A “Fifth Column” for the Periodic Table of Finite Elements

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joint work with
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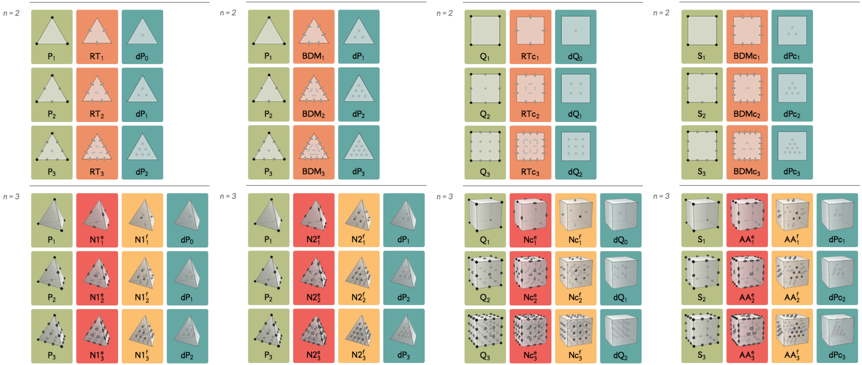

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3. Comparison to other (polygonal) elements

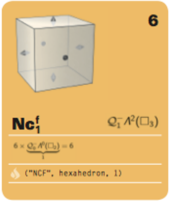
Classification of conforming methods

Conforming finite element method types can be broadly classified by three integers:

n → the spatial dimension of the domain
r → the order of error decay
k → the differential form order of the solution space

An element type is defined in part by its **degrees of freedom**. Typically:

the more degrees of freedom, the greater the computational cost of the method



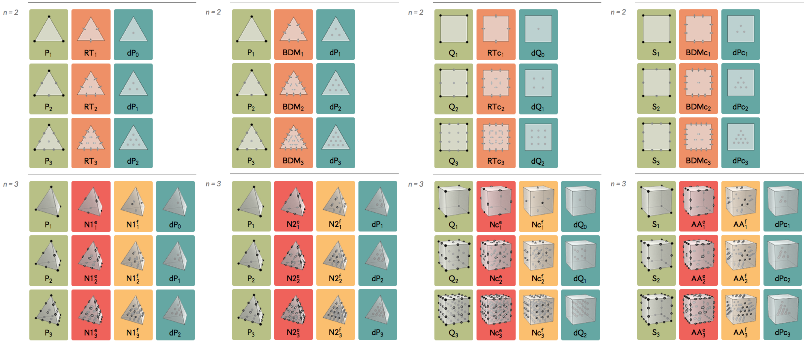
Ex: $\mathcal{Q}_1^-\Lambda^2(\square_3)$ is an element for

$n = 3$ → domains in $\mathbb{R}^3$
$r = 1$ → linear order of error decay
$k = 2$ → conformity in $\Lambda^2(\mathbb{R}^3) \rightsquigarrow H(\mathrm{div})$

$\mathcal{Q}_1^-\Lambda^2(\square_3)$ is part of the $\mathcal{Q}^-$ 'column' of elements,
is defined on geometry $\square_3$ (i.e. a cube),
has a **6** dimensional space of test functions,
and has an associated set of **6** degrees of freedom
that are unisolvent for the test function space.

The ‘Periodic Table of the Finite Elements’

The periodic table of the finite elements *(prepared by Doug Arnold & Anders Logg)*:



Classification of many common conforming finite element types.

n → Domains in $\mathbb{R}^2$ (top half) and in $\mathbb{R}^3$ (bottom half)
r → Order $1, 2, 3$ of error decay (going down columns)
k → Conformity type $k = 0, \ldots, n$ (going across a row)

Geometry types: Simplices (left half) and cubes (right half).

An abbreviated reading list (50 years of theory!)

