A New Family of Conforming Finite Elements on Cubical Meshes

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joint work with
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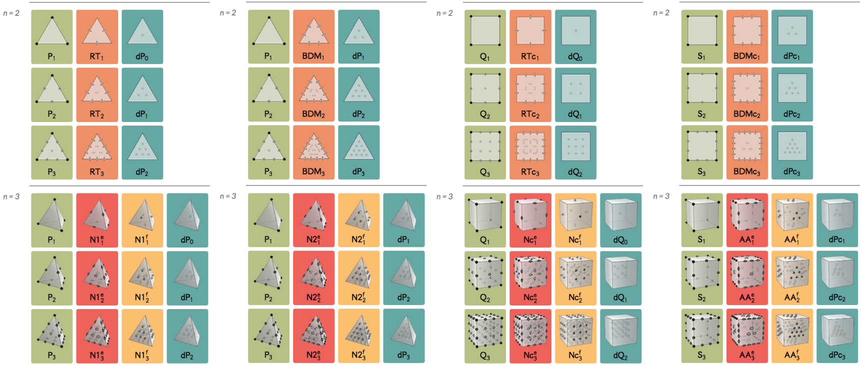

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$H(\text{div})$ / L^2 mixed form of Poisson problem

Derivation of a mixed method for the **Poisson** problem on a domain $\Omega \subset \mathbb{R}^3$:

Given $f : \Omega \to \mathbb{R}$, find a function $p \in H^2(\Omega)$ such that

$$\Delta p + f = 0, \qquad \text{in } \Omega, + \text{B.C.'s}$$

Writing this as a first order system: find $\mathbf{u} \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} \text{div }\mathbf{u} + f &= 0, && \text{in } \Omega, \\ \mathbf{u} - \text{grad } p &= 0, && \text{in } \Omega, \\ (\partial\Omega \text{ conditions}) &= 0 \end{aligned}$$

A **weak form** of these equations: find $\mathbf{u} \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} (\text{div }\mathbf{u}, w) + (f, w) &= 0, && \forall\ w \in L^2 = \Lambda^3(\Omega) \\ (\mathbf{u}, \mathbf{v}) + (p, \text{div }\mathbf{v}) &= 0, && \underbrace{\forall\ \mathbf{v} \in H(\text{div})}_{\text{i.e. } \mathbf{v},\ \text{div }\mathbf{v} \in L^2(\Omega)} = \underbrace{\Lambda^2(\Omega)}_{\text{differential form notation}} \end{aligned}$$

A conforming mixed **finite element** method: find $\mathbf{u}_h \in \Lambda^2_h$ and $p \in \Lambda^3_h$ such that

$$\begin{aligned} (\text{div }\mathbf{u}_h, w_h) + (f, w_h) &= 0 && \forall\ w_h \in \Lambda^3_h \subset L^2(\Omega) \\ (\mathbf{u}_h, \mathbf{v}_h) + (p_h, \text{div }\mathbf{v}_h) &= [\partial\Omega \text{ terms}] && \forall\ \mathbf{v}_h \in \Lambda^2_h \subset H(\text{div}) \\ (\partial\Omega \text{ conditions}) &= 0 \end{aligned}$$

A conforming mixed method for Darcy Flow

Movement of a fluid through porous media modeled via **Darcy flow**:

Given f and g, find pressure p and velocity $\mathbf{u}$ such that:

$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 && \text{in } \Omega, \\ \text{div } \mathbf{u} - f &= 0 && \text{in } \Omega, \\ p &= g && \text{on } \partial\Omega, \end{aligned}$$

where K is a symmetric, uniformly positive definite tensor for $\dfrac{\text{permeability}}{\text{viscosity}}$.

A **weak form** of these equations: find $\mathbf{u} \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} (K^{-1}\mathbf{u}, \mathbf{v}) - (p, \text{div } \mathbf{v}) &= [\partial\Omega \text{ terms}] && \forall\, \mathbf{v} \in H(\text{div}) \\ (\text{div } \mathbf{u}, w) - (f, w) &= 0 && \forall\, w \in L^2(\Omega) \\ (\partial\Omega \text{ conditions}) &= 0 \end{aligned}$$

A conforming mixed **finite element** method: find $\mathbf{u}_h \in \Lambda_h^2$ and $p \in \Lambda_h^3$ such that

$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \text{div } \mathbf{v}_h) &= [\partial\Omega \text{ terms}] && \forall\, \mathbf{v}_h \in \Lambda_h^2 && \subset H(\text{div}) \\ (\text{div } \mathbf{u}_h, w_h) - (f, w_h) &= 0 && \forall\, w_h \in \Lambda_h^3 && \subset L^2(\Omega) \\ (\partial\Omega \text{ conditions}) &= 0 \end{aligned}$$

ARBOGAST, PENCHEVA, WHEELER, YOTOV "A Multiscale Mortar Mixed Finite Element Method"
Multiscale Modeling and Simulation (SIAM) 6:1, 2007.

