

Cubic Serendipity Basis Functions for Isogeometric Analysis

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Tom Hughes 70th Birthday Conference*

What is a serendipity finite element method?

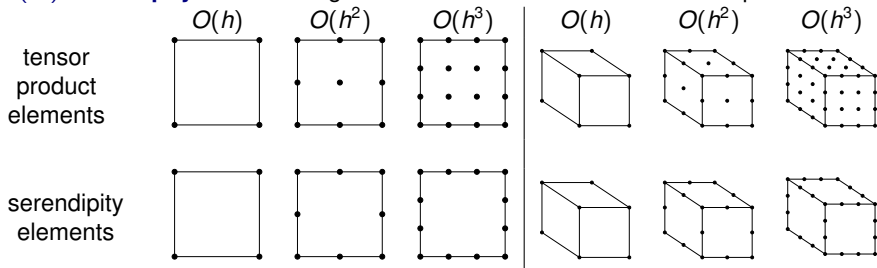
Goal: Efficient, accurate approximation of the solution to a PDE over $\Omega \subset \mathbb{R}^n$.

Standard $O(h^r)$ **tensor product** finite element method in \mathbb{R}^n :

- Mesh Ω by n -dimensional cubes of side length h .
- Set up a linear system involving $(r + 1)^n$ degrees of freedom (DoFs) per cube.
- For unknown continuous solution u and computed discrete approximation u_h :

$$\underbrace{\|u - u_h\|_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C h^r \|u\|_{H^{r+1}(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^{r+1}(\Omega).$$

A $O(h^r)$ **serendipity** FEM converges at the **same rate** with **fewer DoFs** per element:



Example: For $O(h^3)$, $d = 3$, 50% fewer DoFs → $\approx 50\%$ smaller linear system

What is a geometric decomposition?

A **geometric decomposition** for a finite element space is an explicit correspondence:

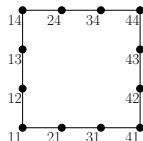
$$\{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, x^3y, xy^3\}$$

monomials



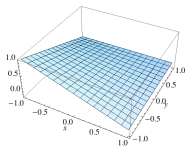
$$\{\vartheta_{11}, \vartheta_{14}, \vartheta_{41}, \vartheta_{44}, \vartheta_{12}, \vartheta_{13}, \vartheta_{42}, \vartheta_{43}, \vartheta_{21}, \vartheta_{31}, \vartheta_{24}, \vartheta_{34}\}$$

basis functions



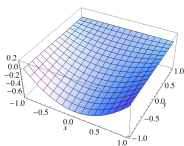
domain points

- Previously known basis functions employ Legendre polynomials
- These functions bear no symmetrical correspondence to the domain points and hence are not useful for isogeometric analysis.



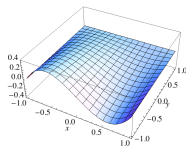
$$\frac{1}{4}(x-1)(y-1)$$

vertex



$$-\frac{1}{4}\sqrt{\frac{3}{2}}(x^2-1)(y-1)$$

edge (quadratic)



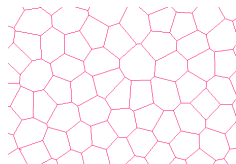
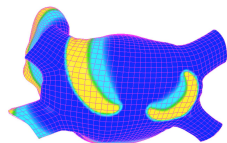
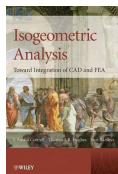
$$-\frac{1}{4}\sqrt{\frac{5}{2}}x(x^2-1)(y-1)$$

edge (cubic)

SZABÓ AND I. BABUŠKA *Finite element analysis*, Wiley Interscience, 1991.

Motivations and Related Topics

Goal: Construct geometric decompositions of serendipity spaces using linear combinations of standard tensor product functions. **Focus:** Cubic Hermites.



- **Isogeometric analysis:** Finding basis functions suitable for both domain description and PDE approximation avoids the expensive computational bottleneck of re-meshing.

COTTRELL, HUGHES, BAZILEVS *Isogeometric Analysis: Toward Integration of CAD and FEA*, Wiley, 2009.

- **Modern mathematics:** Finite Element Exterior Calculus, Discrete Exterior Calculus, Virtual Element Methods. . .