RAVIART, THOMAS, "A mixed finite element method for 2nd order elliptic problems" *Lecture Notes in Mathematics*, 1977 ← *3106 citations, including 100 from 2017!*

NÉDÉLEC, "Mixed finite elements in $\mathbb{R}^3$," *Numerische Mathematik*, 1980

BREZZI, DOUGLAS JR., MARINI, "Two families of mixed finite elements for second order elliptic problems," *Numerische Mathematik*, 1985

NÉDÉLEC, "A new family of mixed finite elements in $\mathbb{R}^3$," *Numerische Mathematik*, 1986

ARNOLD, FALK, WINTHER "Finite element exterior calculus, homological techniques, and applications," *Acta Numerica*, 2006

CHRISTIANSEN, "Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension," *Numerische Mathematik*, 2007

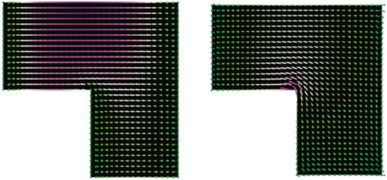
ARNOLD, FALK, WINTHER "Finite element exterior calculus: from hodge theory to numerical stability," *Bulletin of the AMS*, 2010

ARNOLD, AWANOU "The serendipity family of finite elements ", *Found. Comp Math*, 2011

ARNOLD, AWANOU "Finite element differential forms on cubical meshes", *Math Comp.*, 2013

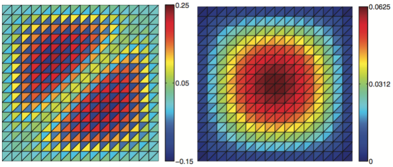
ARNOLD, BOFFI, BONIZZONI "Finite element differential forms on curvillinear meshes and their approximation properties," *Numerische Mathematik*, 2014

The importance of method selection



Vector Poisson problem

- Solutions by the standard non-mixed method (left) and by a mixed method (right).
- Only the second choice shows the correct behavior near the reentrant corner.



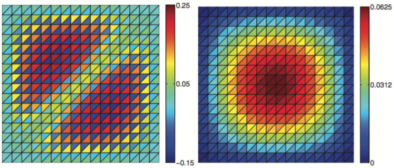
Poisson problem

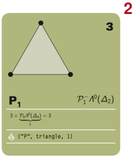
- Solutions by two different choices for the finite element solution spaces in a mixed method.
- Only the second choice looks like the true solution: $x(1-x)y(1-y)$.

Examples and images borrowed from:

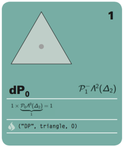
Arnold, Falk, Winther “Finite Element Exterior Calculus: From Hodge Theory to Numerical Stability,” *Bulletin of the AMS*, 47:2, 2010.

Stable choices for mixed methods



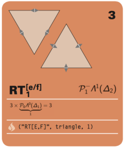


$\subset H^1 \times H^1$

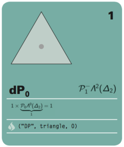


$\subset L^2$

Unstable method, as shown



$\subset H(\mathrm{div})$

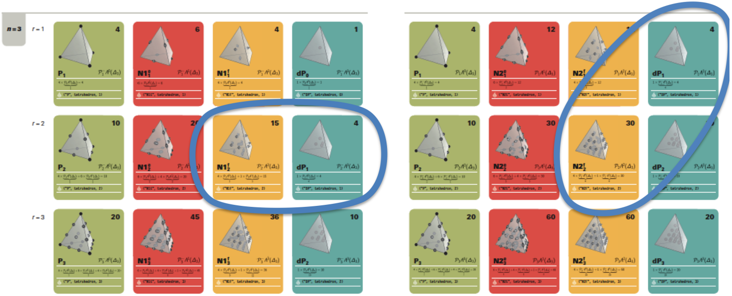


$\subset L^2$

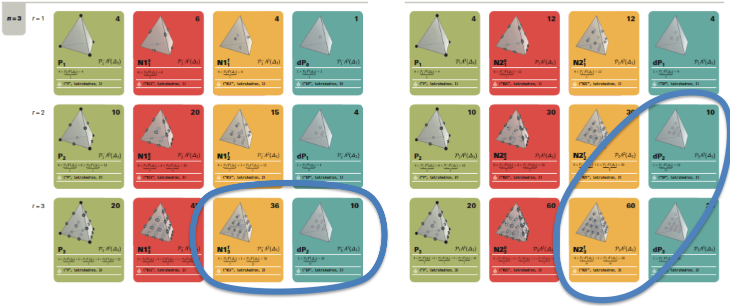
Provably stable method

→ The Periodic Table of Finite Elements lets us ‘read off’ stable pairs visually.

