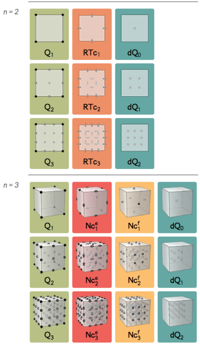
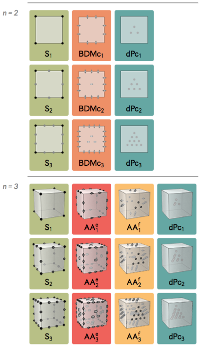
Computational Finite Element Differential Forms on Quadrilateral Meshes

Andrew Gillette - University of Arizona

joint work with
Tyler Kloefkorn, AAAS STP Fellow, hosted at NSF
Victoria Sanders, University of Arizona





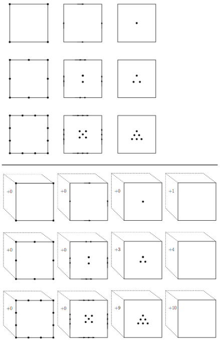
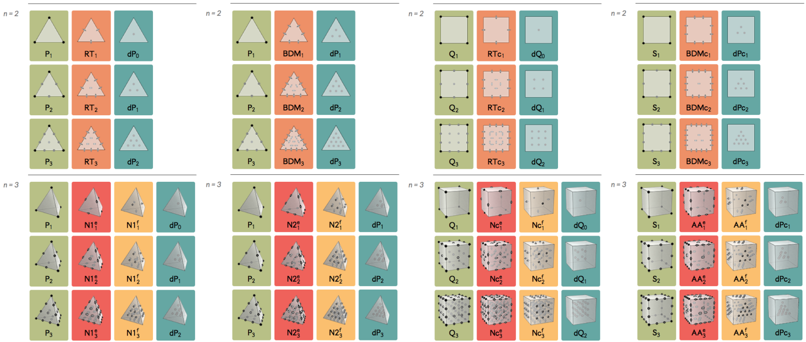


Table of Contents

Outline

The ‘Periodic Table of the Finite Elements’

ARNOLD, LOGG, “Periodic table of the finite elements,” *SIAM News*, 2014.



Classification of many common conforming finite element types.

n $\rightarrow$ Domains in $\mathbb{R}^2$ (top half) and in $\mathbb{R}^3$ (bottom half)
r $\rightarrow$ Order $1, 2, 3$ of error decay (going down columns)
k $\rightarrow$ Conformity type $k = 0, \ldots, n$ (going across a row)

Geometry types: Simplices (left half) and cubes (right half).

An abbreviated reading list (50 years of theory!)

RAVIART, THOMAS, "A mixed finite element method for 2nd order elliptic problems" *Lecture Notes in Mathematics*, 1977 ← *3172 citations, including 150 from 2017!*

NÉDÉLEC, "Mixed finite elements in $\mathbb{R}^3$," *Numerische Mathematik*, 1980

BREZZI, DOUGLAS JR., MARINI, "Two families of mixed finite elements for second order elliptic problems," *Numerische Mathematik*, 1985

NÉDÉLEC, "A new family of mixed finite elements in $\mathbb{R}^3$," *Numerische Mathematik*, 1986

ARNOLD, FALK, WINTHER "Finite element exterior calculus, homological techniques, and applications," *Acta Numerica*, 2006

CHRISTIANSEN, "Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension," *Numerische Mathematik*, 2007

ARNOLD, FALK, WINTHER "Finite element exterior calculus: from hodge theory to numerical stability," *Bulletin of the AMS*, 2010

ARNOLD, AWANOU "The serendipity family of finite elements ", *Found. Comp Math*, 2011

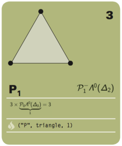
ARNOLD, AWANOU "Finite element differential forms on cubical meshes", *Math Comp.*, 2013

ARNOLD, BOFFI, BONIZZONI "Finite element differential forms on curvillinear meshes and their approximation properties," *Numerische Mathematik*, 2014

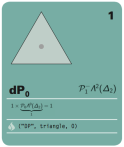
Stable pairs of elements for mixed methods

