERIZE	DISCRETIZE	SOLVE
Real analysis	Geometry & Topology	Linear algebra
PDEs	Combinatorics	Numerical analysis

Examples of research questions related to finite elements:

- Does the PDE have a unique (weak) solution, bounded in some norm?
- How does the discretization affect the quality of the approximate solution?
- Is the solution method optimally efficient?

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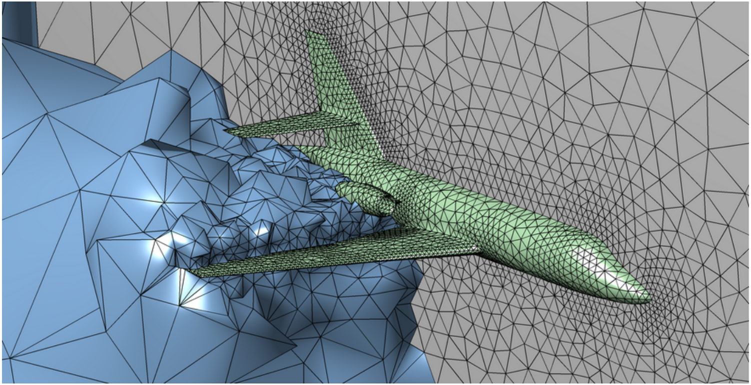
1. Why Polytopal Meshes?
2. The Zoo of Polytopal Finite Element Methods
3. Generalized Barycentric Coordinate Methods
4. Quadratic serendipity elements on polygons
5. Recent applications and open problems

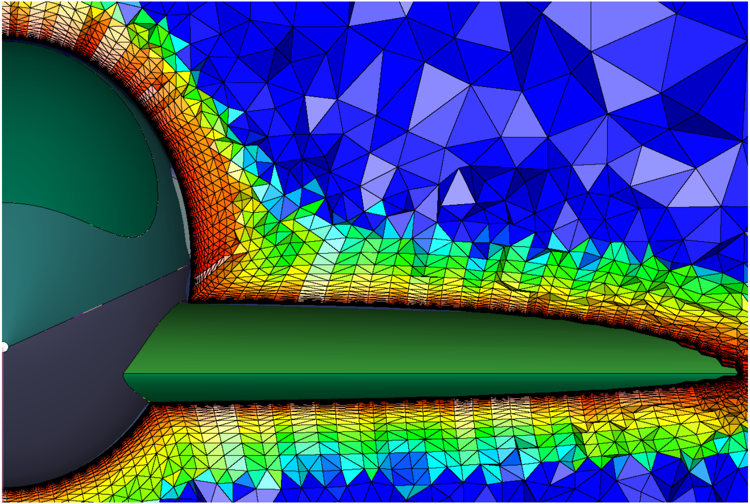
Volume meshing for Computational Fluid Dynamics



*Tetrahedral volume mesh for CFD, using **DistMesh** software.*

(courtesy of Per-Olof Persson)

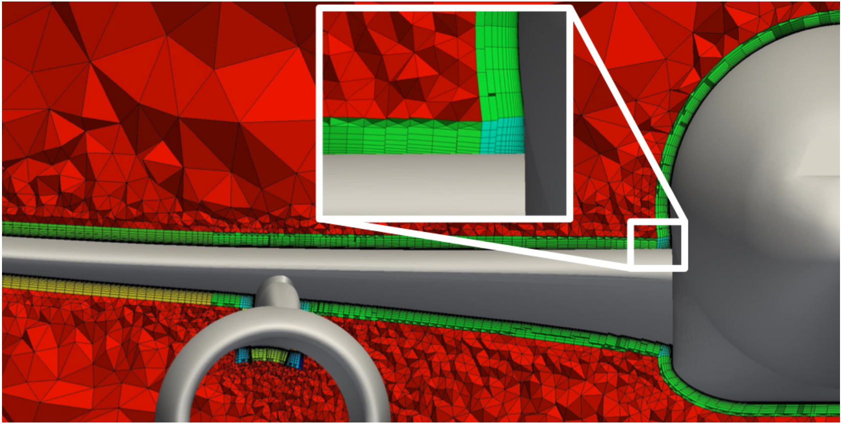
Volume meshing for Computational Fluid Dynamics



Tetrahedral volume mesh for CFD, using ***Pointwise*** *software.*

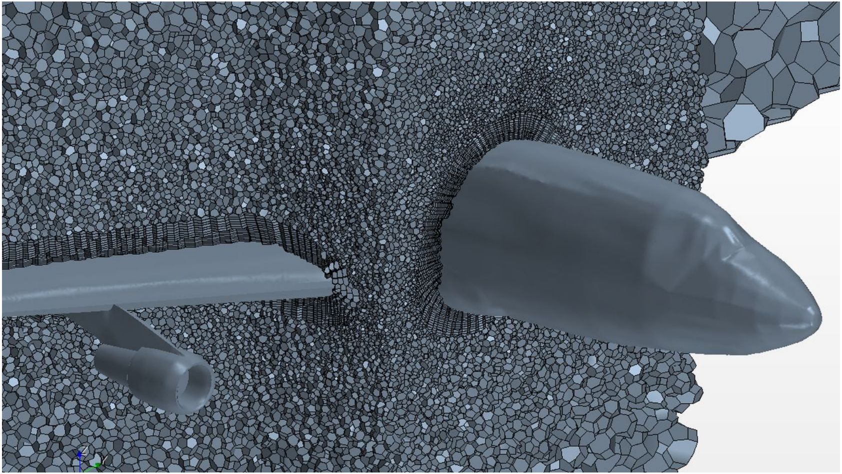
(from blog.pointwise.com)

Volume meshing for Computational Fluid Dynamics



*Hyrbid hex / pyramid / prism / tet mesh for CFD, using **ITI Transcendata** software.*
(from a keynote address at Geometric Modeling and Processing 2015)