Stable pairs for simplicial meshes



Stable pairs for simplicial meshes



Stable pairs for simplicial meshes

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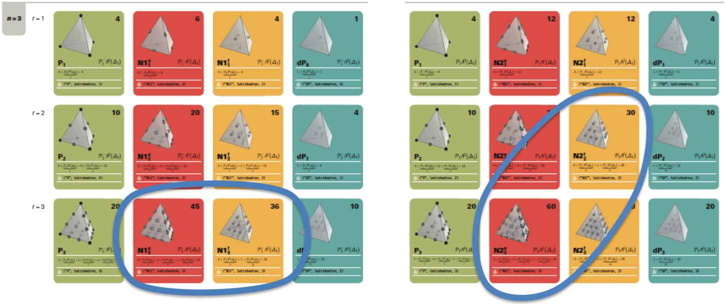

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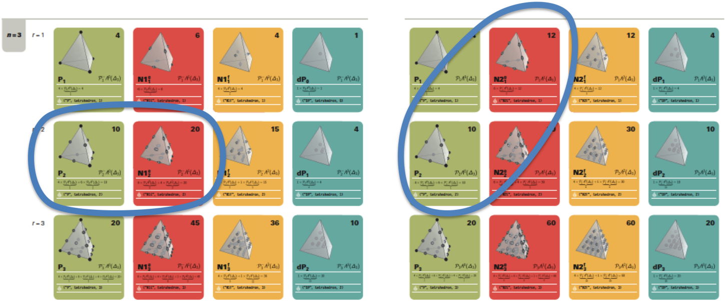

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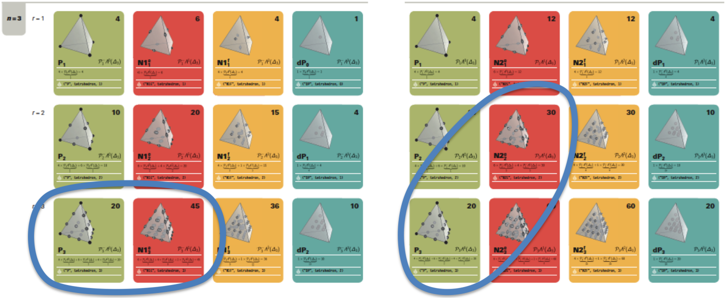
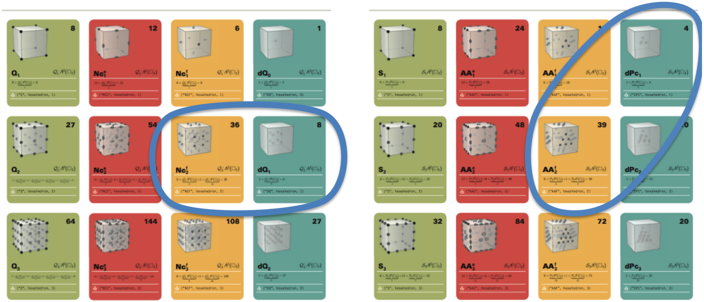


Stable pairs for cubical mehes



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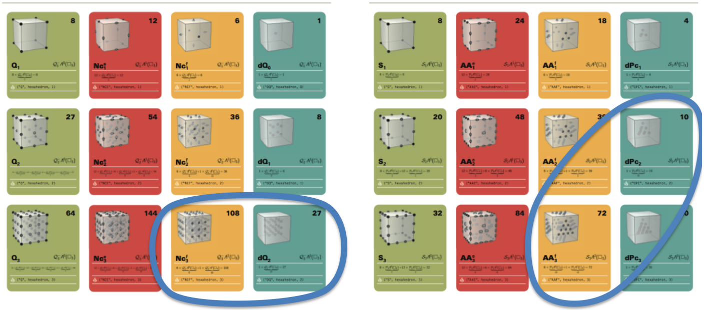

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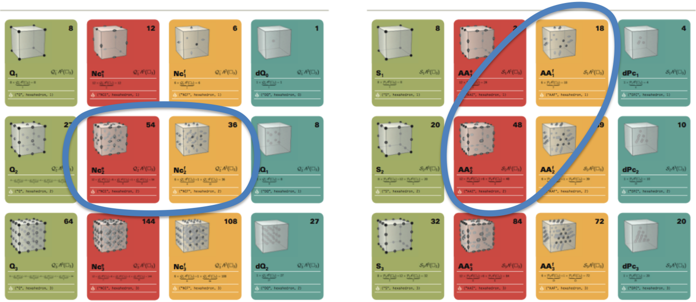

