Modern Directions in Finite Element Theory: Polytope Meshes and Serendipity Methods

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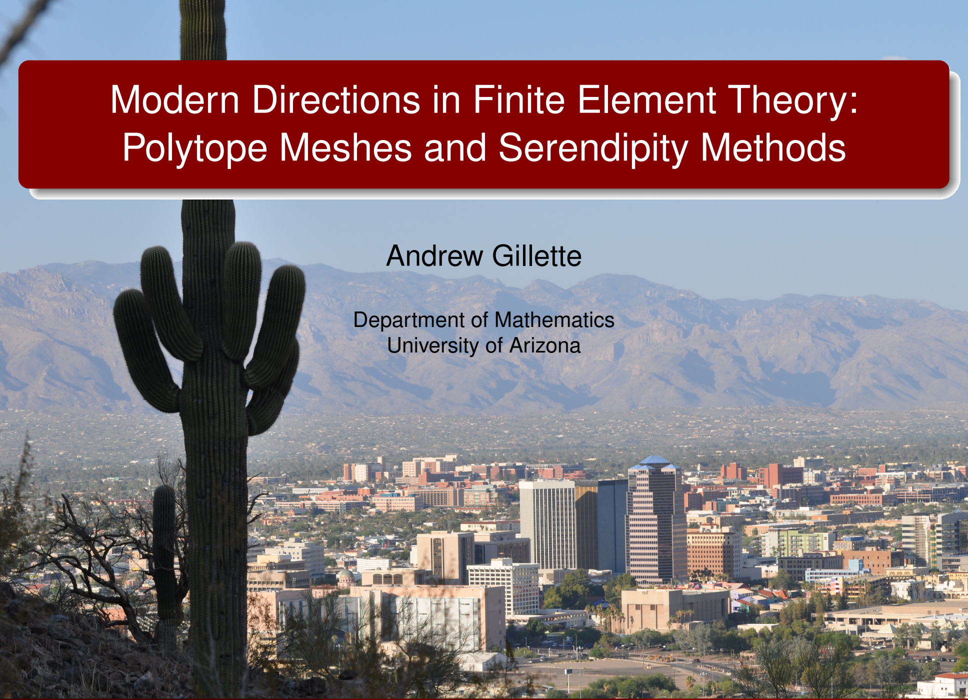


What are *a priori* FEM error estimates?

Poisson's equation in $\mathbb{R}^n$: Given a domain $\mathcal{D} \subset \mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$, find u such that

strong form	$-\Delta u$	$=$	f	$u \in H^2(\mathcal{D})$
weak form	$\int_{\mathcal{D}} \nabla u \cdot \nabla \phi$	$=$	$\int_{\mathcal{D}} f\, \phi$	$\forall \phi \in H^1(\mathcal{D})$
discrete form	$\int_{\mathcal{D}} \nabla u_h \cdot \nabla \phi_h$	$=$	$\int_{\mathcal{D}} f\, \phi_h$	$\forall \phi_h \in V_h \leftarrow$ *finite dim.* $\subset H^1(\mathcal{D})$

Typical **finite element method:**

$\to$ Mesh $\mathcal{D}$ by polytopes $\{P\}$ with vertices $\{\mathbf{v}_i\}$; define $h := \max \operatorname{diam}(P)$.

$\to$ Fix basis functions λ_i with local piecewise support, e.g. barycentric functions.

$\to$ Define u_h such that it uses the λ_i to approximate u, e.g. $u_h := \sum_i u(\mathbf{v}_i)\lambda_i$

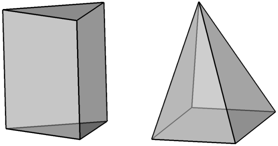
A linear system for u_h can then be derived, admitting an ***a priori* error estimate:**