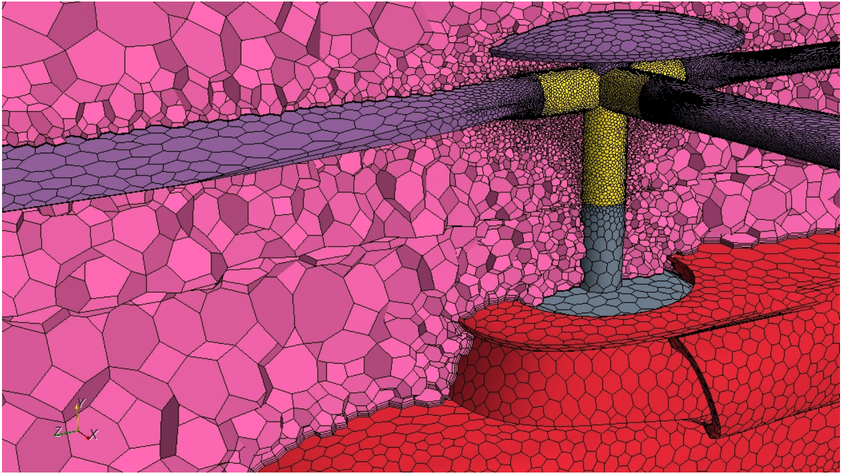
Volume meshing for Computational Fluid Dynamics



Body-aligned prismatic polyhedral meshes for CFD, using ***CD-adapco*** *software.*

(from cd-adapco.com image gallery)

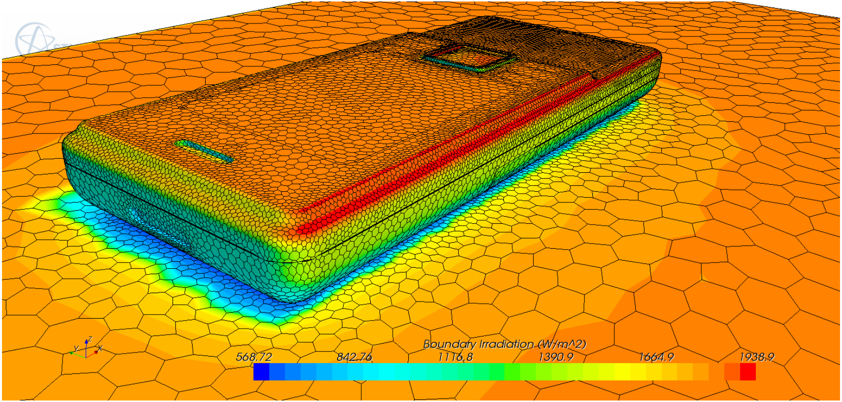
Volume meshing for Computational Fluid Dynamics



Polyhedral mesh of a Bell 407 helicopter and surrounding volume.

(from cd-adapco.com image gallery)

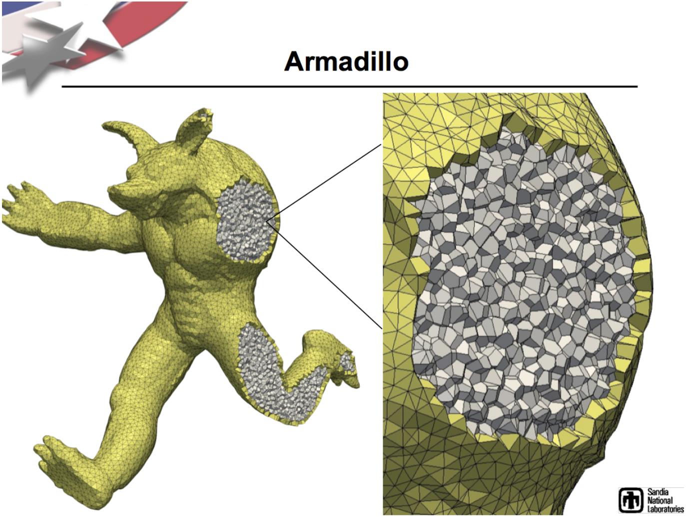
Volume meshing for. . . cel phone design!



A polyhedral mesh used to study heat transfer and cooling of a cell phone.

(from cd-adapco.com image gallery)

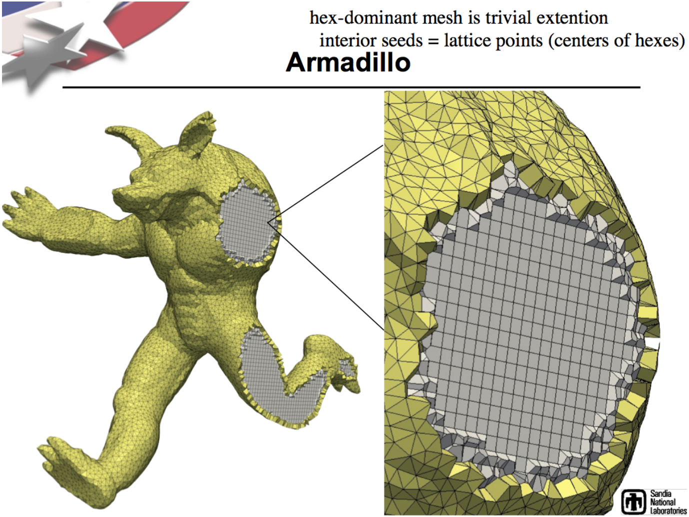
Volume meshing at Sandia National Labs



A polyhedral mesh conforming to a surface triangulation using ***VoroCrust*** *software.*

(from Scott Mitchell, Sandia National Labs)

Volume meshing at Sandia National Labs



hex-dominant mesh is trivial extention
interior seeds = lattice points (centers of hexes)

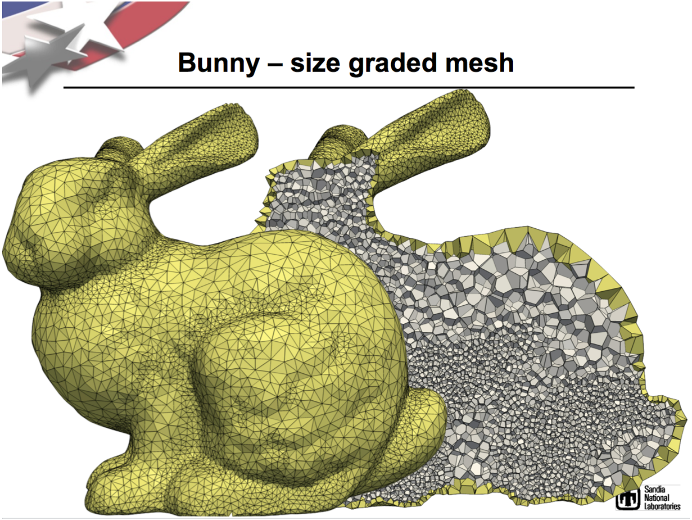
Armadillo

Sandia National Laboratories

*A polyhedral mesh conforming to a surface triangulation using **VoroCrust** software.*