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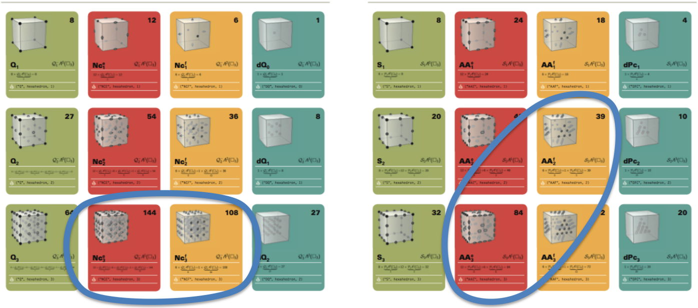

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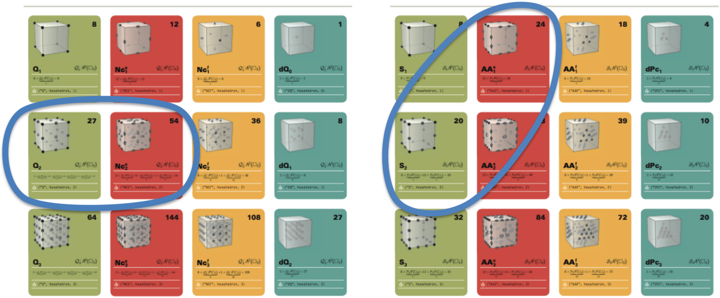
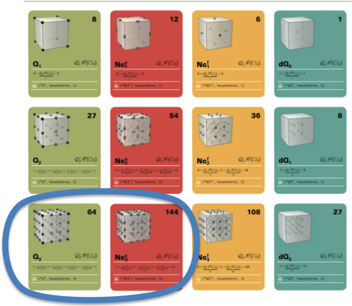
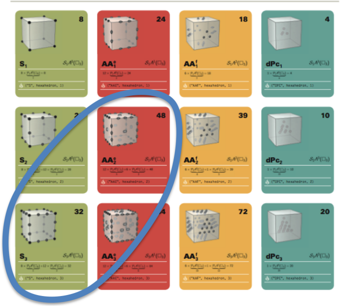


Stable pairs for cubical mehes





Two patterns of choices

The stable pairs just shown come from one of four **sequences** of spaces:

On simplicial meshes in $\mathbb{R}^n$:

$$\mathcal{P}_r^-\Lambda^0 \to \mathcal{P}_r^-\Lambda^1 \to \cdots \to \mathcal{P}_r^-\Lambda^{n-1} \to \mathcal{P}_r^-\Lambda^n \qquad \textbf{‘trimmed’ polynomials}$$

$$\mathcal{P}_r\Lambda^0 \to \mathcal{P}_{r-1}\Lambda^1 \to \cdots \to \mathcal{P}_{r-n+1}\Lambda^{n-1} \to \mathcal{P}_{r-n}\Lambda^n \qquad \textbf{polynomials}$$

On cubical meshes in $\mathbb{R}^n$:

$$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n \qquad \textbf{tensor product}$$

$$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n \qquad \textbf{serendipity}$$

- The ‘minus’ spaces proceed across rows of the PToFE (r stays fixed)
- The ‘regular’ spaces proceed on SW-NE diagonals of the PToFE (r decreases)
- Mysteriously, the degree of freedom count for mixed methods from the $\mathcal{P}_r^-$ spaces is smaller than those from the $\mathcal{P}_r$ spaces, while the opposite is true for the $\mathcal{Q}_r^-$ and $\mathcal{S}_r$ spaces.

Outline

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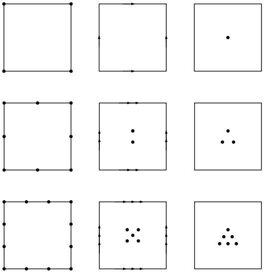
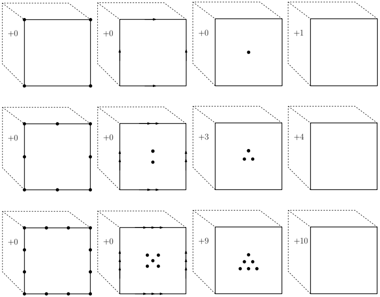
2. Trimmed serendipity spaces: the “5th Column”
3. Comparison to other (polygonal) elements

The 5th column: Trimmed serendipity spaces





A new column for the PToFE:
the **trimmed serendipity** elements.