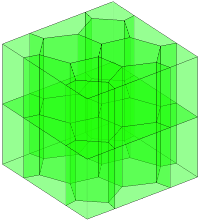
$$\underbrace{||u - u_h||_{H^1(P)}}_{\text{approximation error}} \leq \underbrace{C\, h^{\mathbf{p}}\, |u|_{H^{\mathbf{p+1}}(P)}}_{\text{optimal error bound}}, \quad \forall u \in H^{\mathbf{p+1}}(P),$$

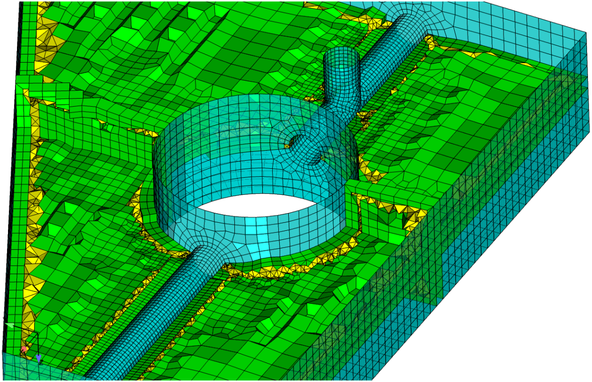
provided that the λ_i span all degree p polynomials on each polytope P.

Two trends in contemporary finite element research

1 **Polygonal / polyhedral domain meshes:** Greater geometric flexibility can alleviate known difficulties with simplicial and cubical elements.



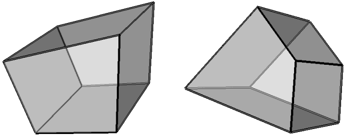


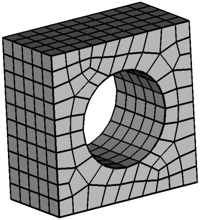


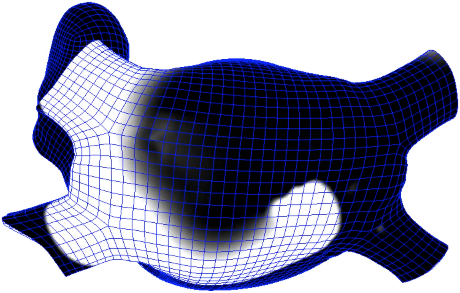
↑ *Image from MeshGems software website*

Two trends in contemporary finite element research

1 **Polygonal / polyhedral domain meshes:** Greater geometric flexibility can alleviate known difficulties with simplicial and cubical elements.





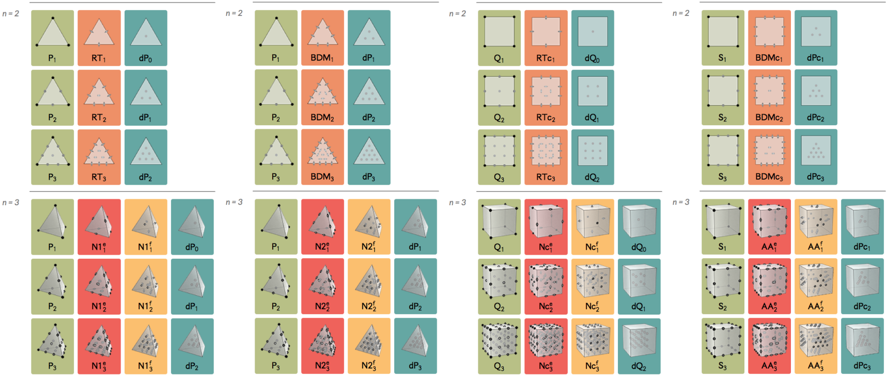


↑ *Heart mesh made using Continuity software, National Biomedical Computation Resource, UCSD*

← *Hole mesh made using CUBIT Geometry and Mesh Generation Toolkit, Sandia National Labs*

Two trends in contemporary finite element research

2 **'Serendipity' higher order methods:** Long observed but only recently formalized theory completing a 'Periodic Table of Finite Elements'



→ Viewable online at `femtable.org`

→ Scientific content prepared by Doug Arnold and Anders Logg

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The generalized barycentric coordinate approach