(from Scott Mitchell, Sandia National Labs)

Volume meshing at Sandia National Labs



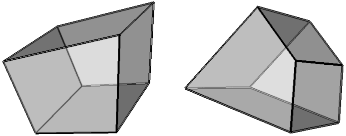
Bunny – size graded mesh

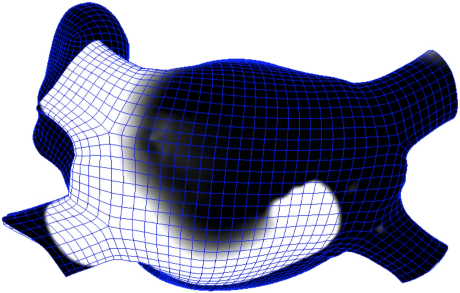
*The requisite Stanford Bunny example using **VoroCrust** software.*

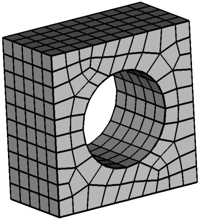
(from Scott Mitchell, Sandia National Labs)

Hexahedral meshing is polyhedral meshing

Meshes of generic hexahedra require a generalized theory of polyhedral discretization, related to but distinct from the theory for perfect tensor product meshes.







↑ *Heart mesh made using Continuity software, National Biomedical Computation Resource, UCSD*

← *Hole mesh made using CUBIT Geometry and Mesh Generation Toolkit, Sandia National Labs*

Outline

1. Why Polytopal Meshes?
2. The Zoo of Polytopal Finite Element Methods
3. Generalized Barycentric Coordinate Methods
4. Quadratic serendipity elements on polygons
5. Recent applications and open problems

The Zoo of Polytopal Finite Element Methods

$$
\begin{array}{ccccc}
\text{VEM} & \text{FES} & \text{WG} & \text{GS} & \text{PDG} \\
\text{HFV} & \ddots & \vdots & \iddots & \text{CDG} \\
\text{MFD} & \cdots & \boxed{\text{FEM}} & \cdots & \text{GBC} \\
\text{DGA} & \iddots & \vdots & \ddots & \text{DEC} \\
\text{CDO} & \text{HHO} & \text{DPG} & \text{HDG} & *\,*\,*
\end{array}
$$

Some of the areas of active research (listed alphabetically by acronym):

GBC = Generalized barycentric coordinate methods
HDG = Hybrid Discontinuous Galerkin methods
HHO = Hybrid Higher-Order methods
MFD = Mimetic finite difference
VEM = Virtual element methods
WG = Weak Galerkin methods

“Polytopal Element Methods in Mathematics and Engineering”

→ Special NSF-funded workshop held at Georgia Tech in Oct 2015

→ Slides from talks: `http://www.poems15.gatech.edu/`

How the zoo started. . .

MFD · · · $\boxed{\text{FEM}}$ · · · GBC

MFD = **Mimetic finite differences** (since the 1960s)

→ Idea is to mimic key properites like conservation laws, symmetry of solutions, duality of variables, etc.
→ 'Straightforward' to implement.
→ Difficult to analyze error.

GBC = **Generalized barycentric coordinates** (since the 1970s)

→ Idea is to generalize basis functions for simple shapes to general polygons and polyhedra.
→ More obstacles to implementation.
→ Easier to analyze error.

One of my main interests: Error analysis for GBC methods
→ relates to the 'quality' of polyhedral meshing.

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The generalized barycentric coordinate approach

Let P be a convex polytope with vertex set V. We say that

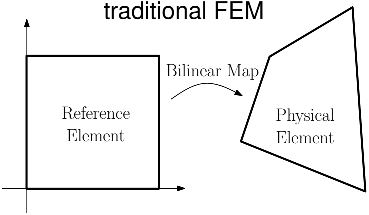
$\lambda_{\mathbf{v}} : P \to \mathbb{R}$ are **generalized barycentric coordinates (GBCs)** on P

if they satisfy $\lambda_{\mathbf{v}} \geq 0$ on P and $L = \sum_{\mathbf{v} \in V} L(\mathbf{v_v})\lambda_{\mathbf{v}}, \quad \forall\, L : P \to \mathbb{R}$ linear.

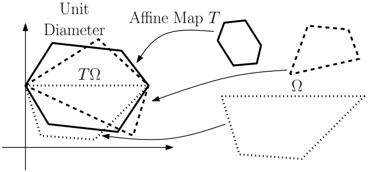
Familiar properties are implied by this definition:

$$\underbrace{\sum_{\mathbf{v}\in V} \lambda_{\mathbf{v}} \equiv 1}_{\text{partition of unity}} \qquad \underbrace{\sum_{\mathbf{v}\in V} \mathbf{v}\lambda_{\mathbf{v}}(\mathbf{x}) = \mathbf{x}}_{\text{linear precision}} \qquad \underbrace{\lambda_{\mathbf{v}_i}(\mathbf{v}_j) = \delta_{ij}}_{\text{interpolation}}$$

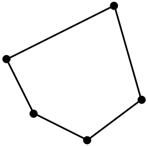
traditional FEM



family of **GBC** reference elements



Geometry dependence for finite elements



We consider ***a priori* error estimates** for finite element methods.