$\boxed{\mathcal{S}_r^-\Lambda^k(\square_n)}$ denotes

approximation order *r*,
subset of *k*-form space $\Lambda^k(\Omega)$,
use on meshes of *n*-dim'l cubes.

Defined for any $n \geq 1$, $0 \leq k \leq n$, $r \geq 1$

Identical or analogous properties to all the other colummns in the table.

The advantage of the $\mathcal{S}_r^-\Lambda^k$ spaces is that they have fewer degrees of freedom for mixed methods than their tensor product and serendipity alternatives.

Key properties of the trimmed serendipity spaces

$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n$ **tensor product**

$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n$ **serendipity**

$\mathcal{S}_r^-\Lambda^0 \to \mathcal{S}_r^-\Lambda^1 \to \cdots \to \mathcal{S}_r^-\Lambda^{n-1} \to \mathcal{S}_r^-\Lambda^n$ **trimmed serendipity**

Subcomplex: $d\mathcal{S}_r^-\Lambda^k \subset \mathcal{S}_r^-\Lambda^{k+1}$

Exactness: The above sequence is exact.
i.e. the image of incoming map = kernel of outgoing map

Inclusion: $\mathcal{S}_r\Lambda^k \subset \mathcal{S}_{r+1}^-\Lambda^k \subset \mathcal{S}_{r+1}\Lambda^k$

Trace: $\mathrm{tr}_f\mathcal{S}_r^-\Lambda^k(\mathbb{R}^n) \subset \mathcal{S}_r^-\Lambda^k(f)$, for any $(n-1)$-hyperplane f in $\mathbb{R}^n$

Special cases:

$$\mathcal{S}_r^-\Lambda^0 = \mathcal{S}_r\Lambda^0$$
$$\mathcal{S}_r^-\Lambda^n = \mathcal{S}_{r-1}\Lambda^n$$
$$\mathcal{S}_r^-\Lambda^k + d\mathcal{S}_{r+1}\Lambda^{k-1} = \mathcal{S}_r\Lambda^k.$$

Replace '$\mathcal{S}$' by '$\mathcal{P}$' $\rightsquigarrow$ key properties about the first two columns for $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$!

Polynomial spaces and degrees of freedom

$\mathcal{S}_r^- \Lambda^k(\square_n)$ is a space of differential k-forms whose coefficients are polynomials in $\mathbb{R}^n$.

$$\mathcal{S}_r^- \Lambda^k = \mathcal{P}_r^- \Lambda^k \oplus \mathcal{J}_r \Lambda^k \oplus d\mathcal{J}_r \Lambda^{k-1}$$

Polynomial coefficients in each summand:

$\mathcal{P}_r^- \Lambda^k$: anything up to degree $r-1$ and some degree r

$\mathcal{J}_r \Lambda^k$: certain polynomials whose degree is between $r+1$ and $r+n-k-1$

$d\mathcal{J}_r \Lambda^{k-1}$: certain polynomials whose degree is between r and $r+n-k-2$

The 'regular' serendtipity space has an analogous decomposition:

$$\mathcal{S}_r \Lambda^k = \mathcal{P}_r \Lambda^k \oplus \mathcal{J}_r \Lambda^k \oplus d\mathcal{J}_{r+1} \Lambda^{k-1}$$

– – – – – – – – – – – – –

The **degrees of freedom** associated to a d-dimensional sub-face f of an n-dimensional cube $\square_n$ are (for any $k \leq d \leq \min\{n, \lfloor r/2 \rfloor + k\}$):

$$u \longmapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \underbrace{\mathcal{P}_{r-2(d-k)-1} \Lambda^{d-k}(f)}_{\text{indexing space for } \mathcal{S}_{r-1}\Lambda^k(f)} \oplus d\mathcal{H}_{r-2(d-k)+1} \Lambda^{d-k-1}(f),$$