ARNOLD, AWANOU *The serendipity family of finite elements*, Found. Comp. Math, 2011.

DA VEIGA, BREZZI, CANGIANI, MANZINI, RUSSO *Basic Principles of Virtual Element Methods*, M3AS, 2013.

- **Flexible Domain Meshing:** Serendipity type elements for Voronoi meshes provide computational benefits without need of tensor product structure.

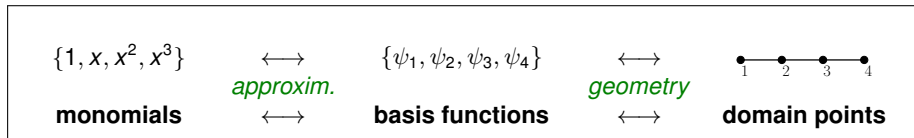
RAND, GILLETTE, BAJAJ *Quadratic Serendipity Finite Elements on Polygons Using Generalized Barycentric Coordinates*, Mathematics of Computation, in press.

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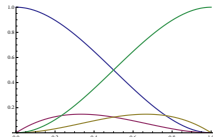
- 1 The Cubic Case: Hermite Functions, Serendipity Spaces
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Cubic Hermite Geometric Decomposition: 1D



Cubic Hermite Basis
on $[0, 1]$

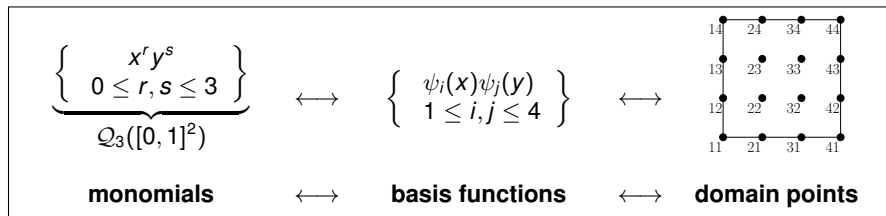
$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} := \begin{bmatrix} 1 - 3x^2 + 2x^3 \\ x - 2x^2 + x^3 \\ x^2 - x^3 \\ 3x^2 - 2x^3 \end{bmatrix}$$



Approximation: $x^r = \sum_{i=1}^4 \varepsilon_{r,i} \psi_i$, for $r = 0, 1, 2, 3$, where $[\varepsilon_{r,i}] = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -3 & 1 \end{pmatrix}$

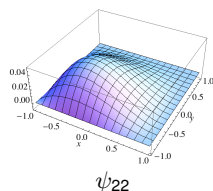
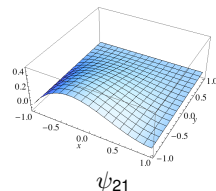
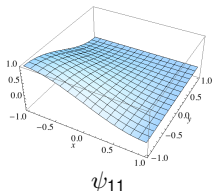
Geometry: $u = u(0)\psi_1 + u'(0)\psi_2 - u'(1)\psi_3 + u(1)\psi_4$, $\forall u \in \underbrace{\mathcal{P}_3([0, 1])}_{\text{cubic polynomials}}$

Cubic Hermite Geometric Decomposition: 2D



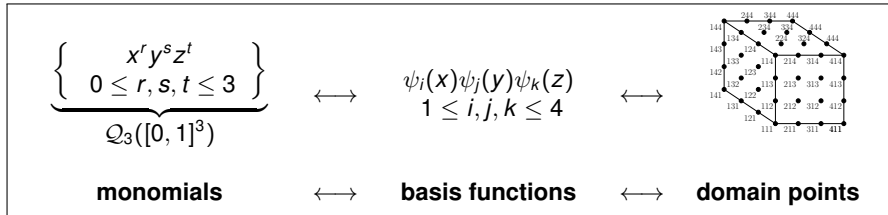
Approximation: $x^r y^s = \sum_{i=1}^4 \sum_{j=1}^4 \varepsilon_{r,i} \varepsilon_{s,j} \psi_{ij}$, for $0 \leq r, s \leq 3$, $\varepsilon_{r,i}$ as in 1D.