P,	a polygon or polytope geometry.
u,	a function from P to $\mathbb{R}$, known only at the vertices of P.
$\mathcal{I}u$,	an interpolant for u over P.
C,	a constant depending on the geometry of P

$$\underbrace{|u-\mathcal{I}u|_{H^1(P)}}_{\text{error between function and its interpolation}} \leq \underbrace{C \cdot \operatorname{diam}(P)\ |u|_{H^2(P)}}_{\text{bound depends on size and geometry of } P}, \quad \underbrace{\forall u \in H^2(P)}_{\text{estimate holds for any } u \text{ with bounded 2nd derivatives}}$$

Scaling P so that $\operatorname{diam}(P) = 1$, we examine how C depends on the geometry of P;

In particular, when is this bound ***sharp?***

The 40 year history of the triangular case

$$|u - \mathcal{I}u|_{H^1(P)} \leq C \cdot \operatorname{diam}(P)\ |u|_{H^2(P)}, \quad \forall u \in H^2(P)$$

If P is a triangle, the **maximum angle condition** refers to the fact that C can be large when P has a large interior angle.

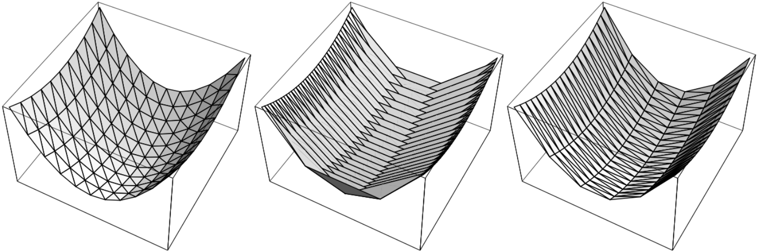


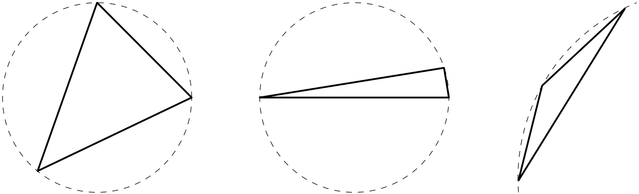
Figure from: Shewchuk: What is a good linear element? *Int'l Meshing Roundtable*, 2002.

Babuška, Aziz: On the angle condition in the finite element method, *SINUM*, 1976.
Jamet: Estimations d'erreur pour des éléments finis. . . *RAIRO Analyse Numérique.*, 1976.
Rippa, Schiff: Minimum energy triangulations for elliptic problems, *CMAME*, 1990.
Shenk: Uniform error estimates for certain narrow Lagrange finite elements, *Math. Comp.*, 1994.
Warren et al: Barycentric coordinates for convex sets, *Adv. Comp. Math.*, 2007.

The triangular case, revisited

The **aspect ratio** of a polygon P is $\dfrac{\text{diam}(P)}{\text{max radius of an inscribed circle}}$.

The **circumradius** of a triangle T is $R_T :=$ radius of circle through the vertices of T.



aspect ratio	small	**large**	large
circumradius	small	**small**	large
C	small	**small**	large

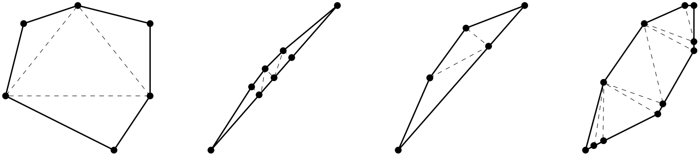
On triangles, a large aspect ratio does not necessarily imply that C is large. . .

. . . but a large circumradius always does.

Polygonal shape quality measures

The **aspect ratio** of a polygon P is $\dfrac{\mathrm{diam}(P)}{\text{max radius of an inscribed circle}}$.

The **circumradius** of a triangle T is $R_T :=$ radius of circle through the vertices of T.



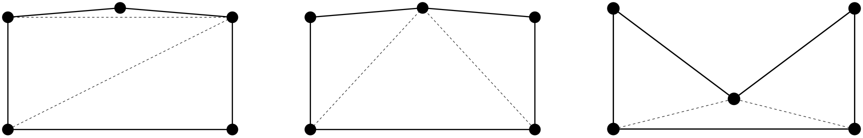
polygon aspect ratio	good	**bad**	bad	okay?
quality of triangulation	good	**good**	bad	okay?

On polygons, a large aspect ratio does not correlate in a clear way with the circumradii of triangulations of its vertices

Polygonal shape quality measures

The **aspect ratio** of a polygon P is $\dfrac{\mathrm{diam}(P)}{\text{max radius of an inscribed circle}}$.

The **circumradius** of a triangle T is $R_T :=$ radius of circle through the vertices of T.



not Delaunay

max R_T large

Delaunay

max R_T small

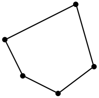
(constrained) Delaunay

max R_T large
(and no alternative triangulation possible!)

Over all possible triangulations of the vertices of a polygon, the **constrained Delaunay triangulation** will have the minimal maximum circumradius.

Two related interpolants on polygons

The **harmonic interpolant** is the solution to



$$\begin{cases} \Delta\left(\mathcal{I}_P u\right) = 0, & \text{on } P, \\ \mathcal{I}_P u = g_u. & \text{on } \partial P. \end{cases}$$

where $g_u : \partial\Omega \to \mathbb{R}$ is the piecewise linear function equal to u at the vertices of P.

By **Dirichlet's principle**, $\mathcal{I}_P u$ is also the minimizer of H^1 semi-norm:

$$\mathcal{I}_P u = \operatorname{argmin}\left\{ |v|_{H^1(P)} \;:\; v = g_u \text{ on } \partial P \right\}$$

The **triangulation interpolant**, denoted $\mathcal{I}_{\mathcal{T}} u$, is the piecewise linear interpolation of g_u with respect to a triangulation $\mathcal{T}$ of the vertices of P.

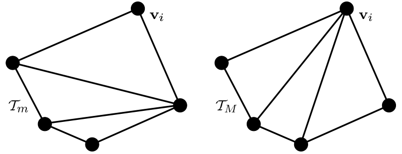
Theorem: There exists a constant $C > 0$ such that for any polygon P, and all triangulations $\mathcal{T}$ of P,

$$|u - \mathcal{I}_P u|_{H^1(P)} \leq C \left(\max_{T \in \mathcal{T}} R_T \right) |u|_{H^2(P)}, \qquad \forall u \in H^2(P),$$

where R_T denotes the circumradius of triangle T.