Picking elements from the table for a mixed method for the Poisson problem:

2

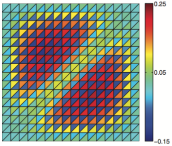


$\subset H^1 \times H^1$

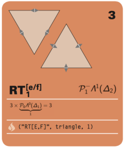


$\subset L^2$

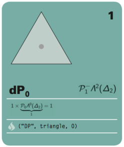
$\Rightarrow$



Unstable method

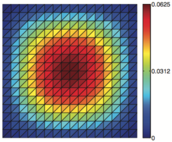


$\subset H(\text{div})$



$\subset L^2$

$\Rightarrow$



Provably stable method

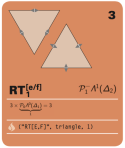
converges to

$u = x(1-x)y(1-y)$

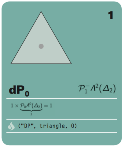
Example and images on right from:

ARNOLD, FALK, WINTHER "Finite Element Exterior Calculus..." *Bulletin of the AMS*, 47:2, 2010.

Method selection and cochain complexes

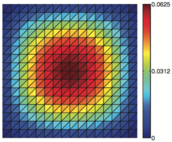


$\subset H(\mathrm{div})$



$\subset L^2$

$\Rightarrow$



Provably stable method

converges to
$\mathrm{u} = x(1-x)y(1-y)$

Stable pairs of elements for mixed Hodge-Laplacian problems are found by choosing consecutive spaces in compatible discretizations of the L^2 deRham Diagram.

$$H^1 \xrightarrow[\text{grad}]{\nabla} H(\text{curl}) \xrightarrow[\text{curl}]{\nabla\times} H(\text{div}) \xrightarrow[\text{div}]{\nabla\cdot} L^2$$

	H^1	$H(\text{curl})$	$H(\text{div})$	L^2
vector Poisson	σ	μ		
Maxwell's eqn's		h	b	
Darcy / Poisson			$\mathbf{u}$	p

Stable pairs are found from consecutive entries in a cochain complex.

Exact cochain complexes found in the table

On an n-simplex in $\mathbb{R}^n$:

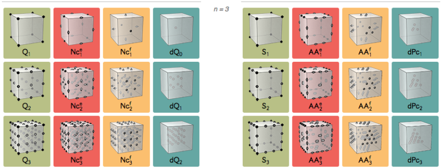
$$\mathcal{P}_r^-\Lambda^0 \to \mathcal{P}_r^-\Lambda^1 \to \cdots \to \mathcal{P}_r^-\Lambda^{n-1} \to \mathcal{P}_r^-\Lambda^n \qquad \text{'trimmed' polynomials}$$

$$\mathcal{P}_r\Lambda^0 \to \mathcal{P}_{r-1}\Lambda^1 \to \cdots \to \mathcal{P}_{r-n+1}\Lambda^{n-1} \to \mathcal{P}_{r-n}\Lambda^n \qquad \text{polynomials}$$

On an n-dimensional cube in $\mathbb{R}^n$:

$$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n \qquad \text{tensor product}$$

$$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n \qquad \text{serendipity}$$



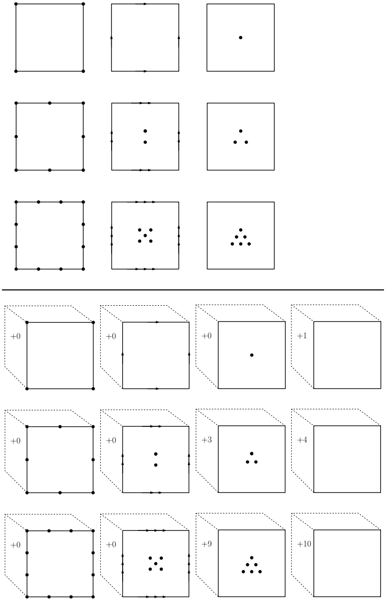
The 'minus' spaces proceed across rows of the PToFE (**r is fixed**) while the 'regular' spaces proceed along diagonals (**r decreases**)

Mysteriously, the degree of freedom count for mixed methods from the $\mathcal{P}_r^-$ spaces is smaller than those from the $\mathcal{P}_r$ spaces, while the opposite is true for the $\mathcal{Q}_r^-$ and $\mathcal{S}_r$ spaces.