Let P be a convex polytope with vertex set V. We say that

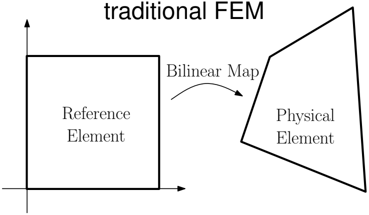
$$\lambda_{\mathbf{v}} : P \to \mathbb{R} \quad \text{are} \quad \textbf{generalized barycentric coordinates (GBCs)} \text{ on } P$$

if they satisfy $\quad \lambda_{\mathbf{v}} \geq 0$ on P and $L = \sum_{\mathbf{v} \in V} L(\mathbf{v}_{\mathbf{v}}) \lambda_{\mathbf{v}}, \quad \forall\, L : P \to \mathbb{R}$ linear.

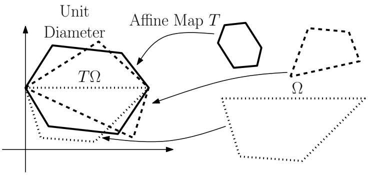
Familiar properties are implied by this definition:

$$\underbrace{\sum_{\mathbf{v}\in V} \lambda_{\mathbf{v}} \equiv 1}_{\text{partition of unity}} \qquad \underbrace{\sum_{\mathbf{v}\in V} \mathbf{v}\lambda_{\mathbf{v}}(\mathbf{x}) = \mathbf{x}}_{\text{linear precision}} \qquad \underbrace{\lambda_{\mathbf{v}_i}(\mathbf{v}_j) = \delta_{ij}}_{\text{interpolation}}$$

traditional FEM



family of **GBC** reference elements



Anatomy of an error estimate

In the case of functions $\lambda_{\mathbf{v}}$ associated to vertices of a *polygonal* mesh, we have:

$$\underbrace{\left\| u - \sum_{\mathbf{v}} u(\mathbf{v})\lambda_{\mathbf{v}} \right\|_{H^1(P)}}_{\substack{\text{approximation error} \\ \text{in value and derivative}}} \leq \underbrace{\text{diam}(P) \quad C_{\textit{proj-linear}} \quad C_{\textit{proj-}\lambda}}_{\text{constants}} \quad \underbrace{|u|_{H^2(P)}}_{\substack{\text{2nd order} \\ \text{oscillation} \\ \text{in } u}}$$

$\text{diam}(P)$	=	diameter of polygon P.
$C_{\textit{proj-linear}}$	$\approx$	operator norm of projection $u \longmapsto$ linear polynomials on P (from Bramble-Hilbert Lemma)
$C_{\textit{proj-}\lambda}$	$\approx$	operator norm of projection $u \longmapsto \sum_{\mathbf{v}} u(\mathbf{v})\lambda_{\mathbf{v}}$

Key relationship between geometry and error estimates

$C_{\textit{proj-}\lambda} = 1 + C_S(1 + \Lambda)$ where C_S is a Sobolev embedding constant and

$$\Lambda := \sup_{\mathbf{x} \in P} \sum_{\mathbf{v} \in V} |\nabla \lambda_{\mathbf{v}}(\mathbf{x})|$$

The triangular case

$$\Lambda := \sup_{\mathbf{x}\in P} \sum_{\mathbf{v}\in V} |\nabla \lambda_{\mathbf{v}}(\mathbf{x})|$$

If P is a triangle, Λ can be large when P has a large interior angle.

$\rightarrow$ This is often called the **maximum angle condition** for finite elements.