Stable pairs of finite element spaces

$$\begin{aligned}(\mathbf{u}_h, \mathbf{v}_h) + (p_h, \text{div } \mathbf{v}_h) &= [\partial\Omega \text{ terms}] && \forall\ \mathbf{v}_h \in \Lambda_h^2 && \subset H(\text{div}) \\ (\text{div } \mathbf{u}_h, w_h) + (f, w_h) &= 0 && \forall\ w_h \in \Lambda_h^3 && \subset L^2(\Omega)\end{aligned}$$

Given a selection for the finite element spaces $(\Lambda_h^2, \Lambda_h^3)$,
the method is said to be **stable** if the error in the computed solution $(\mathbf{u}_h, p_h)$ is within a constant multiple C of the minimal *possible* error. That is:

$$||\mathbf{u} - \mathbf{u}_h||_{H(\text{div})} + ||p - p_h||_{L^2} \leq C \left(\inf_{\mathbf{w} \in \Lambda_h^2} ||\mathbf{u} - \mathbf{w}||_{H(\text{div})} + \inf_{q \in \Lambda_h^3} ||p - q||_{L^2} \right) \qquad (*)$$

Brezzi's theorem establishes the following sufficient criteria for $(*)$:

$$(\mathbf{w}, \mathbf{w}) \geq c\,||\mathbf{w}||^2_{H(\text{div})}, \qquad \forall\ \mathbf{w} \in \mathbf{Z}_h := \left\{ \mathbf{w} \in \Lambda_h^2 \ :\ (\text{div } \mathbf{w}, q) = 0,\ \ \forall\ q \in \Lambda_h^3 \right\},$$

$$\sup_{\mathbf{w} \in \Lambda_h^2} \frac{(\text{div } \mathbf{w}, q)}{||\mathbf{w}||_{H(\text{div})}} \geq c\,||q||_{L^2}, \qquad \forall\ q \in \Lambda_h^3.$$

If the pair $(\Lambda_h^2, \Lambda_h^3)$ satisfies these two criteria it is called a **stable pair**.

BREZZI, "On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers," RAIRO, 1974.

H^1 / $H(\text{curl})$ mixed form of vector Poisson problem

Derivation of a mixed method for the **vector Poisson** problem on a domain $\Omega \subset \mathbb{R}^3$:

$$-\text{grad div}\,\mu + \text{curl curl}\,\mu = \text{f}, \qquad + \partial\Omega \textit{ conditions}$$

$$\begin{aligned} \sigma + \text{div}\,\mu &= 0, &\quad + \partial\Omega \textit{ conditions} \\ \text{grad}\,\sigma + \text{curl curl}\,\mu &= \text{f}, \end{aligned}$$

A **weak form** of these equations: find $\sigma \in H^1(\Omega)$ and $\mu \in H(\text{curl})$ such that

$$\begin{aligned} (\sigma, \tau) - (\mu, \text{grad}\,\tau) &= 0 & \forall \tau \in H^1(\Omega) &= \Lambda^0(\Omega) \\ (\text{grad}\,\sigma, v) + (\text{curl}\,\mu, \text{curl}\,v) &= (f, v), & \underbrace{\forall v \in H(\text{curl})}_{\text{i.e. } \omega,\ \text{curl}\,\omega \in L^2(\Omega)} &= \underbrace{\Lambda^1(\Omega)}_{\text{differential form notation}} \end{aligned}$$

A conforming mixed **finite element** method: find $\sigma_h \in \Lambda_h^0$ and $\mu_h \in \Lambda_h^1$ such that

$$\begin{aligned} (\sigma_h, \tau_h) - (\mu_h, \text{grad}\,\tau_h) &= 0, & \forall \tau_h \in \Lambda_h^0 &\subset H^1(\Omega) \\ (\text{grad}\,\sigma_h, v_h) + (\text{curl}\,\mu_h, \text{curl}\,v_h) &= (f, v_h), & \forall v_h \in \Lambda_h^1 &\subset H(\text{curl}) \end{aligned}$$

Applications to computational electromagnetism, image processing, visualization, ...

H(curl) and H(div) in Maxwell's equations

domain	contractible 3-manifold $\Omega \subset \mathbb{R}^3$ with boundary Γ
variables	b (magnetic field / magnetic induction)
	h (magnetizing field / auxiliary magnetic field)
input	j (current density field)
equations	$\underbrace{\nabla \cdot b = 0}_{\text{Gauss' law}} \quad \underbrace{\mu b = h}_{\text{constitutive relation}} \quad \underbrace{\nabla \times h = j}_{\text{Ampère's law}}$
boundary	$\Gamma = \Gamma^e \cup \Gamma^h, \quad \hat{n} \cdot b = 0 \text{ on } \Gamma^e, \quad \hat{n} \times h = 0 \text{ on } \Gamma^h$

The L^2 **deRham Diagram** relates the key functional spaces for domains in $\mathbb{R}^3$

$$H^1 \xrightarrow[\text{grad}]{\nabla} H(\text{curl}) \xrightarrow[\text{curl}]{\nabla\times} H(\text{div}) \xrightarrow[\text{div}]{\nabla\cdot} L^2$$

	H^1	$H(\text{curl})$	$H(\text{div})$	L^2
vector Poisson	σ	μ		
Maxwell's eqn's		h	b	
Darcy / Poisson			$\mathbf{u}$	p

An abbreviated reading list (50 years of theory!)