Outline

1. Four well-known families of finite element differential forms
2. Trimmed serendipity spaces: a new, smaller family of elements
3. Computational bases for serendipity-type spaces
4. Extension to generic quads and hexes

The 5th column: Trimmed serendipity spaces



A new column for the PToFE:
the **trimmed serendipity** elements.

$\boxed{\mathcal{S}_r^- \Lambda^k(\square_n)}$ denotes

approximation order r,
subset of k-form space $\Lambda^k(\Omega)$,
use on meshes of n-dim'l cubes.

Defined for any $n \geq 1$, $0 \leq k \leq n$, $r \geq 1$

Identical or analogous properties to all the other colummns in the table.

The advantage of the $\mathcal{S}_r^- \Lambda^k$ spaces is that they have fewer degrees of freedom for mixed methods than their tensor product and serendipity counterparts.

Key properties of the trimmed serendipity spaces

$$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n$$ **tensor product**

$$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n$$ **serendipity**

$$\mathcal{S}_r^-\Lambda^0 \to \mathcal{S}_r^-\Lambda^1 \to \cdots \to \mathcal{S}_r^-\Lambda^{n-1} \to \mathcal{S}_r^-\Lambda^n$$ **trimmed serendipity**

Subcomplex: $d\mathcal{S}_r^-\Lambda^k \subset \mathcal{S}_r^-\Lambda^{k+1}$

Exactness: The above sequence is exact.
i.e. the image of incoming map = kernel of outgoing map

Inclusion: $\mathcal{S}_r\Lambda^k \subset \mathcal{S}_{r+1}^-\Lambda^k \subset \mathcal{S}_{r+1}\Lambda^k$

Trace: $\mathrm{tr}_f\mathcal{S}_r^-\Lambda^k(\mathbb{R}^n) \subset \mathcal{S}_r^-\Lambda^k(f)$, for any $(n-1)$-hyperplane f in $\mathbb{R}^n$

Special cases:
$\mathcal{S}_r^-\Lambda^0 = \mathcal{S}_r\Lambda^0$
$\mathcal{S}_r^-\Lambda^n = \mathcal{S}_{r-1}\Lambda^n$
$\mathcal{S}_r^-\Lambda^k + d\mathcal{S}_{r+1}\Lambda^{k-1} = \mathcal{S}_r\Lambda^k$.

Replace '$\mathcal{S}$' by '$\mathcal{P}$' $\rightsquigarrow$ key properties about the first two columns for $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$!

Dimension count and comparison

Formula for counting degrees of freedom of $\mathcal{S}_r^- \Lambda^k(\square_n)$:

$$\sum_{d=k}^{\min\{n,\lfloor r/2\rfloor+k\}} 2^{n-d}\binom{n}{d}\left(\binom{r-d+2k-1}{r-d+k-1}\binom{r-d+k-1}{d-k}+\binom{r-d+2k}{k}\binom{r-d+k-1}{d-k-1}\right)$$

	k	r=1	2	3	4	5	6	7
n=2	0	4	8	12	17	23	30	38
	1	4	10	17	26	37	50	65
	2	1	3	6	10	15	21	28
n=3	0	8	20	32	50	74	105	144
	1	12	36	66	111	173	255	360
	2	6	21	45	82	135	207	301
	3	1	4	10	20	35	56	84
n=4	0	16	48	80	136	216	328	480
	1	32	112	216	392	656	1036	1563
	2	24	96	216	422	746	1227	1910
	3	8	36	94	200	375	644	1036
	4	1	5	15	35	70	126	210

Mixed Method dimension comparison 1

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \operatorname{div} \mathbf{u} - f &= 0 \end{aligned}$$

We compare degree of freedom counts among the three families for use on meshes of affinely-mapped squares or cubes, when a conforming method with (at least) order r decay in the approximation of p, $\mathbf{u}$, and div $\mathbf{u}$ is desired.