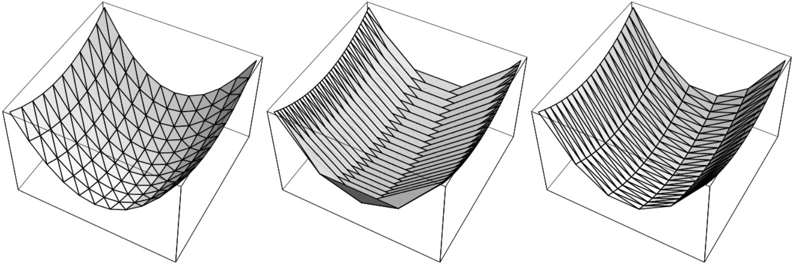
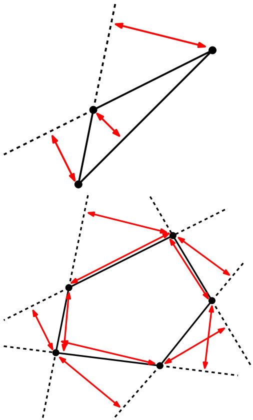


Figure from: SHEWCHUK *What is a good linear element?* Int'l Meshing Roundtable, 2002.

BABUŠKA, AZIZ *On the angle condition in the finite element method*, SIAM J. Num. An., 1976.

JAMET *Estimations d'erreur pour des éléments finis droits presque dégénérés*, ESAIM:M2AN, 1976.

Motivation



Observe that on triangles of fixed diameter:

$$|\nabla\lambda_{\mathbf{v}}| \text{ large} \iff \text{interior angle at } \mathbf{v} \text{ is large}$$
$$\iff \text{the altitude "at } \mathbf{v}\text{" is small}$$

For Wachspress coordinates, we generalize to polygons:

$$|\nabla\lambda_{\mathbf{v}}| \text{ large} \iff \text{the "altitude" at } \mathbf{v} \text{ is small}$$

and then to **simple** polytopes.

(A simple *d*-dimensional polytope has exactly *d* faces at each vertex)

Given a simple convex *d*-dimensional polytope *P*, let

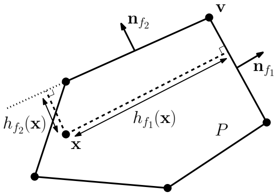
$$h_* := \text{minimum distance from a vertex to a hyper-plane of a non-incident face.}$$

Then $$\sup_{\mathbf{x}\in P}\sum_{\mathbf{v}\in V}|\nabla\lambda_{\mathbf{v}}(\mathbf{x})| =: \Lambda \text{ is large} \iff h_* \text{ is small}$$

Upper bound for simple convex polytopes

Theorem [Floater, G., Sukumar]

Let P be a simple convex polytope in $\mathbb{R}^d$ and let $\lambda_{\mathbf{v}}$ be **generalized Wachspress coordinates**. Then $\Lambda \leq \dfrac{2d}{h_*}$ where $h_* = \min_f \min_{\mathbf{v} \notin f} \operatorname{dist}(\mathbf{v}, f)$



$$\mathbf{p}_f(\mathbf{x}) := \frac{\mathbf{n}_f}{h_f(\mathbf{x})} =$$ normal to face f, scaled by the reciprocal of the distance from $\mathbf{x}$ to f

$$w_{\mathbf{v}}(\mathbf{x}) := \det(\mathbf{p}_{f_1}(\mathbf{x}), \cdots, \mathbf{p}_{f_d}(\mathbf{x}))$$

= volume formed by the d vectors $\{\mathbf{p}_{f_i}(\mathbf{x})\}$ for the d faces incident to $\mathbf{v}$

The **generalized Wachspress coordinates** are defined by

$$\lambda_{\mathbf{v}}(\mathbf{x}) := \frac{w_{\mathbf{v}}(\mathbf{x})}{\sum_{\mathbf{u}} w_{\mathbf{u}}(\mathbf{x})}$$

Proof sketch for upper bound

To prove: $\sup_{\mathbf{x}} \sum_{\mathbf{v}} |\nabla \lambda_{\mathbf{v}}(\mathbf{x})| =: \Lambda \leq \frac{2d}{h_*}$ where $h_* := \min_f \min_{\mathbf{v} \notin f} h_f(\mathbf{v})$.