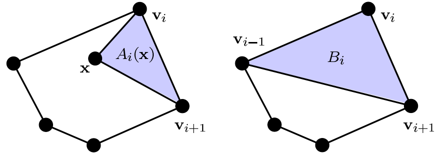
G., Rand Interpolation Error Estimates for Harmonic Coordinates On Polytopes, arXiv:1504.00599 2015.

Many other barycentric coordinates are available . . .

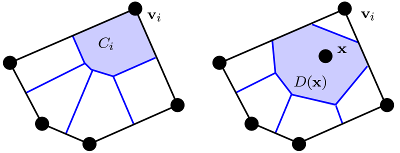


- Triangulation
  $\Rightarrow$ FLOATER, HORMANN, KÓS, *A general construction of barycentric coordinates over convex polygons*, 2006

$$0 \leq \lambda_i^{\mathcal{T}_m}(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq \lambda_i^{\mathcal{T}_M}(\mathbf{x}) \leq 1$$

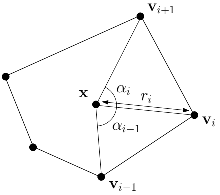


- Wachspress
  $\Rightarrow$ WACHSPRESS, *A Rational Finite Element Basis*, 1975.
  $\Rightarrow$ WARREN, *Barycentric coordinates for convex polytopes*, 1996.

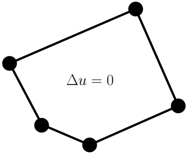


- Sibson / Laplace
  $\Rightarrow$ SIBSON, *A vector identity for the Dirichlet tessellation*, 1980.
  $\Rightarrow$ HIYOSHI, SUGIHARA, *Voronoi-based interpolation with higher continuity*, 2000.

Many other barycentric coordinates are available . . .



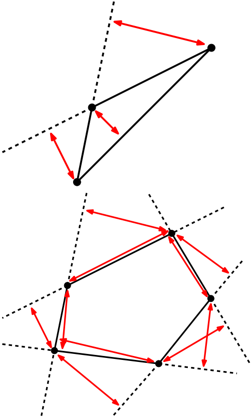
- Mean value
  $\Rightarrow$ FLOATER, *Mean value coordinates*, 2003.
  $\Rightarrow$ FLOATER, KÓS, REIMERS, *Mean value coordinates in 3D*, 2005.



- Harmonic
  $\Rightarrow$ WARREN, SCHAEFER, HIRANI, DESBRUN, *Barycentric coordinates for convex sets*, 2007.
  $\Rightarrow$ CHRISTIANSEN, *A construction of spaces of compatible differential forms on cellular complexes*, 2008.

Many more papers could be cited (maximum entropy coordinates, moving least squares coordinates, surface barycentric coordinates, etc...)

Graident bounds for the Wachspress coordinates



Observe that on triangles of fixed diameter:

$$\begin{aligned} |\nabla\lambda_{\mathbf{v}}| \text{ large} &\iff \text{interior angle at } \mathbf{v} \text{ is large} \\ &\iff \text{the altitude “at } \mathbf{v}\text{” is small} \end{aligned}$$

For Wachspress coordinates, we generalize to polygons:

$$|\nabla\lambda_{\mathbf{v}}| \text{ large} \iff \text{the “altitude” at } \mathbf{v} \text{ is small}$$

and then to **simple** polytopes.

(A simple *d*-dimensional polytope
has exactly *d* faces at each vertex)

Given a simple convex *d*-dimensional polytope *P*, let

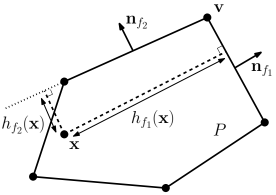
$h_* :=$ minimum distance from a vertex to a hyper-plane of a non-incident face.

$$\text{Then} \quad \sup_{\mathbf{x}\in P} \sum_{\mathbf{v}\in V} |\nabla\lambda_{\mathbf{v}}(\mathbf{x})| =: \Lambda \text{ is large} \iff h_* \text{ is small}$$

Upper bound for simple convex polytopes

Theorem [Floater, G., Sukumar]

Let P be a simple convex polytope in $\mathbb{R}^d$ and let $\lambda_{\mathbf{v}}$ be **generalized Wachspress coordinates**. Then $\Lambda \leq \dfrac{2d}{h_*}$ where $h_* = \min_f \min_{\mathbf{v} \notin f} \operatorname{dist}(\mathbf{v}, f)$



$$\mathbf{p}_f(\mathbf{x}) := \frac{\mathbf{n}_f}{h_f(\mathbf{x})} = \text{normal to face } f\text{, scaled by the reciprocal of the distance from } \mathbf{x} \text{ to } f$$

$$w_{\mathbf{v}}(\mathbf{x}) := \det(\mathbf{p}_{f_1}(\mathbf{x}), \cdots, \mathbf{p}_{f_d}(\mathbf{x}))$$

$$= \text{volume formed by the } d \text{ vectors } \{\mathbf{p}_{f_i}(\mathbf{x})\} \text{ for the } d \text{ faces incident to } \mathbf{v}$$

The **generalized Wachspress coordinates** are defined by

$$\lambda_{\mathbf{v}}(\mathbf{x}) := \frac{w_{\mathbf{v}}(\mathbf{x})}{\sum_{\mathbf{u}} w_{\mathbf{u}}(\mathbf{x})}$$

Proof sketch for upper bound

To prove: $\sup_{\mathbf{x}} \sum_{\mathbf{v}} |\nabla\lambda_{\mathbf{v}}(\mathbf{x})| =: \Lambda \leq \frac{2d}{h_*}$ where $h_* := \min_f \min_{\mathbf{v}\notin f} h_f(\mathbf{v})$.