Total # of degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1\rvert + \lvert\mathcal{Q}_r^-\Lambda^2\rvert$	$\lvert\mathcal{S}_r\Lambda^1\rvert + \lvert\mathcal{S}_{r-1}\Lambda^2\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1\rvert + \lvert\mathcal{S}_r^-\Lambda^2\rvert$
1	4+1 = 5	8+1 = 9	4+1 = 5
2	12+4 = 16	14+3 = 17	10+3 = 13
3	24+9 = 33	22+6 = 28	17+6 = 23

Total # of degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2\rvert + \lvert\mathcal{Q}_r^-\Lambda^3\rvert$	$\lvert\mathcal{S}_r\Lambda^2\rvert + \lvert\mathcal{S}_{r-1}\Lambda^3\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2\rvert + \lvert\mathcal{S}_r^-\Lambda^3\rvert$
1	6+1 = 7	18+1 = 19	6+1 = 7
2	36+8 = 44	39+4 = 43	21+4 = 25
3	108+27 = 135	72+10 = 82	45+10 = 55

Mixed Method dimension comparison 2

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$$

The number of interior degrees of freedom is reduced from tensor product, to serendipity, to trimmed serendipity:

of **interior** degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1_0\rvert + \lvert\mathcal{Q}_r^-\Lambda^2_0\rvert$	$\lvert\mathcal{S}_r\Lambda^1_0\rvert + \lvert\mathcal{S}_{r-1}\Lambda^2_0\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1_0\rvert + \lvert\mathcal{S}_r^-\Lambda^2_0\rvert$
1	0+1 = 1	0+1 = 1	0+1 = 1
2	4+4 = 8	2+3 = 5	2+3 = 5
3	12+9 = 21	6+6 = 12	5+6 = 11

of **interior** degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2_0\rvert + \lvert\mathcal{Q}_r^-\Lambda^3_0\rvert$	$\lvert\mathcal{S}_r\Lambda^2_0\rvert + \lvert\mathcal{S}_{r-1}\Lambda^3_0\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2_0\rvert + \lvert\mathcal{S}_r^-\Lambda^3_0\rvert$
1	0+1 = 1	0+1 = 1	0+1 = 1
2	12+8 = 20	3+4 = 7	3+4 = 7
3	54+27 = 81	12+10 = 22	9+10 = 19

Mixed Method dimension comparison 3

Mixed method for Darcy problem:

$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$$

Assuming interior degrees of freedom could be dealt with efficiently (e.g. by static condensation), trimmed serendipity elements *still* have the fewest DoFs:

of **interface** (edge) degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1(\partial\square_2)\rvert$	$\lvert\mathcal{S}_r\Lambda^1(\partial\square_2)\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1(\partial\square_2)\rvert$
1	4	8	4
2	8	12	8
3	12	16	12

of **interface** (edge+face) degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2(\partial\square_3)\rvert$	$\lvert\mathcal{S}_r\Lambda^2(\partial\square_3)\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2(\partial\square_3)\rvert$
1	6	18	6
2	24	36	18
3	54	60	36

Outline

1. Four well-known families of finite element differential forms
2. Trimmed serendipity spaces: a new, smaller family of elements
3. Computational bases for serendipity-type spaces
4. Extension to generic quads and hexes

Building a computational basis

dx	dy	dz
$-yz$	xz	0
0	$-xz$	xy
yz	xz	xy
$2xy$	x^2	0
$2xz$	0	x^2
y^2	$2xy$	0
0	$2yz$	y^2
z^2	0	$2xz$
0	z^2	$2yz$
$2xyz$	x^2z	x^2y
y^2z	$2xyz$	xy^2
yz^2	xz^2	$2xyz$

Goal: find a computational basis for $\mathcal{S}_1\Lambda^1(\square_3)$:

- Must be $H(\text{curl})$-conforming
- Must have 24 functions, 2 associated to each edge of cube
- Must recover constant and linear approx. on each edge
- The approximation space contains:

  (1) Any polynomial coefficient of at most linear order:
  $\{1, x, y, z\} \times \{dx, dy, dz\} \;\to$ 12 forms

  (2) Certain forms with quadratic or cubic order coefficients
  shown in table at left $\to$ 12 forms

- For constants, use "obvious" functions:

$$\{(y \pm 1)(z \pm 1)dx,\;\; (x \pm 1)(z \pm 1)dy,\;\; (x \pm 1)(y \pm 1)dz\}$$

e.g. $(y+1)(z+1)dx$ evaluates to zero on every edge
except $\{y = 1, z = 1\}$ where it is $\equiv 4 \to$ constant approx.