1. Bound $|\nabla \lambda_{\mathbf{v}}|$ by summations over faces incident and not incident to $\mathbf{v}$.

$$|\nabla \lambda_{\mathbf{v}}| \leq \lambda_{\mathbf{v}} \sum_{f \in F_{\mathbf{v}}} \frac{1}{h_f} \left(1 - \sum_{\mathbf{u} \in f} \lambda_{\mathbf{u}} \right) + \lambda_{\mathbf{v}} \sum_{f \notin F_{\mathbf{v}}} \frac{1}{h_f} \left(\sum_{\mathbf{u} \in f} \lambda_{\mathbf{u}} \right)$$

2. Summing over $\mathbf{v}$ gives a constant bound.

$$\sum_{\mathbf{v}} |\nabla \lambda_{\mathbf{v}}| \leq 2 \sum_{f \in F} \frac{1}{h_f} \left(1 - \sum_{\mathbf{u} \in f} \lambda_{\mathbf{u}} \right) \left(\sum_{\mathbf{u} \in f} \lambda_{\mathbf{u}} \right)$$

3. Write $h_f(\mathbf{x})$ using $\lambda_{\mathbf{v}}$ (possible since h_f is linear) and derive the bound.

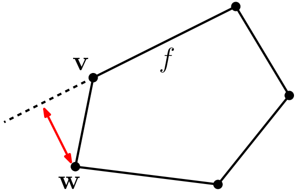
$$\Lambda \leq 2 \sum_{f \in F} \left(\sum_{\mathbf{u} \in f} \lambda_{\mathbf{u}} \right) \frac{1}{h_*} = 2 \sum_{\mathbf{v} \in V} |\{ f \ : \ f \ni \mathbf{v} \}| \lambda_{\mathbf{v}} \frac{1}{h_*} = \frac{2d}{h_*}$$

Lower bound for polytopes

Theorem [Floater, G., Sukumar]

Let P be a simple convex polytope in $\mathbb{R}^d$ and let $\lambda_{\mathbf{v}}$ be **any** generalized barycentric coordinates on P. Then

$$\frac{1}{h_*} \leq \Lambda$$



Proof sketch:

1. Show that $h_* = h_f(\mathbf{w})$, for some particular face f of P and vertex $\mathbf{w} \notin f$.
2. Let $\mathbf{v}$ be the vertex in f closest to $\mathbf{w}$. Show that

$$|\nabla \lambda_{\mathbf{w}}(\mathbf{v})| = \frac{1}{h_f(\mathbf{w})}$$

3. Conclude the result, since

$$\Lambda \geq |\nabla \lambda_{\mathbf{w}}(\mathbf{v})| = \frac{1}{h_f(\mathbf{w})} = \frac{1}{h_*}$$

Upper and lower bounds on polytopes

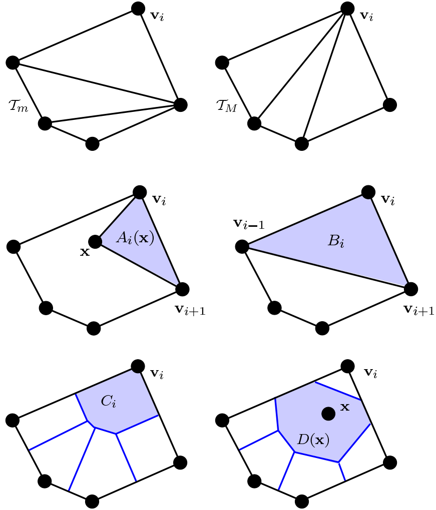
For a polytope $P \subset \mathbb{R}^d$, define $\Lambda := \sup_{\mathbf{x}\in P} \sum_{\mathbf{v}} |\nabla \lambda_{\mathbf{v}}(\mathbf{x})|$.