RAVIART, THOMAS, "A mixed finite element method for 2nd order elliptic problems" Lecture Notes in Mathematics, 1977. $\leftarrow$ *2910 citations, more than 100 of which are from 2016!*

NÉDÉLEC, "Mixed finite elements in $\mathbb{R}^3$," Numerische Mathematik, 1980

BREZZI, DOUGLAS JR., MARINI, "Two families of mixed finite elements for second order elliptic problems," Numerische Mathematik, 1985.

NÉDÉLEC, "A new family of mixed finite elements in $\mathbb{R}^3$," Numerische Mathematik, 1986

ARNOLD, FALK, WINTHER "Finite element exterior calculus, homological techniques, and applications," *Acta Numerica*, 2006

CHRISTIANSEN, "Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension," Numerische Mathematik, 2007.

ARNOLD, FALK, WINTHER "Finite element exterior calculus: from hodge theory to numerical stability," *Bulletin of the AMS*, 2010.

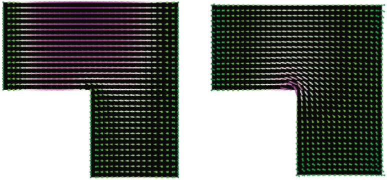
ARNOLD, AWANOU "The serendipity family of finite elements ", *Found. Comp Math*, 2011

ARNOLD, AWANOU "Finite element differential forms on cubical meshes", *Math Comp.*, 2013

ARNOLD, BOFFI, BONIZZONI "Finite element differential forms on curvillinear meshes and their approximation properties," Numerische Mathematik, 2014

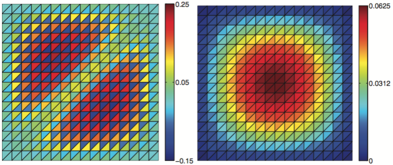
ARBOGAST, CORREA "Two families of H(div) mixed finite elements on quadrilaterals of minimal dimension," ICES Report, UT Austin, 2015

The importance of method selection



Vector Poisson problem

- Solutions by the standard non-mixed method (left) and by a mixed method (right).
- Only the second choice shows the correct behavior near the reentrant corner.



Poisson problem

- Solutions by two different choices for the finite element solution spaces in a mixed method.
- Only the second choice looks like the true solution is $x(1-x)y(1-y)$.

Examples and images borrowed from:

ARNOLD, FALK, WINTHER "Finite Element Exterior Calculus: From Hodge Theory to Numerical Stability," *Bulletin of the AMS*, 47:2, 2010.

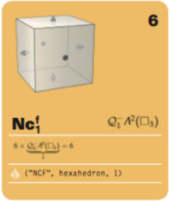
Outline

Unified notation via differential forms

Finite element method types can be broadly classified by three integers:

n	$\rightarrow$	the spatial dimension of the domain
r	$\rightarrow$	the order of error decay
k	$\rightarrow$	the differential form order of the solution space

An element type is defined in part by its **degrees of freedom**;
the more degrees of freedom, the greater the computational cost of the method.



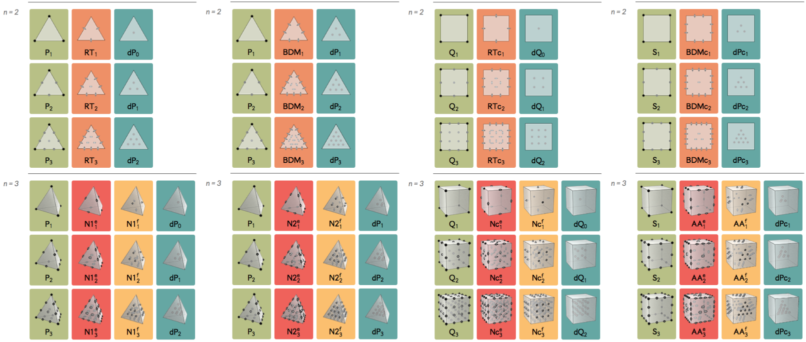
Ex: $\mathcal{Q}_1^-\Lambda^2(\square_3)$ is an element for

$n = 3$	$\rightarrow$	domains in $\mathbb{R}^3$
$r = 1$	$\rightarrow$	linear order of error decay
$k = 2$	$\rightarrow$	conformity in $\Lambda^2(\mathbb{R}^3) \rightsquigarrow H(\mathrm{div})$

$\mathcal{Q}_1^-\Lambda^2(\square_3)$ is part of the $\mathcal{Q}^-$ 'column' of elements,
is defined on geometry $\square_3$ (i.e. a cube),
has a **6** dimensional space of test functions,
and has an associated set of **6** degrees of freedom
that are unisolvent for the test function space.

The ‘Periodic Table of the Finite Elements’

The periodic table of the finite elements *(prepared by Doug Arnold & Anders Logg)*:

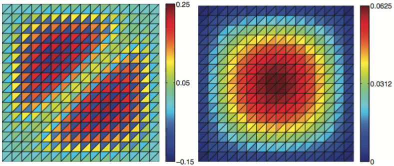


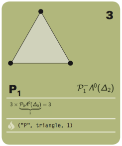
Classification of many common conforming finite element types.