Dimension count and comparison

Formula for counting degrees of freedom of $\mathcal{S}_r^-\Lambda^k(\square_n)$:

$$\sum_{d=k}^{\min\{n,\lfloor r/2\rfloor+k\}} 2^{n-d}\binom{n}{d}\left(\binom{r-d+2k-1}{r-d+k-1}\binom{r-d+k-1}{d-k}+\binom{r-d+2k}{k}\binom{r-d+k-1}{d-k-1}\right)$$

	k	r=1	2	3	4	5	6	7
n=2	0	4	8	12	17	23	30	38
	1	4	10	17	26	37	50	65
	2	1	3	6	10	15	21	28
n=3	0	8	20	32	50	74	105	144
	1	12	36	66	111	173	255	360
	2	6	21	45	82	135	207	301
	3	1	4	10	20	35	56	84
n=4	0	16	48	80	136	216	328	480
	1	32	112	216	392	656	1036	1563
	2	24	96	216	422	746	1227	1910
	3	8	36	94	200	375	644	1036
	4	1	5	15	35	70	126	210

Mixed Method dimension comparison 1

Mixed method for Darcy problem: $$\begin{array}{rcl}\mathbf{u}+K\nabla p &=& 0\\ \operatorname{div}\ \mathbf{u}-f &=& 0\end{array}$$

We compare degree of freedom counts among the three families for use on meshes of affinely-mapped squares or cubes, when a conforming method with (at least) order r decay in the approximation of p, $\mathbf{u}$, and div $\mathbf{u}$ is desired.

Total # of degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1\rvert+\lvert\mathcal{Q}_r^-\Lambda^2\rvert$	$\lvert\mathcal{S}_r\Lambda^1\rvert+\lvert\mathcal{S}_{r-1}\Lambda^2\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1\rvert+\lvert\mathcal{S}_r^-\Lambda^2\rvert$
1	4+1 = 5	8+1 = 9	4+1 = 5
2	12+4 = 16	14+3 = 17	10+3 = 13
3	24+9 = 33	22+6 = 28	17+6 = 23

Total # of degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2\rvert+\lvert\mathcal{Q}_r^-\Lambda^3\rvert$	$\lvert\mathcal{S}_r\Lambda^2\rvert+\lvert\mathcal{S}_{r-1}\Lambda^3\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2\rvert+\lvert\mathcal{S}_r^-\Lambda^3\rvert$
1	6+1 = 7	18+1 = 19	6+1 = 7
2	36+8 = 44	39+4 = 43	21+4 = 25
3	108+27 = 135	72+10 = 82	45+10 = 55

Mixed Method dimension comparison 2

Mixed method for Darcy problem:

$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \operatorname{div}\ \mathbf{u} - f &= 0 \end{aligned}$$

The number of interior degrees of freedom is reduced from tensor product, to serendipity, to trimmed serendipity:

of **interior** degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1_{\mathbf{0}}\rvert + \lvert\mathcal{Q}_r^-\Lambda^2_{\mathbf{0}}\rvert$	$\lvert\mathcal{S}_r\Lambda^1_{\mathbf{0}}\rvert + \lvert\mathcal{S}_{r-1}\Lambda^2_{\mathbf{0}}\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1_{\mathbf{0}}\rvert + \lvert\mathcal{S}_r^-\Lambda^2_{\mathbf{0}}\rvert$
1	0+1 = 1	0+1 = 1	0+1 = 1
2	4+4 = 8	2+3 = 5	2+3 = 5
3	12+9 = 21	6+6 = 12	5+6 = 11

of **interior** degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2_{\mathbf{0}}\rvert + \lvert\mathcal{Q}_r^-\Lambda^3_{\mathbf{0}}\rvert$	$\lvert\mathcal{S}_r\Lambda^2_{\mathbf{0}}\rvert + \lvert\mathcal{S}_{r-1}\Lambda^3_{\mathbf{0}}\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2_{\mathbf{0}}\rvert + \lvert\mathcal{S}_r^-\Lambda^3_{\mathbf{0}}\rvert$
1	0+1 = 1	0+1 = 1	0+1 = 1
2	12+8 = 20	3+4 = 7	3+4 = 7
3	54+27 = 81	12+10 = 22	9+10 = 19

Mixed Method dimension comparison 3

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \operatorname{div}\ \mathbf{u} - f &= 0 \end{aligned}$$

Assuming interior degrees of freedom could be dealt with efficiently (e.g. by sum factorization), trimmed serendipity elements still have the fewest DoFs:

of **interface** (edge) degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1(\partial\Box_2)\rvert$	$\lvert\mathcal{S}_r\Lambda^1(\partial\Box_2)\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1(\partial\Box_2)\rvert$
1	4	8	4
2	8	12	8
3	12	16	12

of **interface** (edge+face) degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2(\partial\Box_3)\rvert$	$\lvert\mathcal{S}_r\Lambda^2(\partial\Box_3)\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2(\partial\Box_3)\rvert$
1	6	18	6
2	24	36	18
3	54	60	36

Minimality of finite element systems

Theorem [Christiansen, G]

Suppose that A is a finite element system on Ω_n and that B is a compatible finite element system containing A. Suppose that

$$\dim B_0^k(\Omega_n) = \dim A_0^k(\Omega_n) + \dim \mathrm{H}^{k+1}\left(A_0^\bullet(\Omega_n)\right).$$

Then B has minimal dimension among compatible finite elt systems containing A.