simple convex polytope in $\mathbb{R}^d$	$\dfrac{1}{h_*}$	$\leq$	Λ	$\leq$	$\dfrac{2d}{h_*}$
d-simplex in $\mathbb{R}^d$	$\dfrac{1}{h_*}$	$\leq$	Λ	$\leq$	$\dfrac{d+1}{h_*}$
hyper-rectangle in $\mathbb{R}^d$	$\dfrac{1}{h_*}$	$\leq$	Λ	$\leq$	$\dfrac{d+\sqrt{d}}{h_*}$
regular k-gon in $\mathbb{R}^2$	$\dfrac{2(1+\cos(\pi/k))}{h_*}$	$\leq$	Λ	$\leq$	$\dfrac{4}{h_*}$

Note that $\lim_{k\to\infty} 2(1+\cos(\pi/k)) = 4$, so the bound is **sharp** in $\mathbb{R}^2$.

Floater, G, Sukumar *Gradient bounds for Wachspress coordinates on polytopes*, SIAM J. Numerical Analysis, 2014.

Many other barycentric coordinates are available . . .



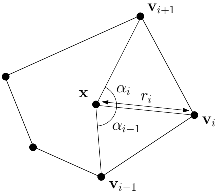
- Triangulation
  ⇒ FLOATER, HORMANN, KÓS, *A general construction of barycentric coordinates over convex polygons*, 2006

  $$0 \leq \lambda_i^{\mathcal{T}_m}(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq \lambda_i^{\mathcal{T}_M}(\mathbf{x}) \leq 1$$

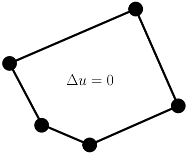
- Wachspress
  ⇒ WACHSPRESS, *A Rational Finite Element Basis*, 1975.
  ⇒ WARREN, *Barycentric coordinates for convex polytopes*, 1996.

- Sibson / Laplace
  ⇒ SIBSON, *A vector identity for the Dirichlet tessellation*, 1980.
  ⇒ HIYOSHI, SUGIHARA, *Voronoi-based interpolation with higher continuity*, 2000.

Many other barycentric coordinates are available . . .



- Mean value
  $\Rightarrow$ Floater, *Mean value coordinates*, 2003.
  $\Rightarrow$ Floater, Kós, Reimers, *Mean value coordinates in 3D*, 2005.



- Harmonic
  $\Rightarrow$ Warren, Schaefer, Hirani, Desbrun, *Barycentric coordinates for convex sets*, 2007.
  $\Rightarrow$ Christiansen, *A construction of spaces of compatible differential forms on cellular complexes*, 2008.

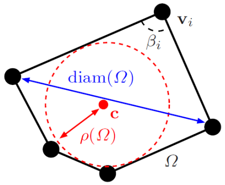
Many more papers could be cited (maximum entropy coordinates, moving least squares coordinates, surface barycentric coordinates, etc...)

Geometric criteria for convergence estimates

For other types of coordinates (on polygons only) we consider additional geometric measures.

Let $\rho(\Omega)$ denote the radius of the largest inscribed circle.
The **aspect ratio** γ is defined by

$$\gamma = \frac{\text{diam}(\Omega)}{\rho(\Omega)} \in (2, \infty)$$



Three possible geometric conditions on a polygonal mesh:

G1.	BOUNDED ASPECT RATIO:	$\exists\ \gamma^* < \infty$	such that	$\gamma < \gamma^*$
G2.	MINIMUM EDGE LENGTH:	$\exists\ d_* > 0$	such that	$\lvert\mathbf{v}_i - \mathbf{v}_{i-1}\rvert > d_*$
G3.	MAXIMUM INTERIOR ANGLE:	$\exists\ \beta^* < \pi$	such that	$\beta_i < \beta^*$

Polygonal Finite Element Optimal Convergence

Theorem [G, Rand, Bajaj]

In the table, any necessary geometric criteria to achieve the ***a priori* linear error estimate** are denoted by N. The set of geometric criteria denoted by S in each row ***taken together*** are sufficient to guarantee the estimate.