1. Bound $|\nabla\lambda_{\mathbf{v}}|$ by summations over faces incident and not incident to **v**.

$$|\nabla\lambda_{\mathbf{v}}| \leq \lambda_{\mathbf{v}} \sum_{f\in F_{\mathbf{v}}} \frac{1}{h_f}\left(1-\sum_{\mathbf{u}\in f}\lambda_{\mathbf{u}}\right) + \lambda_{\mathbf{v}} \sum_{f\notin F_{\mathbf{v}}} \frac{1}{h_f}\left(\sum_{\mathbf{u}\in f}\lambda_{\mathbf{u}}\right)$$

2. Summing over **v** gives a constant bound.

$$\sum_{\mathbf{v}} |\nabla\lambda_{\mathbf{v}}| \leq 2\sum_{f\in F} \frac{1}{h_f}\left(1-\sum_{\mathbf{u}\in f}\lambda_{\mathbf{u}}\right)\left(\sum_{\mathbf{u}\in f}\lambda_{\mathbf{u}}\right)$$

3. Write $h_f(\mathbf{x})$ using $\lambda_{\mathbf{v}}$ (possible since h_f is linear) and derive the bound.

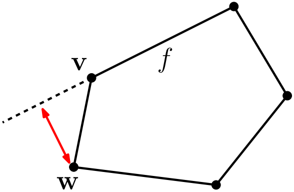
$$\Lambda \leq 2\sum_{f\in F}\left(\sum_{\mathbf{u}\in f}\lambda_{\mathbf{u}}\right)\frac{1}{h_*} = 2\sum_{\mathbf{v}\in V} |\{f \ : \ f \ni \mathbf{v}\}| \lambda_{\mathbf{v}} \frac{1}{h_*} = \frac{2d}{h_*}$$

Lower bound for polytopes

Theorem [Floater, G., Sukumar]

Let P be a simple convex polytope in $\mathbb{R}^d$ and let $\lambda_{\mathbf{v}}$ be **any** generalized barycentric coordinates on P. Then

$$\frac{1}{h_*} \leq \Lambda$$



Proof sketch:

1. Show that $h_* = h_f(\mathbf{w})$, for some particular face f of P and vertex $\mathbf{w} \notin f$.
2. Let $\mathbf{v}$ be the vertex in f closest to $\mathbf{w}$. Show that
$$|\nabla\lambda_{\mathbf{w}}(\mathbf{v})| = \frac{1}{h_f(\mathbf{w})}$$
3. Conclude the result, since
$$\Lambda \geq |\nabla\lambda_{\mathbf{w}}(\mathbf{v})| = \frac{1}{h_f(\mathbf{w})} = \frac{1}{h_*}$$

Upper and lower bounds on polytopes

For a polytope $P \subset \mathbb{R}^d$, define $\Lambda := \sup_{\mathbf{x} \in P} \sum_{\mathbf{v}} |\nabla \lambda_{\mathbf{v}}(\mathbf{x})|$.

simple convex polytope in $\mathbb{R}^d$	$\frac{1}{h_*}$	$\leq$	Λ	$\leq$	$\frac{2d}{h_*}$
d-simplex in $\mathbb{R}^d$	$\frac{1}{h_*}$	$\leq$	Λ	$\leq$	$\frac{d+1}{h_*}$
hyper-rectangle in $\mathbb{R}^d$	$\frac{1}{h_*}$	$\leq$	Λ	$\leq$	$\frac{d+\sqrt{d}}{h_*}$
regular k-gon in $\mathbb{R}^2$	$\frac{2(1+\cos(\pi/k))}{h_*}$	$\leq$	Λ	$\leq$	$\frac{4}{h_*}$

Note that $\lim_{k \to \infty} 2(1+\cos(\pi/k)) = 4$, so the bound is **sharp** in $\mathbb{R}^2$.

FLOATER, G, SUKUMAR *Gradient bounds for Wachspress coordinates on polytopes*, SIAM J. Numerical Analysis, 2014.

Polygonal Finite Element Optimal Convergence

Theorem [G, Rand, Bajaj]

In the table, any necessary geometric criteria to achieve the ***a priori* linear error estimate** are denoted by N. The set of geometric criteria denoted by S in each row ***taken together*** are sufficient to guarantee the estimate.

		G1 (aspect ratio)	**G2** (min edge length)	**G3** (max interior angle)
Triangulated	λ^{Tri}	-	-	S,N
Wachspress	λ^{Wach}	S	S	S,N
Sibson	λ^{Sibs}	S	S	-
Mean Value	λ^{MV}	S	S	-
Harmonic	λ^{Har}	S	-	-

G, Rand, Bajaj *Error Estimates for Generalized Barycentric Interpolation*
Advances in Computational Mathematics, 37:3, 417-439, 2012

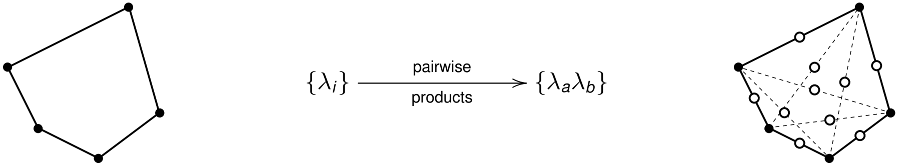
Rand, G, Bajaj *Interpolation Error Estimates for Mean Value Coordinates*,
Advances in Computational Mathematics, 39:2, 327-347, 2013.

Outline

1. Why Polytopal Meshes?
2. The Zoo of Polytopal Finite Element Methods
3. Generalized Barycentric Coordinate Methods
4. Quadratic serendipity elements on polygons
5. Recent applications and open problems

From linear to quadratic elements

A naïve quadratic element is formed by products of linear **GBCs**:



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\}$$

Why is this naïve?

- For a k-gon, this construction gives $k + \binom{k}{2}$ basis functions $\lambda_a\lambda_b$
- The space of quadratic polynomials is only dimension 6: $\{1, x, y, xy, x^2, y^2\}$
- Conforming to a linear function on the boundary requires 2 degrees of freedom per edge $\Rightarrow$ *only $2k$ functions needed!*

Problem Statement

Construct $2k$ basis functions associated to the vertices and edge midpoints of an arbitrary k-gon such that a quadratic convergence estimate is obtained.

Polygonal Quadratic Serendipity Elements

We define matrices $\mathbb{A}$ and $\mathbb{B}$ to reduce the naïve quadratic basis.