Geometry:



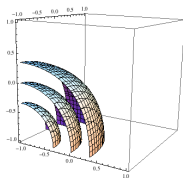
$$u = u|_{(0,0)} \psi_{11} + \partial_x u|_{(0,0)} \psi_{21} + \partial_y u|_{(0,0)} \psi_{12} + \partial_x \partial_y u|_{(0,0)} \psi_{22} + \dots, \quad \forall u \in \mathcal{Q}_3([0, 1]^2)$$

Cubic Hermite Geometric Decomposition: 3D

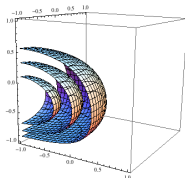


Approximation: $x^r y^s z^t = \sum_{i=1}^4 \sum_{j=1}^4 \sum_{k=1}^4 \varepsilon_{r,i} \varepsilon_{s,j} \varepsilon_{t,k} \psi_{ijk}$, for $0 \leq r, s, t \leq 3$, $\varepsilon_{r,i}$ as in 1D.

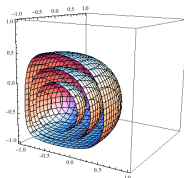
Geometry: Contours of level sets of the basis functions:



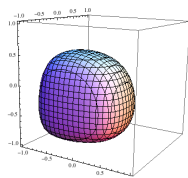
ψ_{111}



ψ_{112}



ψ_{212}



ψ_{222}

Two families of finite elements on cubical meshes

$\mathcal{Q}_r^- \Lambda^k([0, 1]^n) \rightarrow$ tensor product spaces (\leq degree r in each variable)

early work: [RAVIART, THOMAS 1976](#), [NEDELEC 1980](#)

more recently: [ARNOLD, BOFFI, BONIZZONI arXiv:1212.6559, 2012](#)

$\mathcal{S}_r \Lambda^k([0, 1]^n) \rightarrow$ serendipity finite element spaces (superlinear degree r)

early work: [STRANG, FIX *An analysis of the finite element method* 1973](#)

more recently: [ARNOLD, AWANOU FoCM 11:3, 2011](#), and [arXiv:1204.2595, 2012](#).

The **superlinear** degree of a polynomial ignores linearly-appearing variables.

$$n = 2 : \underbrace{\{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, x^3y, xy^3, x^2y^2, x^3y^2, x^2y^3, x^3y^3\}}_{\mathcal{S}_3 \Lambda^0([0, 1]^2) \text{ (dim=12)}} \underbrace{\{x^3y^2, x^2y^3, x^3y^3\}}_{\mathcal{Q}_3 \Lambda^0([0, 1]^2) \text{ (dim=16)}}$$

$$n = 3 : \underbrace{\{1, \dots, xyz, x^3y, x^3z, y^3z, \dots, x^3yz, xy^3z, xyz^3, x^3y^2, \dots, x^3y^3z^3\}}_{\mathcal{S}_3 \Lambda^0([0, 1]^3) \text{ (dim=32)}} \underbrace{\{x^3y^2z^3, x^3y^3z^3\}}_{\mathcal{Q}_3 \Lambda^0([0, 1]^3) \text{ (dim=64)}}$$

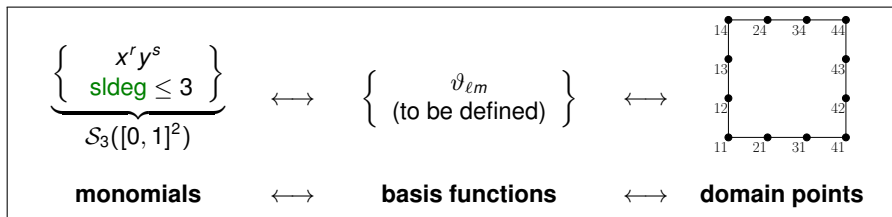
$\mathcal{Q}_r^- \Lambda^k$ and $\mathcal{S}_r \Lambda^k$ and have the **same** key mathematical properties needed for FEEC
(degree, inclusion, trace, subcomplex, unisolvence, commuting projections)

but for fixed $k \geq 0, r, n \geq 2$ the serendipity spaces have **fewer** degrees of freedom

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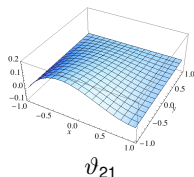
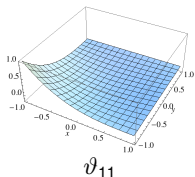
Cubic Hermite Serendipity Geom. Decomp: 2D

Theorem [G, 2012]: A Hermite-like geometric decomposition of $\mathcal{S}_3([0, 1]^2)$ exists.