Here, $\mathrm{H}^{k+1}\left(A_0^\bullet(\Omega_n)\right)$ denotes the $k+1$ homology group of the system $A_0^\bullet$, where the subscript 0 indicates vanishing trace on all $n-1$ dimensional subfaces.

$A_0^\bullet(\Omega_n)$	$B_0^k(\Omega_n)$	interpretation
$\mathcal{P}_{r-1}\Lambda^k(\triangle_n)$	$\mathcal{P}_r^-\Lambda^k(\triangle_n)$	trimmed polynomials minimal, r fixed
$\mathcal{P}_{r-k}\Lambda^k(\square_n)$	$\mathcal{S}_r\Lambda^k(\square_n)$	serendipity minimal, r decreasing
$\mathcal{Q}_r^-\Lambda^k(\square_n)$	$\mathrm{TNT}_r\Lambda^k(\square_n)$	TNT minimal, tensor product degree r fixed
$\mathcal{P}_{r-1}\Lambda^k(\square_n)$	???	unknown space minimal, polynomial degree r fixed

CHRISTIANSEN, G. "Constructions of some minimal finite element systems."
Mathematical Modelling and Numerical Analysis, 2016.

Minimality of trimmed serendipity family

$A_0^\bullet(\Omega_n)$	$B_0^k(\Omega_n)$	interpretation
$\mathcal{P}_{r-1}\Lambda^k(\triangle_n)$	$\mathcal{P}_r^-\Lambda^k(\triangle_n)$	trimmed polynomials minimal on n-simplices, r fixed
$\mathcal{P}_{r-1}\Lambda^k(\square_n)$	$\mathcal{S}_r^-\Lambda^k(\square_n)$	trimmed serendipity space minimal on n-cubes, r fixed

Theorem [G, Kloefkorn]

$\mathcal{S}_r^-\Lambda^\bullet(\square_n)$ is a minimal compatible finite element system containing $\mathcal{P}_{r-1}\Lambda^\bullet(\square_n)$.

Idea of the proof:

We want: $\dim \mathcal{S}_r^-\Lambda_0^k(\square_n) = \dim \mathcal{P}_{r-1}\Lambda^k(\square_n) + \dim \mathrm{H}^{k+1}\left(\mathcal{P}_{r-1}\Lambda^\bullet(\square_n)\right)$

We have $\dim \mathcal{P}_{r-1}\Lambda_0^k(\square_n) = \dim \mathcal{P}_{r-2(n-k)-1}\Lambda^{n-k}(\square_n)$

$\dim \mathrm{H}^{k+1}\left(\mathcal{P}_{r-1}\Lambda_0^\bullet(\square_n)\right) = \dim d\mathcal{H}_{r+2k-2n+1}\Lambda^{n-k-1}(\square_n)$,

$\dim \mathcal{S}_r^-\Lambda_0^k(\square_n)$ = # of DoFs for interior of $\square_n$ = sum of above

G., Kloefkorn "Trimmed Serendipity Finite Element Differential Forms" `arXiv:1607.00571`, 2016.

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2. Trimmed serendipity spaces: the “5th Column”
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Comparison to Arbogast–Correa elements

ARBOGAST, CORREA "Two families of H(div) mixed finite elements on quadrilaterals of minimal dimension," *SIAM J. Numerical Analysis*, 2016

A pair of spaces are defined:

$$\mathbf{V}^r_{AC} := \mathbb{P}^2_r \oplus \mathbf{x}\tilde{\mathbb{P}}_r \oplus \mathbb{S}_r, \qquad \text{and} \qquad W^r_{AC} := \mathbb{P}_r.$$