$n \rightarrow$ Domains in $\mathbb{R}^2$ (top half) and in $\mathbb{R}^3$ (bottom half)
$r \rightarrow$ Order $1, 2, 3$ of error decay (going down columns)
$k \rightarrow$ Conformity type $k = 0, \ldots, n$ (going across a row)

Geometry types: Simplices (left half) and cubes (right half).

Stable choices for mixed methods





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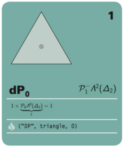
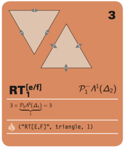
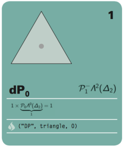






$\subset H^1 \times H^1$ $\subset L^2$ $\subset H(\text{div})$ $\subset L^2$

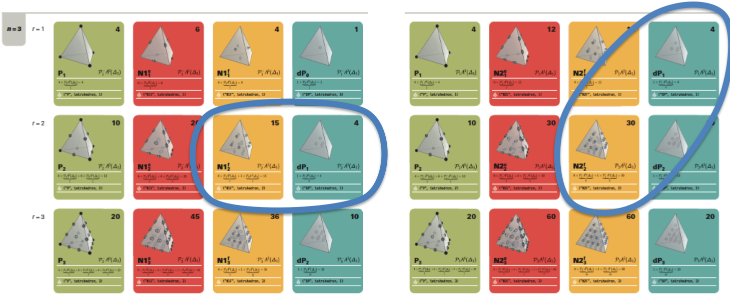
Unstable method, as shown

Provably stable method

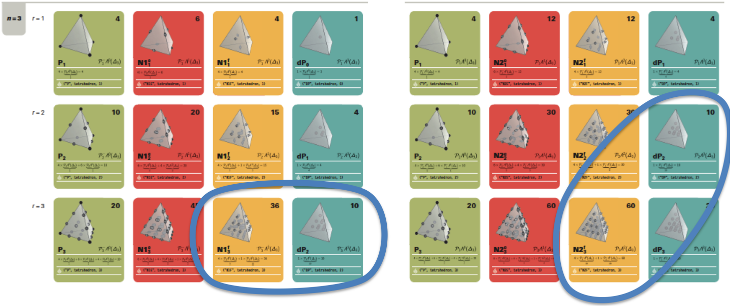
→ The term **stable pair** of elements for mixed methods has a precise mathematical meaning (see e.g. Arnold, Falk, Winther paper).

→ The Periodic Table of Finite Elements lets us ‘read off’ stable pairs visually.