Approximation: $x^r y^s = \sum_{\ell m} \varepsilon_{r,i} \varepsilon_{s,j} v_{\ell m}$, for **superlinear degree** $(x^r y^s) \leq 3$

Geometry:



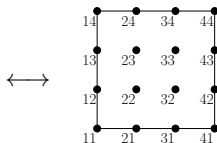
$$\begin{aligned}
 u &= u|_{(0,0)} v_{11} \\
 &+ \partial_x u|_{(0,0)} v_{21} \\
 &+ \partial_y u|_{(0,0)} v_{12} \\
 &+ \dots \\
 \forall u &\in \mathcal{S}_3([0, 1]^2)
 \end{aligned}$$

Cubic Hermite Serendipity Geom. Decomp: 2D

Proof Overview:

- 1 Fix index sets and basis orderings based on domain points:

V = vertices (11, 14, ...)
 E = edges (12, 13, ...)
 D = interior (22, 23, ...)



$$[\vartheta_{\ell m}] := [\vartheta_{11}, \vartheta_{14}, \vartheta_{41}, \vartheta_{44}, \vartheta_{12}, \vartheta_{13}, \vartheta_{42}, \vartheta_{43}, \vartheta_{21}, \vartheta_{31}, \vartheta_{24}, \vartheta_{34}],$$

$$[\psi_{ij}] := [\underbrace{\psi_{11}, \psi_{14}, \psi_{41}, \psi_{44}}_{\text{indices in } V}, \underbrace{\psi_{12}, \psi_{13}, \psi_{42}, \psi_{43}, \psi_{21}, \psi_{31}, \psi_{24}, \psi_{34}}_{\text{indices in } E}, \underbrace{\psi_{22}, \psi_{23}, \psi_{32}, \psi_{33}}_{\text{indices in } D}]$$

- 2 Define a 12×16 matrix \mathbb{H} with entries $h_{ij}^{\ell m}$ so that $\ell m \in V \cup E$, $ij \in V \cup E \cup D$.
- 3 Define the serendipity basis functions $\vartheta_{\ell m}$ via

$$[\vartheta_{\ell m}] := \mathbb{H}[\psi_{ij}]$$

and show that the **approximation** and **geometry** properties hold.

Cubic Hermite Serendipity Geom. Decomp: 2D

Proof Details:

2 Define a 12×16 matrix \mathbb{H} with entries $h_{ij}^{\ell m}$ so that $\ell m \in V \cup E$, $ij \in V \cup E \cup D$.

$$\mathbb{H} := \left[\begin{array}{c|c} & \begin{matrix} -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{matrix} \\ \hline \text{II} \\ (12 \times 12 \text{ identity matrix}) \end{array} \right]$$

3 Define $[\vartheta_{\ell m}] := \mathbb{H}[\psi_{ij}]$. The **geometry** property holds since for $\ell m \in V \cup E$,

$$\vartheta_{\ell m} = \underbrace{\psi_{\ell m}}_{\text{bicubic Hermite}} + \underbrace{\sum_{ij \in D} h_{ij}^{\ell m} \psi_{ij}}_{\text{zero on boundary}} \implies \vartheta_{\ell m} \equiv \psi_{\ell m} \text{ on edges}$$

Cubic Hermite Serendipity Geom. Decomp: 2D

Proof Details:

To prove that the **approximation** property holds, observe:

$$[\vartheta_{\ell m}] := \mathbb{H}[\psi_{ij}] \quad \text{implies} \quad \sum_{ij} h_{ij}^{\ell m} \psi_{ij} = \vartheta_{\ell m}$$

For all (r, s) pairs such that $\text{slddeg}(x^r y^s) \leq 3$, the matrix entries in column ij satisfy

$$\varepsilon_{r,i\varepsilon s,j} = \sum_{\ell m \in VUE} \varepsilon_{r,\ell\varepsilon s,m} h_{ij}^{\ell m}$$