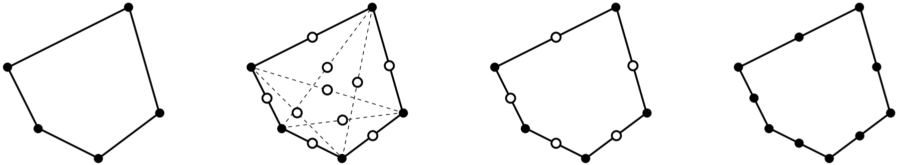
filled dot = **Interpolatory** domain point

= all functions in the set evaluate to 0

except the associated function which evaluates to 1

open dot = non-interpolatory domain point

= partition of unity satisfied, but not a nodal basis



$$\begin{tikzcd}\{\lambda_i\} \arrow[r, "\text{pairwise}", "\text{products}"'] & \{\lambda_a\lambda_b\} \arrow[r, "\mathbb{A}"] & \{\xi_{ij}\} \arrow[r, "\mathbb{B}"] & \{\psi_{ij}\}\end{tikzcd}$$

k	$k+\binom{k}{2}$	$2k$	$2k$
Linear	Quadratic	Serendipity	Lagrange

From quadratic to serendipity

The bases are ordered as follows:

ξ_{ii} and $\lambda_a\lambda_a$ = basis functions associated with vertices
$\xi_{i(i+1)}$ and $\lambda_a\lambda_{a+1}$ = basis functions associated with edge midpoints
$\lambda_a\lambda_b$ = basis functions associated with interior diagonals,
i.e. $b \notin \{a-1, a, a+1\}$

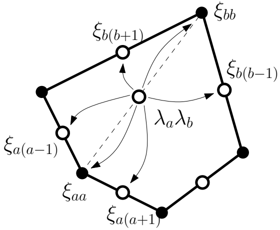
Serendipity basis functions ξ_{ij} are a linear combination of pairwise products $\lambda_a\lambda_b$:

$$\begin{bmatrix} \xi_{ii} \\ \vdots \\ \xi_{i(i+1)} \end{bmatrix} = \mathbb{A} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix} = \begin{bmatrix} c_{11}^{11} & \cdots & c_{ab}^{11} & \cdots & c_{(n-2)n}^{11} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{ij} & \cdots & c_{ab}^{ij} & \cdots & c_{(n-2)n}^{ij} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{n(n+1)} & \cdots & c_{ab}^{n(n+1)} & \cdots & c_{(n-2)n}^{n(n+1)} \end{bmatrix} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix}$$

From quadratic to serendipity

We ***require*** the serendipity basis to have quadratic approximation power:

$$\begin{aligned}
\textbf{Constant precision:} \quad 1 &= \sum_i \xi_{ii} + 2\xi_{i(i+1)} \\
\textbf{Linear precision:} \quad \mathbf{x} &= \sum_i \mathbf{v}_i\xi_{ii} + 2\mathbf{v}_{i(i+1)}\xi_{i(i+1)} \\
\textbf{Quadratic precision:} \quad \mathbf{x}\mathbf{x}^T &= \sum_i \mathbf{v}_i\mathbf{v}_i^T\xi_{ii} + (\mathbf{v}_i\mathbf{v}_{i+1}^T + \mathbf{v}_{i+1}\mathbf{v}_i^T)\xi_{i(i+1)}
\end{aligned}$$



Theorem [Rand, G, Bajaj]

Constants $\{c_{ij}^{ab}\}$ exist for **any** convex polygon such that the resulting basis $\{\xi_{ij}\}$ satisfies constant, linear, and quadratic precision requirements.

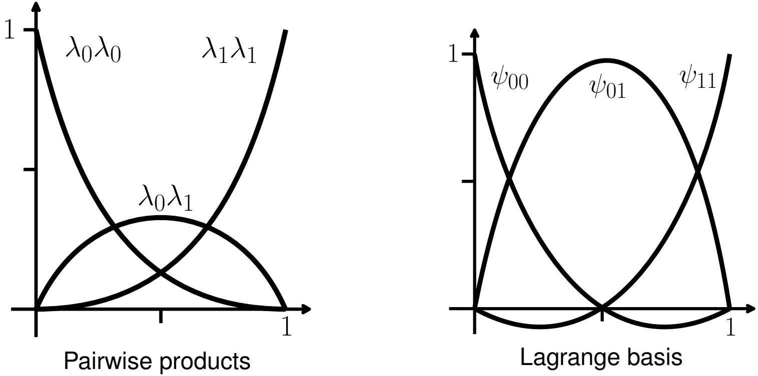
Proof: We produce a coefficient matrix $\mathbb{A}$ with the structure

$$\mathbb{A} := \left[\ \mathbb{I}\ \middle|\ \mathbb{A}'\ \right]$$

where $\mathbb{A}'$ has only six non-zero entries per column and show that the resulting functions satisfy the six precision equations.

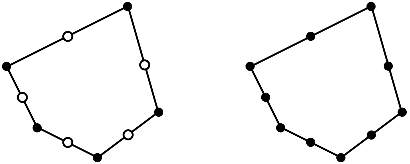
Pairwise products vs. Lagrange basis

Even in 1D, pairwise products of barycentric functions do not form a Lagrange basis at interior degrees of freedom:



Translation between these two bases is straightforward and generalizes to the higher dimensional case.