Stable pairs for simplicial meshes

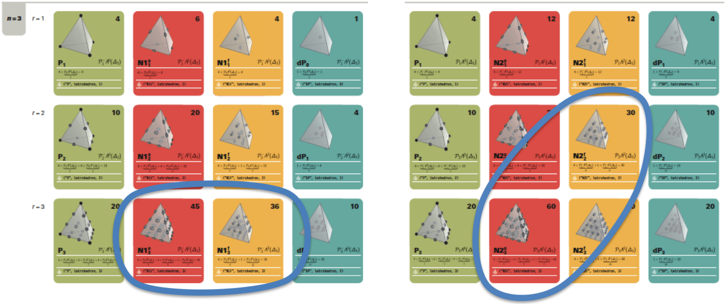


Stable pairs for simplicial meshes

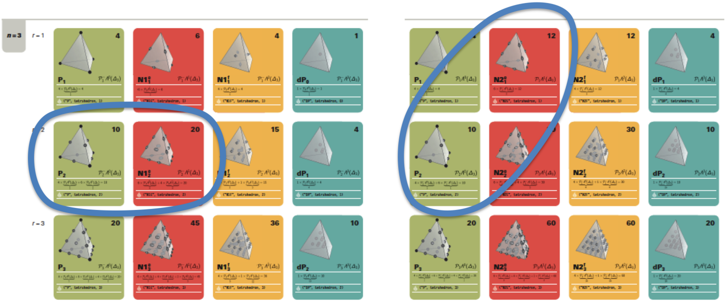


Stable pairs for simplicial meshes

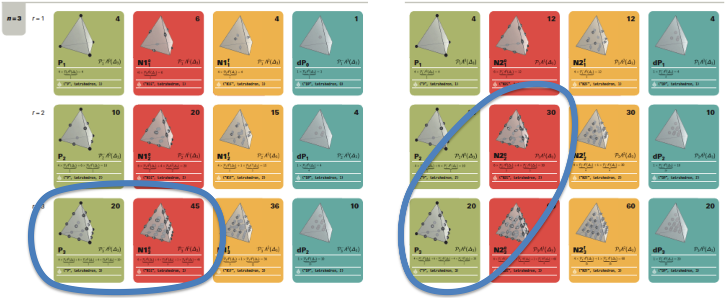
Stable pairs for simplicial meshes



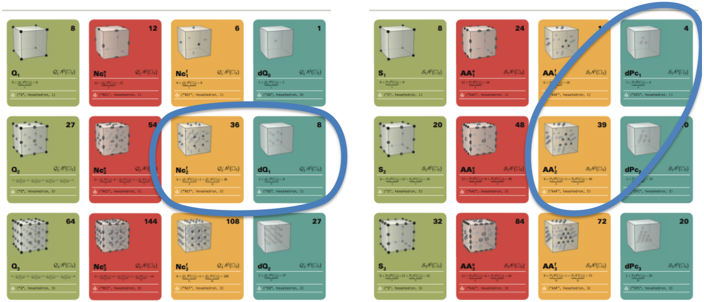
Stable pairs for simplicial meshes



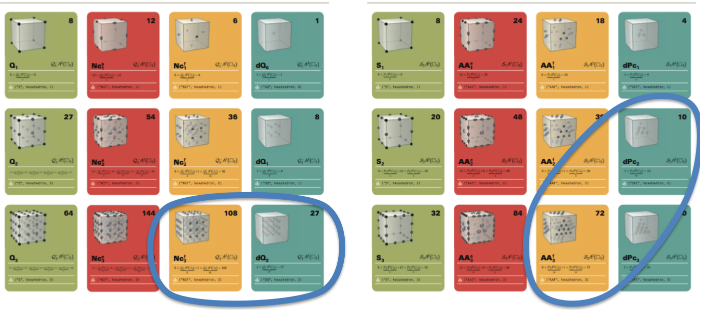
Stable pairs for simplicial meshes



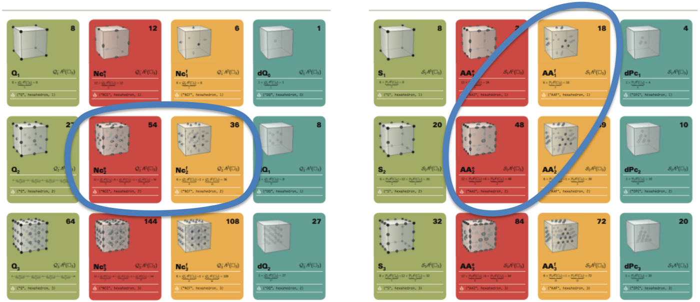
Stable pairs for cubical mehes



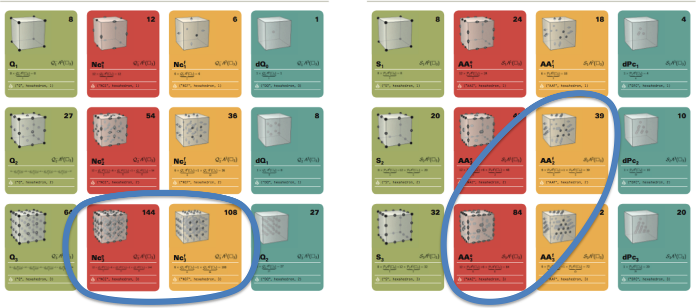
Stable pairs for cubical mehes



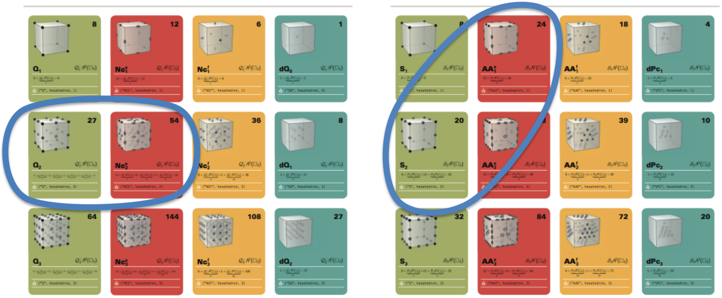
Stable pairs for cubical mehes



Stable pairs for cubical mehes



Stable pairs for cubical mehes



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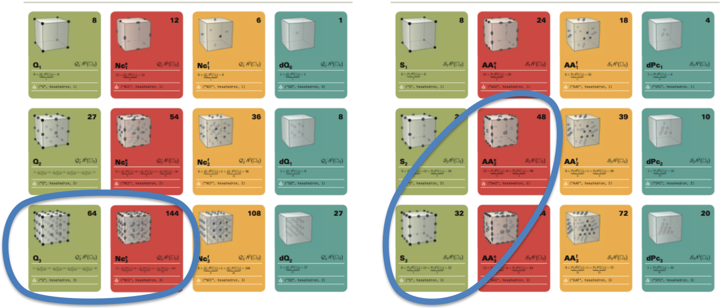

Two patterns of choices

The stable pairs just shown come from one of four **sequences** of spaces:

On simplicial meshes in $\mathbb{R}^n$:

$$\mathcal{P}_r^-\Lambda^0 \to \mathcal{P}_r^-\Lambda^1 \to \cdots \to \mathcal{P}_r^-\Lambda^{n-1} \to \mathcal{P}_r^-\Lambda^n$$ **'trimmed' polynomials**

$$\mathcal{P}_r\Lambda^0 \to \mathcal{P}_{r-1}\Lambda^1 \to \cdots \to \mathcal{P}_{r-n+1}\Lambda^{n-1} \to \mathcal{P}_{r-n}\Lambda^n$$ **polynomials**

On cubical meshes in $\mathbb{R}^n$:

$$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n$$ **tensor product**

$$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n$$ **serendipity**

- The 'minus' spaces proceed across rows of the PToFE (r stays fixed)
- The 'regular' spaces proceed on SW-NE diagonals of the PToFE (r decreases)
- Mysteriously, the degree of freedom count for mixed methods from the $\mathcal{P}_r^-$ spaces is smaller than those from the $\mathcal{P}_r$ spaces, while the opposite is true for the $\mathcal{Q}_r^-$ and $\mathcal{S}_r$ spaces.