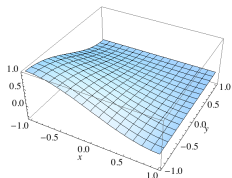
Substitute these into the Hermite 2D approximation property:

$$\begin{aligned} x^r y^s &= \sum_{ij \in VUEUD} \varepsilon_{r,i\varepsilon s,j} \psi_{ij} = \sum_{ij} \sum_{\ell m} \varepsilon_{r,\ell\varepsilon s,m} h_{ij}^{\ell m} \psi_{ij} \\ &= \sum_{\ell m} \varepsilon_{r,\ell\varepsilon s,m} \sum_{ij} h_{ij}^{\ell m} \psi_{ij} = \sum_{\ell m} \varepsilon_{r,\ell\varepsilon s,m} \vartheta_{\ell m} \end{aligned}$$

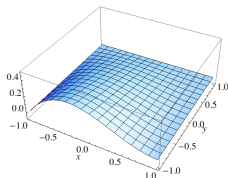
Hence $[\vartheta_{\ell m}]$ is a basis for $\mathcal{S}_2([0, 1]^2)$, completing the geometric decomposition. \square

Hermite Style Serendipity Functions (2D)

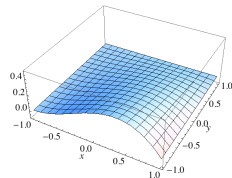
$$[\vartheta_{\ell m}] = \begin{bmatrix} \vartheta_{11} \\ \vartheta_{14} \\ \vartheta_{41} \\ \vartheta_{44} \\ \vartheta_{12} \\ \vartheta_{13} \\ \vartheta_{42} \\ \vartheta_{43} \\ \vartheta_{21} \\ \vartheta_{31} \\ \vartheta_{24} \\ \vartheta_{34} \end{bmatrix} = \begin{bmatrix} -(x-1)(y-1)(-2+x+x^2+y+y^2) \\ (x-1)(y+1)(-2+x+x^2-y+y^2) \\ (x+1)(y-1)(-2-x+x^2+y+y^2) \\ -(x+1)(y+1)(-2-x+x^2-y+y^2) \\ -(x-1)(y-1)^2(y+1) \\ (x-1)(y-1)(y+1)^2 \\ (x+1)(y-1)^2(y+1) \\ -(x+1)(y-1)(y+1)^2 \\ -(x-1)^2(x+1)(y-1) \\ (x-1)(x+1)^2(y-1) \\ (x-1)^2(x+1)(y+1) \\ -(x-1)(x+1)^2(y+1) \end{bmatrix} \cdot \frac{1}{8}$$



ϑ_{11}



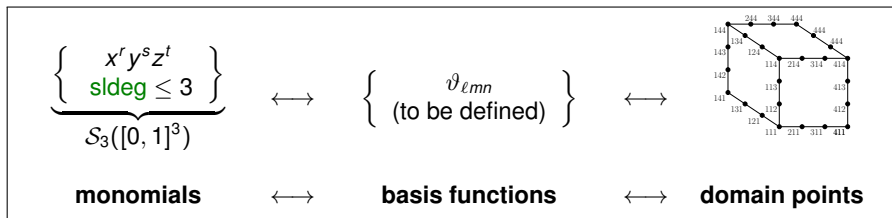
ϑ_{21}



ϑ_{31}

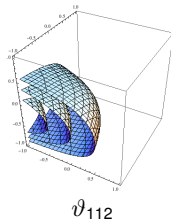
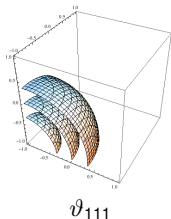
Cubic Hermite Serendipity Geom. Decomp: 3D

Theorem [G, 2012]: A Hermite-like geometric decomposition of $\mathcal{S}_3([0, 1]^3)$ exists.