On a square element, use the "flat" operator to convert to differential forms:

$$\begin{aligned}
\operatorname{rot}\left(\mathbb{P}^2_r \oplus \mathbf{x}\tilde{\mathbb{P}}_r\right)^\flat &= \mathcal{P}_r\Lambda^1(\square_2) \oplus \kappa\mathcal{H}_r\Lambda^0(\square_2) = \mathcal{P}^-_{r+1}\Lambda^1(\square_2) && \subset \mathcal{S}^-_{r+1}\Lambda^1(\square_2)\\
\operatorname{rot}(\mathbb{S}_r)^\flat &= \operatorname{rot}\underbrace{\left(\text{span}\left\{\text{curl}\,(x^{r-1}(1-x^2)y),\ \text{curl}\,(xy^{r-1}(1-y^2))\right\}\right)^\flat}_{\text{“supplemental” vectors}} && \subset \mathcal{S}^-_{r+1}\Lambda^1(\square_2)\\
\left(W^r_{AC}\right)^\flat &= (\mathbb{P}_r)^\flat = \mathcal{P}_r\Lambda^2(\square_2) && = \mathcal{S}^-_{r+1}\Lambda^2(\square_2)
\end{aligned}$$

A simple linear independence argument shows that in fact $\operatorname{rot}\mathbf{V}^r_{AC} = \mathcal{S}^-_{r+1}\Lambda^1(\square_2)$.

Remark: The AC elements can be used on non-affinely mapped quadrilaterals!

Comparison to Cockburn–Fu elements

COCKBURN, FU "A systematic construction of finite element commuting exact sequences" *arXiv:1605.00132*, 2016.

Multiple sequences of spaces are defined, on various geometries.
On a square element, use the "flat" operator to convert their sequence $S_{2,r}^{\square_3}$ to differential forms:

$$\begin{tikzcd}
\mathscr{P}_{r+1} \oplus \delta H^{3,l}_{r+1} \arrow[r] \arrow[d, "\flat"] & \boldsymbol{\mathscr{P}}_r \oplus \mathbf{x} \times \tilde{\boldsymbol{\mathscr{P}}}_r \oplus \nabla \delta H^{3,l}_{r+1} \oplus \delta E^{3,l}_{r+1} \arrow[r] \arrow[d, "\flat"] & \boldsymbol{\mathscr{P}}_r \oplus \mathbf{x} \tilde{\mathscr{P}}_r \oplus \nabla \times \delta E^{3,l}_{r+1} \arrow[r] \arrow[d, "\flat"] & \mathscr{P}_r \arrow[d, "\flat"] \\
\mathcal{S}^-_{r+1}\Lambda^0(\square_3) \arrow[r] & \mathcal{S}^-_{r+1}\Lambda^1(\square_3) \arrow[r] & \mathcal{S}^-_{r+1}\Lambda^2(\square_3) \arrow[r] & \mathcal{S}^-_{r+1}\Lambda^3(\square_3)
\end{tikzcd}$$

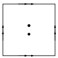
The direct sum decompositions correspond exactly to our formula:

$$\mathcal{S}_r^-\Lambda^k = \mathcal{P}_r^-\Lambda^k \oplus \mathcal{J}_r\Lambda^k \oplus d\mathcal{J}_r\Lambda^{k-1}$$

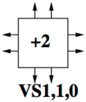
Remark: The CF elements are also defined on many other standard and general geometries!

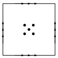
Comparison to serendipity virtual elements

BEIRÃO DA VEIGA, BREZZI, MARINI, RUSSO "Serendipity face and edge VEM spaces"
Rendiconti Lincei-Matematica e Applicazioni, 2017.

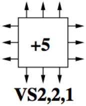


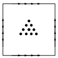
$\mathcal{S}_2^-\Lambda^1(\square_2)$



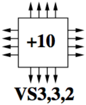


$\mathcal{S}_3^-\Lambda^1(\square_2)$





$\mathcal{S}_4^-\Lambda^1(\square_2)$



A set of virtual element $H(\mathrm{div})$-conforming spaces are defined that "preserve $\mathbf{P}_r + \mathbf{x}P_r$," denoted $VEMS^f_{r,r,r-1}$.

When defined on a square geometry, the spaces have the same degree of freedom counts as the trimmed serendipity spaces $\mathcal{S}_{r+1}^-\Lambda^1(\square_2)$.

Remark: Virtual element spaces are defined on generic quadrilaterals, and more general polygons as well.

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Slides and Pre-prints

`http://math.arizona.edu/~agillette/`