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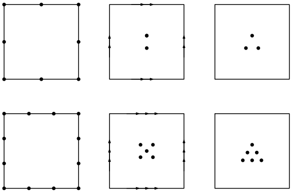
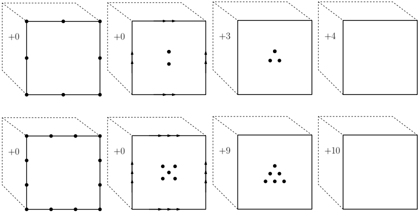
The 5th column: Trimmed serendipity spaces

$n = 2$



$n = 3$



A new column for the PToFE:
the **trimmed serendipity** elements.

$\boxed{\mathcal{S}_r^- \Lambda^k(\square_n)}$ denotes
approximation order r,
subset of k-form space $\Lambda^k(\Omega)$,
use on meshes of n-dim'l cubes.

Defined for any $n \geq 1$, $0 \leq k \leq n$, $r \geq 1$

Identical or analogous properties to all the other colummns in the table.

The advantage of the $\mathcal{S}_r^- \Lambda^k$ spaces is that they have **minimal dimension**, in a certain sense to be described, and thus allow potential computational benefits.

Key properties of the trimmed serendipity spaces

$$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n$$ **tensor product**

$$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n$$ **serendipity**

$$\mathcal{S}_r^-\Lambda^0 \to \mathcal{S}_r^-\Lambda^1 \to \cdots \to \mathcal{S}_r^-\Lambda^{n-1} \to \mathcal{S}_r^-\Lambda^n$$ **trimmed serendipity**

Subcomplex: $d\mathcal{S}_r^-\Lambda^k \subset \mathcal{S}_r^-\Lambda^{k+1}$

Exactness: The above sequence is exact.
i.e. the image of incoming map = kernel of outgoing map

Inclusion: $\mathcal{S}_r\Lambda^k \subset \mathcal{S}_{r+1}^-\Lambda^k \subset \mathcal{S}_{r+1}\Lambda^k$

Trace: $\mathrm{tr}_f\mathcal{S}_r^-\Lambda^k(\mathbb{R}^n) \subset \mathcal{S}_r^-\Lambda^k(f)$, for any $(n-1)$-hyperplane f in $\mathbb{R}^n$

Special cases:
$\mathcal{S}_r^-\Lambda^0 = \mathcal{S}_r\Lambda^0$
$\mathcal{S}_r^-\Lambda^n = \mathcal{S}_{r-1}\Lambda^n$
$\mathcal{S}_r^-\Lambda^k + d\mathcal{S}_{r+1}\Lambda^{k-1} = \mathcal{S}_r\Lambda^k.$

Replace '$\mathcal{S}$' by '$\mathcal{P}$' $\rightsquigarrow$ key properties about the first two columns for $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$!

Polynomial spaces and degrees of freedom

$\mathcal{S}_r^-\Lambda^k(\square_n)$ is a space of differential k-forms whose coefficients are polynomials in $\mathbb{R}^n$.

$$\mathcal{S}_r^-\Lambda^k = \mathcal{P}_r^-\Lambda^k \oplus \mathcal{J}_r\Lambda^k \oplus d\mathcal{J}_r\Lambda^{k-1}$$

Polynomial coefficients in each summand:

$\mathcal{P}_r^-\Lambda^k$: anything up to degree $r-1$ and some degree r

$\mathcal{J}_r\Lambda^k$: certain polynomials whose degree is between $r+1$ and $r+n-k-1$

$d\mathcal{J}_r\Lambda^{k-1}$: certain polynomials whose degree is between r and $r+n-k-2$

The 'regular' serendtipity space has an analogous decomposition:

$$\mathcal{S}_r\Lambda^k = \mathcal{P}_r\Lambda^k \oplus \mathcal{J}_r\Lambda^k \oplus d\mathcal{J}_{r+1}\Lambda^{k-1}$$

– – – – – – – – – – – – –

The **degrees of freedom** associated to a d-dimensional sub-face f of an n-dimensional cube $\square_n$ are (for any $k \le d \le \min\{n, \lfloor r/2 \rfloor + k\}$):

$$u \longmapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \underbrace{\mathcal{P}_{r-2(d-k)-1}\Lambda^{d-k}(f)}_{\text{indexing space for } \mathcal{S}_{r-1}\Lambda^k(f)} \oplus d\mathcal{H}_{r-2(d-k)+1}\Lambda^{d-k-1}(f),$$