Approximation: $x^r y^s z^t = \sum_{\ell mn} \varepsilon_{r,i} \varepsilon_{s,j} \varepsilon_{t,k} \vartheta_{\ell mn}$, for $\text{superlinear degree}(x^r y^s z^t) \leq 3$

Geometry:



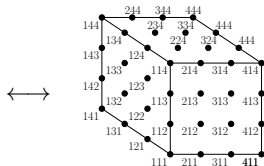
$$\begin{aligned}
 u &= u|_{(0,0,0)} \vartheta_{111} \\
 &+ \partial_x u|_{(0,0,0)} \vartheta_{211} \\
 &+ \partial_y u|_{(0,0,0)} \vartheta_{121} \\
 &+ \partial_z u|_{(0,0,0)} \vartheta_{112} \\
 &+ \dots \\
 \forall u &\in \mathcal{S}_3([0, 1]^3)
 \end{aligned}$$

Cubic Hermite Serendipity Geom. Decomp: 3D

Proof Overview:

- 1 Fix index sets and basis orderings based on domain points:

V = vertices (111, ...)
 E = edges (112, ...)
 F = face interior (122, ...)
 M = volume interior (222, ...)



$$[\vartheta_{\ell mn}] := [\vartheta_{111}, \dots, \vartheta_{444}, \vartheta_{112}, \dots, \vartheta_{443}],$$

$$[\psi_{ijk}] := [\underbrace{\psi_{111}, \dots, \psi_{444}}_{\text{indices in } V}, \underbrace{\psi_{112}, \dots, \psi_{443}}_{\text{indices in } E}, \underbrace{\psi_{122}, \dots, \psi_{433}}_{\text{indices in } F}, \underbrace{\psi_{222}, \dots, \psi_{333}}_{\text{indices in } M}]$$

- 2 Define a 32×64 matrix \mathbb{W} with entries $h_{ijk}^{\ell mn}$ (where $\ell mn \in V \cup E$)

- 3 Define the serendipity basis functions $\vartheta_{\ell mn}$ via

$$[\vartheta_{\ell mn}] := \mathbb{W}[\psi_{ijk}]$$

and show that the **approximation** and **geometry** properties hold.

Cubic Hermite Serendipity Geom. Decomp: 3D

Proof Details:

2 Define a 32×64 matrix \mathbb{W} with entries $h_{ijk}^{\ell mn}$ so that $\ell mn \in V \cup E$

$$\mathbb{W} := \left[\begin{array}{c|c} \mathbb{I} & \text{specific full rank} \\ (32 \times 32 \text{ identity matrix}) & \text{32} \times \text{32 matrix} \\ & \text{with entries -1, 0, or 1} \end{array} \right]$$

3 Define $[\vartheta_{\ell mn}] := \mathbb{W}[\psi_{ijk}]$.

→ Confirm directly that $[\vartheta_{\ell mn}]$ restricts to $[\vartheta_{\ell m}]$ on faces.

→ Similar proof technique confirms **geometry** and **approximation** properties.

$$[\vartheta_{\ell mn}] = \begin{bmatrix} \vartheta_{111} \\ \vartheta_{114} \\ \vdots \\ \vartheta_{442} \\ \vartheta_{443} \end{bmatrix} = \begin{bmatrix} (x-1)(y-1)(z-1)(-2+x+x^2+y+y^2+z+z^2) \\ -(x-1)(y-1)(z+1)(-2+x+x^2+y+y^2-z+z^2) \\ \vdots \\ (x+1)(y+1)(z-1)^2(z+1) \\ -(x+1)(y+1)(z-1)(z+1)^2 \end{bmatrix} \cdot \frac{1}{16}$$

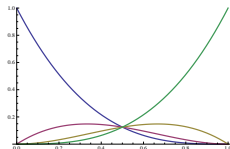
Complete list and more details in my paper:

GILLETTE *Hermite and Bernstein Style Basis Functions for Cubic Serendipity Spaces on Squares and Cubes*, arXiv:1208.5973, 2012

Cubic Bernstein Serendipity Geom. Decomp: 2D, 3D

**Cubic
Bernstein Basis**
on $[0, 1]$

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} := \begin{bmatrix} (1-x)^3 \\ (1-x)^2 x \\ (1-x)x^2 \\ x^3 \end{bmatrix}$$

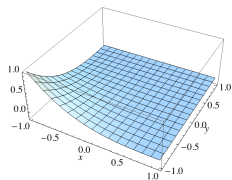


Theorem [G, 2012]: Bernstein-like geometric decompositions of $\mathcal{S}_3([0, 1]^2)$ and $\mathcal{S}_3([0, 1]^3)$ exist.