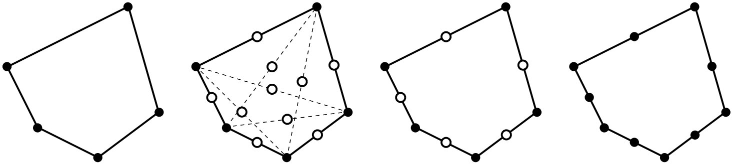
From serendipity to Lagrange



$$\{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

$$[\psi_{ij}] = \begin{bmatrix} \psi_{11} \\ \psi_{22} \\ \vdots \\ \psi_{nn} \\ \psi_{12} \\ \psi_{23} \\ \vdots \\ \psi_{n1} \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 1 & & & & & -1 & & \cdots & & -1 \\ & 1 & & & & -1 & -1 & \cdots & & \\ & & \ddots & & & & \ddots & \ddots & & \\ & & & \ddots & & & & \ddots & \ddots & \\ & & & & 1 & & & & -1 & -1 \\ \hline & & & & & 4 & & & & \\ & & & & & & 4 & & & \\ & & 0 & & & & & \ddots & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 4 \end{array}\right] \begin{bmatrix} \xi_{11} \\ \xi_{22} \\ \vdots \\ \xi_{nn} \\ \xi_{12} \\ \xi_{23} \\ \vdots \\ \xi_{n1} \end{bmatrix} = \mathbb{B}[\xi_{ij}].$$

Serendipity Theorem



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\} \xrightarrow{\mathbb{A}} \{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

Theorem [Rand, G, Bajaj]

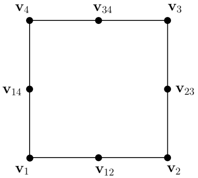
Given bounds on polygonal geometric quality:

- $||\mathbb{A}||$ is uniformly bounded,
- $||\mathbb{B}||$ is uniformly bounded, and
- span$\{\psi_{ij}\} \supset \mathcal{P}_2(\mathbb{R}^2)$ = quadratic polynomials in x and y

We obtain the quadratic *a priori* error estimate: $\boxed{||u - u_h||_{H^1(\Omega)} \leq C\, h^{\mathbf{2}}\, |u|_{H^3(\Omega)}}$

RAND, G, BAJAJ *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Math. Comp., 2011

Special case of a square



Bilinear functions are barycentric coordinates:

$$\begin{aligned}\lambda_1 &= (1-x)(1-y)\\ \lambda_2 &= x(1-y)\\ \lambda_3 &= xy\\ \lambda_4 &= (1-x)y\end{aligned}$$

Compute $[\xi_{ij}] := \left[\ \mathbb{I}\ \middle|\ \mathbb{A}'\ \right][\lambda_a\lambda_b]$

$$\begin{bmatrix}\xi_{11}\\ \xi_{22}\\ \xi_{33}\\ \xi_{44}\\ \xi_{12}\\ \xi_{23}\\ \xi_{34}\\ \xi_{14}\end{bmatrix} = \begin{bmatrix}1 & & 0 & -1 & 0\\ 0 & & 0 & 0 & -1\\ 0 & & 0 & -1 & 0\\ 0 & \cdots & 0 & 0 & -1\\ 0 & \cdots & 0 & 1/2 & 1/2\\ 0 & & 0 & 1/2 & 1/2\\ 0 & & 0 & 1/2 & 1/2\\ 0 & & 1 & 1/2 & 1/2\end{bmatrix}\begin{bmatrix}\lambda_1\lambda_1\\ \lambda_2\lambda_2\\ \lambda_3\lambda_3\\ \lambda_4\lambda_4\\ \lambda_1\lambda_2\\ \lambda_2\lambda_3\\ \lambda_3\lambda_4\\ \lambda_1\lambda_4\end{bmatrix} = \begin{bmatrix}(1-x)(1-y)(1-x-y)\\ x(1-y)(x-y)\\ xy(-1+x+y)\\ (1-x)y(y-x)\\ (1-x)x(1-y)\\ x(1-y)y\\ (1-x)xy\\ (1-x)(1-y)y\end{bmatrix}$$

$$\text{span}\left\{\xi_{ii},\xi_{i(i+1)}\right\} = \text{span}\left\{1,x,y,x^2,y^2,xy,x^2y,xy^2\right\} =: \mathcal{S}_2(I^2)$$

Hence, this provides a computational basis for the serendipity space $\mathcal{S}_2(I^2)$ defined in

ARNOLD, AWANOU *The serendipity family of finite elements*, Found. Comp. Math, 2011.

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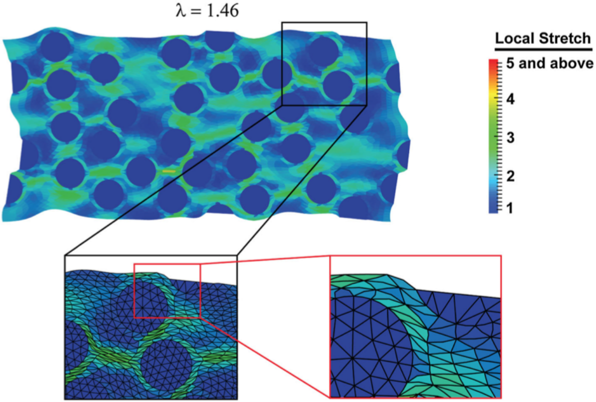
Elasticity modeling

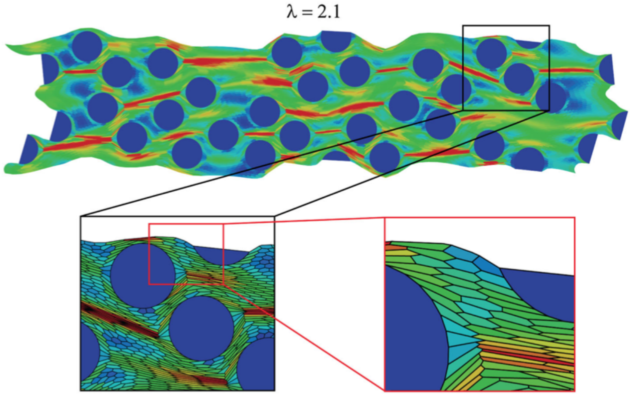
Standard triangular FEM cannot model maximal stretch factors due to numerical errors from the deformation.



(from Chi, Talischi, Lopez-Pamies, Paulino, "Polygonal finite elements for finite elasticity." International Journal for Numerical Methods in Engineering, 2015)

Elasticity modeling

The flexibility of polyhedral meshes (using GBCs!) allows greater shape deformation and more realistic stretch factors.



Chi et al. "Polygonal finite elements for finite elasticity."
Talischi et al. "Gradient correction for polygonal and polyhedral finite elements."

International Journal for Numerical Methods in Engineering, 2015

Open research directions

- Applications
- Quadrature
- Polyhedral meshing

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Thanks for the invitation to speak!

Slides and pre-prints: `http://math.arizona.edu/~agillette/`