Dimension count and comparison

Formula for counting degrees of freedom of $\mathcal{S}_r^- \Lambda^k(\square_n)$:

$$\sum_{d=k}^{\min\{n,\lfloor r/2\rfloor+k\}} 2^{n-d}\binom{n}{d}\left(\binom{r-d+2k-1}{r-d+k-1}\binom{r-d+k-1}{d-k}+\binom{r-d+2k}{k}\binom{r-d+k-1}{d-k-1}\right)$$

	k	r=1	2	3	4	5	6	7
n=2	0	4	8	12	17	23	30	38
	1	4	10	17	26	37	50	65
	2	1	3	6	10	15	21	28
n=3	0	8	20	32	50	74	105	144
	1	12	36	66	111	173	255	360
	2	6	21	45	82	135	207	301
	3	1	4	10	20	35	56	84
n=4	0	16	48	80	136	216	328	480
	1	32	112	216	392	656	1036	1563
	2	24	96	216	422	746	1227	1910
	3	8	36	94	200	375	644	1036
	4	1	5	15	35	70	126	210

Outline

New $H(\mathrm{curl})$ and $H(\mathrm{div})$ elements in $\mathbb{R}^3$

The degrees of freedom for $\mathcal{S}_r^-\Lambda^1(\square_3)$ (i.e. $H(\mathrm{curl})$-conforming in $\mathbb{R}^3$):

$$u \longmapsto \int_e u \cdot \vec{t}\, p, \qquad p \in \mathcal{P}_{r-1}(e), \quad e \text{ an edge of } \square_3 \text{ with unit tangent } \vec{t},$$

$$u \longmapsto \int_f (u \times \hat{n}) \cdot \vec{p}, \qquad \vec{p} \in [\mathcal{P}_{r-3}(f)]^2 \oplus \mathrm{grad}\, \mathcal{H}_{r-1}\Lambda^0(f),$$

$$f \text{ a face of } \square_3 \text{ with unit normal } \hat{n},$$

$$u \longmapsto \int_{\square_3} u \cdot \vec{p}, \qquad \vec{p} \in [\mathcal{P}_{r-5}(\square_3)]^3 \oplus \mathrm{curl}\, \mathcal{H}_{r-3}\Lambda^1(\square_3).$$

The degrees of freedom for $\mathcal{S}_r^-\Lambda^2(\square_3)$ (i.e. $H(\mathrm{div})$-conforming in $\mathbb{R}^3$):

$$u \longmapsto \int_f u \cdot \hat{n}\, p, \qquad p \in \mathcal{P}_{r-1}(f), \; f \text{ a face of } \square_3 \text{ with unit normal } \hat{n},$$

$$u \longmapsto \int_{\square_3} u \cdot p, \qquad \vec{p} \in [\mathcal{P}_{r-3}(\square_3)]^3 \oplus \mathrm{grad}\, \mathcal{H}_{r-1}\Lambda^0(\square_3).$$

For $k = 1$ or 2, we have $\quad \dim \mathcal{S}_r^-\Lambda^k(\square_3) < \dim \mathcal{S}_r\Lambda^k(\square_3)$

and $\quad \dim \mathcal{S}_r^-\Lambda^k(\square_3) \leq \dim \mathcal{Q}_r^-\Lambda^k(\square_3)$, equal iff $r = 1$.

Mixed Method dimension comparison

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \operatorname{div}\ \mathbf{u} - f &= 0 \end{aligned}$$

We compare degree of freedom counts among the three families for use on meshes of affinely-mapped squares or cubes, when a conforming method with (at least) order r decay in the approximation of p, $\mathbf{u}$, and div $\mathbf{u}$ is desired.

Meshes of squares ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1\rvert + \lvert\mathcal{Q}_r^-\Lambda^2\rvert$	$\lvert\mathcal{S}_r\Lambda^1\rvert + \lvert\mathcal{S}_{r-1}\Lambda^2\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1\rvert + \lvert\mathcal{S}_r^-\Lambda^2\rvert$
1	4+1 = 5	8+1 = 9	4+1 = 5
2	12+4 = 16	14+3 = 17	10+3 = 13
3	24+9 = 33	22+6 = 28	17+6 = 23

Meshes of cubes ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2\rvert + \lvert\mathcal{Q}_r^-\Lambda^3\rvert$	$\lvert\mathcal{S}_{r+1}\Lambda^2\rvert + \lvert\mathcal{S}_r\Lambda^3\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2\rvert + \lvert\mathcal{S}_r^-\Lambda^3\rvert$
1	6+1 = 7	18+1 = 19	6+1 = 7
2	36+8 = 44	39+4 = 43	21+4 = 25
3	108+27 = 135	72+10 = 82	45+10 = 55

Minimality of finite element systems

Theorem [Christiansen, G]

Suppose that A is a finite element system on Ω_n and that B is a compatible finite element system containing A. Suppose that

$$\dim B_0^k(\Omega_n) = \dim A_0^k(\Omega_n) + \dim \mathrm{H}^{k+1}\left(A_0^\bullet(\Omega_n)\right).$$

Then B has minimal dimensionn among compatible finite elt systems containing A.