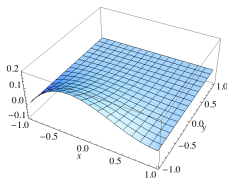
$$[\xi_{\ell m}] = \begin{bmatrix} \xi_{11} \\ \xi_{14} \\ \vdots \\ \xi_{24} \\ \xi_{34} \end{bmatrix} = \begin{bmatrix} (x-1)(y-1)(-2-2x+x^2-2y+y^2) \\ -(x-1)(y+1)(-2-2x+x^2+2y+y^2) \\ \vdots \\ (x-1)^2(x+1)(y+1) \\ -(x-1)(x+1)^2(y+1) \end{bmatrix} \cdot \frac{1}{16}$$

$$[\xi_{\ell mn}] = \begin{bmatrix} \xi_{111} \\ \xi_{114} \\ \vdots \\ \xi_{442} \\ \xi_{443} \end{bmatrix} = \begin{bmatrix} -(x-1)(y-1)(z-1)(-5-2x+x^2-2y+y^2-2z+z^2) \\ (x-1)(y-1)(z+1)(-5-2x+x^2-2y+y^2+2z+z^2) \\ \vdots \\ (x+1)(y+1)(z-1)^2(z+1) \\ -(x+1)(y+1)(z-1)(z+1)^2 \end{bmatrix} \cdot \frac{1}{32}$$

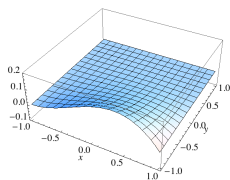
Bernstein Style Serendipity Functions (2D)



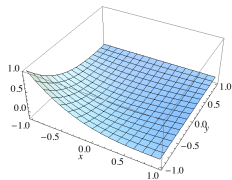
β_{11}



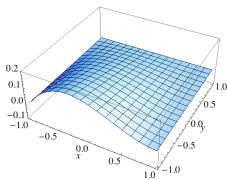
β_{21}



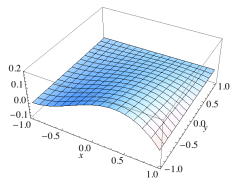
β_{31}



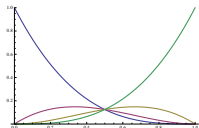
ξ_{11}



ξ_{21}



ξ_{31}

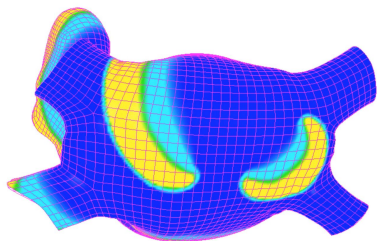
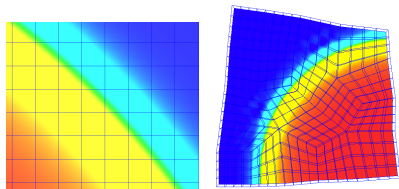


Bicubic Bernstein functions (top) and Bernstein-style serendipity functions (bottom).

→ Note boundary agreement with Bernstein functions.

- 1 The Cubic Case: Hermite Functions, Serendipity Spaces
- 2 Geometric Decompositions of Cubic Serendipity Spaces
- 3 Applications and Future Directions

Application: Cardiac Electrophysiology



→ Cubic Hermite serendipity functions recently incorporated into Continuity software package for cardiac electrophysiology models.

→ Used to solve the *monodomain* equations, a type of reaction-diffusion equation

→ Initial results show agreement of serendipity and standard bicubics on a benchmark problem with a

4x computational speedup in 3D.

→ Fast computation essential to clinical applications and 'real time' simulations

GONZALEZ, VINCENT, G., McCULLOCH *High Order Interpolation Methods in Cardiac Electrophysiology Simulation*, in preparation, 2013.

Future Directions and Open Questions

- Applications to problems using Bernstein tensor product bases
- Analysis of the use of serendipity bases for geometric modeling
- Construction of bases for $\mathcal{S}_r \Lambda^k([0, 1]^n)$ for
 - Higher order scalar cases ($k = 0, r > 3, n = 2, 3$)
 - Higher form order cases ($k > 0$)

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→ Thanks for the invitation to speak!

UC San Diego

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Happy birthday Tom Hughes!

Slides and pre-prints: <http://ccom.ucsd.edu/~agillette>