Also, $(y+1)(z+1)dx$ can be written as linear combo,
by using the first three forms at left to get the $yz\,dx$ term

Building a computational basis

dx	dy	dz
$-yz$	xz	0
0	$-xz$	xy
yz	xz	xy
$2xy$	x^2	0
$2xz$	0	x^2
y^2	$2xy$	0
0	$2yz$	y^2
z^2	0	$2xz$
0	z^2	$2yz$
$2xyz$	x^2z	x^2y
y^2z	$2xyz$	xy^2
yz^2	xz^2	$2xyz$

- For constant approx on edges, we used:

$$\{(y \pm 1)(z \pm 1)dx,\;\; (x \pm 1)(z \pm 1)dy,\;\; (x \pm 1)(y \pm 1)dz\}$$

- Guess for linear approx on edges:

$$\{x(y \pm 1)(z \pm 1)dx,\;\; y(x \pm 1)(z \pm 1)dy,\;\; z(x \pm 1)(y \pm 1)dz\}$$

  e.g. $x(y+1)(z+1)dx$ evaluates to $4x$ on $\{y = 1, z = 1\}$.

- Unfortunately: $\boxed{x(y+1)(z+1)dx \notin \mathcal{S}_1\Lambda(\square_3)!}$

  Why? $x(y+1)(z+1)dx = (xyz + xy + xz + x)dx$

  but *$xyz\ dx$ only appears with other cubic order coefficients!*

- Remedy: add *dy* and *dz* terms that vanish on all edges.

Building a computational basis

dx	dy	dz
$-yz$	xz	0
0	$-xz$	xy
yz	xz	xy
$2xy$	x^2	0
$2xz$	0	x^2
y^2	$2xy$	0
0	$2yz$	y^2
z^2	0	$2xz$
0	z^2	$2yz$
$2xyz$	x^2z	x^2y
y^2z	$2xyz$	xy^2
yz^2	xz^2	$2xyz$

Computational basis element associated to $\{y = 1, z = 1\}$:

$$2x(y+1)(z+1)\ dx + (z+1)(x^2-1)\ dy + (y+1)(x^2-1)\ dz$$

✓ Evaluates to $4x$ on $\{y = 1, z = 1\}$ (linear approx.)

✓ Evaluates to 0 on all other edges

✓ Belongs to the space $\mathcal{S}_1\Lambda(\square_3)$:

$$\begin{array}{rcrcrl} 2xyz\ dx & + & x^2z\ dy & + & x^2y\ dz & \\ 2xy\ dx & + & x^2\ dy & + & 0\ dz & \\ 2xz\ dx & + & 0\ dy & + & x^2\ dz & \\ 2x\ dx & + & (-z-1)dy & + & (-y-1)dz & \leftarrow \text{linear order} \end{array}$$

$\hookrightarrow$ summation and factoring yields the desired form)

There are 11 other such functions, one per edge. We have:

$$\mathcal{S}_1\Lambda(\square_3) = \underbrace{E_0\Lambda^1(\square_3)}_{\text{"obvious" basis for constant approx}} \oplus \underbrace{\tilde{E}_1\Lambda^1(\square_3)}_{\text{modified basis for linear approx}}$$