Here, $\mathrm{H}^{k+1}\left(A_0^\bullet(\Omega_n)\right)$ denotes the $k+1$ homology group of the system $A_0^\bullet$, where the subscript 0 indicates vanishing trace on all $n-1$ dimensional subfaces.

$A_0^\bullet(\Omega_n)$	$B_0^k(\Omega_n)$	interpretation
$\mathcal{P}_{r-1}\Lambda^k(\triangle_n)$	$\mathcal{P}_r^-\Lambda^k(\triangle_n)$	trimmed polynomials minimal, r fixed
$\mathcal{P}_{r-k}\Lambda^k(\square_n)$	$\mathcal{S}_r\Lambda^k(\square_n)$	serendipity minimal, r decreasing
$\mathcal{Q}_r^-\Lambda^k(\square_n)$	$\mathrm{TNT}_r\Lambda^k(\square_n)$	TNT minimal, tensor product degree r fixed
$\mathcal{P}_{r-1}\Lambda^k(\square_n)$	???	unknown space minimal, polynomial degree r fixed

Christiansen, G. "Constructions of some minimal finite element systems." Mathematical Modelling and Numerical Analysis, 2016.

Minimality of trimmed serendipity family

$A_0^\bullet(\Omega_n)$	$B_0^k(\Omega_n)$	interpretation
$\mathcal{P}_{r-1}\Lambda^k(\triangle_n)$	$\mathcal{P}_r^-\Lambda^k(\triangle_n)$	trimmed polynomials minimal on n-simplices, r fixed
$\mathcal{P}_{r-1}\Lambda^k(\square_n)$	$\mathcal{S}_r^-\Lambda^k(\square_n)$	trimmed serendipity space minimal on n-cubes, r fixed

Theorem [G, Kloefkorn]

$\mathcal{S}_r^-\Lambda^\bullet(\square_n)$ is a minimal compatible finite element system containing $\mathcal{P}_{r-1}\Lambda^\bullet(\square_n)$.

Idea of the proof:

We want: $\dim \mathcal{S}_r^-\Lambda_0^k(\square_n) = \dim \mathcal{P}_{r-1}\Lambda^k(\square_n) + \dim \mathrm{H}^{k+1}\left(\mathcal{P}_{r-1}\Lambda^\bullet(\square_n)\right)$

We have

$$\dim \mathcal{P}_{r-1}\Lambda_0^k(\square_n) = \dim \mathcal{P}_{r-2(n-k)-1}\Lambda^{n-k}(\square_n)$$

$$\dim \mathrm{H}^{k+1}\left(\mathcal{P}_{r-1}\Lambda_0^\bullet(\square_n)\right) = \dim d\mathcal{H}_{r+2k-2n+1}\Lambda^{n-k-1}(\square_n),$$

$$\dim \mathcal{S}_r^-\Lambda_0^k(\square_n) = \text{\# of DoFs for interior of } \square_n = \text{sum of above}$$

G., KLOEFKORN "Trimmed Serendipity Finite Element Differential Forms"
arXiv:1607.00571, 2016.

Related work and open questions

Related contemporary work:

ARBOGAST, CORREA "Two families of $H(\text{div})$ mixed finite elements on quadrilaterals of minimal dimension," ICES Report, UT Austin, 2015

BEIRAO DA VEIGA, BREZZI, MARINI, RUSSO "Serendipity nodal VEM spaces" `arXiv:1510.08477`, 2015.

COCKBURN, FU "A systematic construction of finite element commuting exact sequences" `arXiv:1605.00132`, 2016.

Open questions:

- Unisolvence in general case.

  We have proved all cases of immediate relevance to applications:

  $1 \leq n \leq 4$, $0 \leq k \leq n$, $1 \leq r \leq 10$.

  (Extra credit problem available at the end of the talk!)
- Construction of basis functions for implementation
- Extension to non-affine maps of cubes
- Further comparison to other approaches

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Upcoming Events

Advances in Quadrilateral and Hexahedral Finite Elements

Poster session at SIAM CSE in Atlanta, February 2017.

Slides and Pre-prints

`http://math.arizona.edu/~agillette/`

Extra credit

We have shown that

$$\dim \mathcal{J}_r\Lambda^k(\mathbb{R}^n) = \sum_{i=0}^{k}(-1)^i\left[\sum_{d=k-i}^{\min\{n,\lfloor (r+i)/2\rfloor+k-i\}} 2^{n-d}\binom{n}{d}\binom{r-d+2k-i}{d}\binom{d}{k-i} - \left(\sum_{j=0}^{r+i}\binom{n+j-1}{j}\binom{n}{k-i}\right)\right]$$

Prove that

$$\sum_{d=k}^{\min\{n,\lfloor r/2\rfloor+k\}} 2^{n-d}\binom{n}{d}\left(\binom{r-d+2k-1}{r-d+k-1}\binom{r-d+k-1}{d-k}+\binom{r-d+2k}{k}\binom{r-d+k-1}{d-k-1}\right)$$

equals

$$\binom{r+n}{r+k}\binom{r+k-1}{k}+\dim \mathcal{J}_r\Lambda^k(\mathbb{R}^n)+\dim \mathcal{J}_r\Lambda^{k-1}(\mathbb{R}^n)$$