$$\dim\ 24 = 12 + 12$$

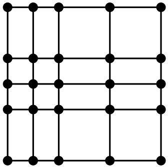
A complete table of computational bases

$n=3$	$k=0$	$k=1$	$k=2$	$k=3$
$\mathcal{S}_r\Lambda^k$	$V\Lambda^0(\square_3)$	$\emptyset$	$\emptyset$	$\emptyset$
	$\bigoplus_{i=0}^{r-2} E_i\Lambda^0(\square_3)$	$\bigoplus_{i=0}^{r-1} E_i\Lambda^1(\square_3) \oplus \tilde{E}_r\Lambda^1(\square_3)$	$\emptyset$	$\emptyset$
	$\bigoplus_{i=4}^{r} F_i\Lambda^0(\square_3)$	$\bigoplus_{i=2}^{r-1} F_i\Lambda^1(\square_3) \oplus \hat{F}_r\Lambda^1(\square_3)$	$\bigoplus_{i=0}^{r-1} F_i\Lambda^2(\square_3) \oplus \tilde{F}_r\Lambda^2(\square_3)$	$\emptyset$
	$\bigoplus_{i=6}^{r} I_i\Lambda^0(\square_3)$	$\bigoplus_{i=4}^{r} I_i\Lambda^1(\square_3)$	$\bigoplus_{i=2}^{r} I_i\Lambda^2(\square_3)$	$\bigoplus_{i=2}^{r} I_i\Lambda^3(\square_3)$
$\mathcal{S}_r^-\Lambda^k$	$V\Lambda^0(\square_3)$	$\emptyset$	$\emptyset$	$\emptyset$
	$\bigoplus_{i=0}^{r-2} E_i\Lambda^0(\square_3)$	$\bigoplus_{i=0}^{r-1} E_i\Lambda^1(\square_3)$	$\emptyset$	$\emptyset$
	$\bigoplus_{i=4}^{r} F_i\Lambda^0(\square_3)$	$\bigoplus_{i=2}^{r-1} F_i\Lambda^1(\square_3) \oplus \tilde{F}_r\Lambda^1(\square_3)$	$\bigoplus_{i=0}^{r-1} F_i\Lambda^2(\square_3)$	$\emptyset$
	$\bigoplus_{i=6}^{r} I_i\Lambda^0(\square_3)$	$\bigoplus_{i=4}^{r-1} I_i\Lambda^1(\square_3) \oplus \tilde{I}_r\Lambda^1(\square_3)$	$\bigoplus_{i=2}^{r-1} I_i\Lambda^2(\square_3) \oplus \tilde{I}_r\Lambda^2(\square_3)$	$\bigoplus_{i=2}^{r-1} I_i\Lambda^3(\square_3)$

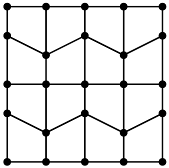
Outline

1. Four well-known families of finite element differential forms
2. Trimmed serendipity spaces: a new, smaller family of elements
3. Computational bases for serendipity-type spaces
4. Extension to generic quads and hexes

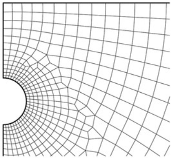
From squares and cubes to quads and hexes



structured mesh with affinely mapped quad elements



structured mesh with non-affinely mapped quad elements



unstructured mesh with non-affinely mapped quads; image from Zhao, Yu, Tao 2013

Ideally: construct bases on unstructured quad meshes by passing basis functions through a non-affine geometry transformation.

Unfortunately: Non-affine maps of serendipity-type elements result in sub-optimal convergence rates (see e.g. Arnold, Boffi, Falk 2002).

A way out: Adjust the mapping procedure and adjust some basis functions to the physical element (see e.g. Arbogast, Correa 2016)

Potential impact: A complete conforming finite element theory that can be applied in software for generic quad/hex meshes (consider: CUBIT, `deal.ii`, FEniCS, etc)

Acknowledgments

Related Publications

G., Kloefkorn "Trimmed Serendipity Finite Element Differential Forms." Mathematics of Computation, to appear. See arXiv:1607.00571

G., Kloefkorn, Sanders "Computational serendipity and tensor product finite element differential forms." Submitted. See arXiv:1806.00031

Research Funding

Supported in part by the National Science Foundation grant DMS-1522289.

Slides and Pre-prints

http://math.arizona.edu/~agillette/

Thanks for your attention!