		G1 (aspect ratio)	**G2** (min edge length)	**G3** (max interior angle)
Triangulated	λ^{Tri}	-	-	S,N
Wachspress	λ^{Wach}	S	S	S,N
Sibson	λ^{Sibs}	S	S	-
Mean Value	λ^{MV}	S	S	-
Harmonic	λ^{Har}	S	-	-

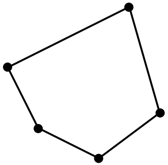
G, RAND, BAJAJ *Error Estimates for Generalized Barycentric Interpolation* Advances in Computational Mathematics, 37:3, 417-439, 2012

RAND, G, BAJAJ *Interpolation Error Estimates for Mean Value Coordinates*, Advances in Computational Mathematics, 39:2, 327-347, 2013.

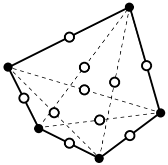
Outline

From linear to quadratic elements

A naïve quadratic element is formed by products of linear **GBCs**:



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\}$$



Why is this naïve?

- For a k-gon, this construction gives $k + \binom{k}{2}$ basis functions $\lambda_a\lambda_b$
- The space of quadratic polynomials is only dimension 6: $\{1, x, y, xy, x^2, y^2\}$
- Conforming to a linear function on the boundary requires 2 degrees of freedom per edge $\Rightarrow$ *only $2k$ functions needed!*

Problem Statement

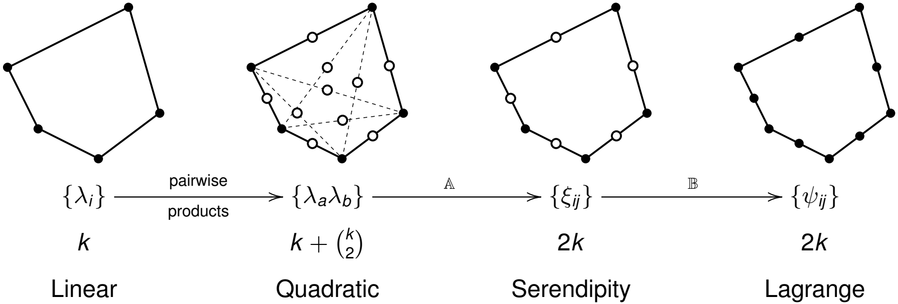
Construct $2k$ basis functions associated to the vertices and edge midpoints of an arbitrary k-gon such that a quadratic convergence estimate is obtained.

Polygonal Quadratic Serendipity Elements

We define matrices $\mathbb{A}$ and $\mathbb{B}$ to reduce the naïve quadratic basis.

filled dot = **Interpolatory** domain point
= all functions in the set evaluate to 0
except the associated function which evaluates to 1

open dot = non-interpolatory domain point
= partition of unity satisfied, but not a nodal basis



From quadratic to serendipity

The bases are ordered as follows:

ξ_{ii}	and	$\lambda_a\lambda_a$	= basis functions associated with vertices
$\xi_{i(i+1)}$	and	$\lambda_a\lambda_{a+1}$	= basis functions associated with edge midpoints
		$\lambda_a\lambda_b$	= basis functions associated with interior diagonals, i.e. $b \notin \{a-1, a, a+1\}$

Serendipity basis functions ξ_{ij} are a linear combination of pairwise products $\lambda_a\lambda_b$:

$$\begin{bmatrix} \xi_{ii} \\ \vdots \\ \xi_{i(i+1)} \end{bmatrix} = \mathbb{A} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix} = \begin{bmatrix} c_{11}^{11} & \cdots & c_{ab}^{11} & \cdots & c_{(n-2)n}^{11} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{ij} & \cdots & c_{ab}^{ij} & \cdots & c_{(n-2)n}^{ij} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{n(n+1)} & \cdots & c_{ab}^{n(n+1)} & \cdots & c_{(n-2)n}^{n(n+1)} \end{bmatrix} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix}$$

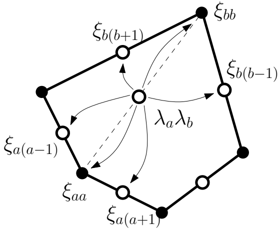
From quadratic to serendipity

We ***require*** the serendipity basis to have quadratic approximation power:

Constant precision: $1 = \sum_i \xi_{ii} + 2\xi_{i(i+1)}$

Linear precision: $\mathbf{x} = \sum_i \mathbf{v}_i \xi_{ii} + 2\mathbf{v}_{i(i+1)} \xi_{i(i+1)}$

Quadratic precision: $\mathbf{x}\mathbf{x}^T = \sum_i \mathbf{v}_i \mathbf{v}_i^T \xi_{ii} + (\mathbf{v}_i \mathbf{v}_{i+1}^T + \mathbf{v}_{i+1} \mathbf{v}_i^T) \xi_{i(i+1)}$



Theorem [Rand, G, Bajaj]

Constants $\{c_{ij}^{ab}\}$ exist for **any** convex polygon such that the resulting basis $\{\xi_{ij}\}$ satisfies constant, linear, and quadratic precision requirements.

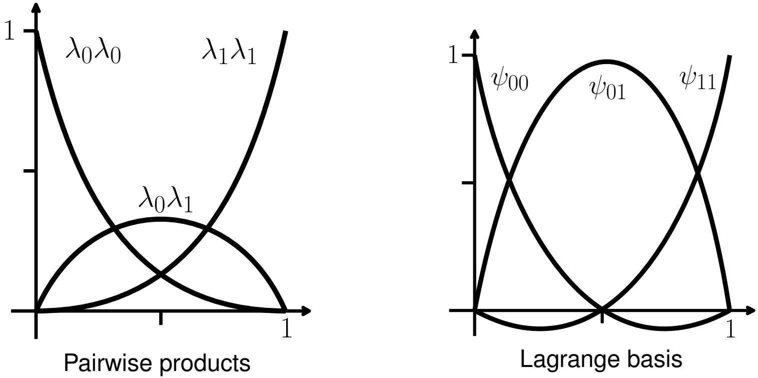
Proof: We produce a coefficient matrix $\mathbb{A}$ with the structure

$$\mathbb{A} := \left[\ \mathbb{I} \ \middle| \ \mathbb{A}' \ \right]$$

where $\mathbb{A}'$ has only six non-zero entries per column and show that the resulting functions satisfy the six precision equations.

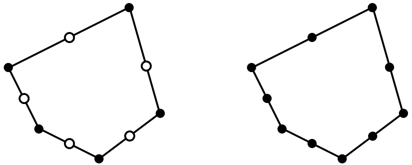
Pairwise products vs. Lagrange basis

Even in 1D, pairwise products of barycentric functions do not form a Lagrange basis at interior degrees of freedom:



Translation between these two bases is straightforward and generalizes to the higher dimensional case.

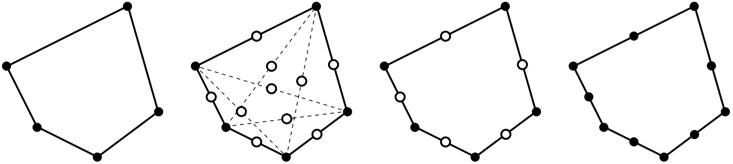
From serendipity to Lagrange



$$\{\xi_{ij}\} \xrightarrow{\quad\quad\mathbb{B}\quad\quad} \{\psi_{ij}\}$$

$$[\psi_{ij}] = \begin{bmatrix} \psi_{11} \\ \psi_{22} \\ \vdots \\ \psi_{nn} \\ \psi_{12} \\ \psi_{23} \\ \vdots \\ \psi_{n1} \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 1 & & & & & -1 & & \cdots & & -1 \\ & 1 & & & & -1 & -1 & \cdots & & \\ & & \ddots & & & & \ddots & \ddots & & \\ & & & \ddots & & & & \ddots & \ddots & \\ & & & & 1 & & & & -1 & -1 \\ \hline & & & & & 4 & & & & \\ & & & & & & 4 & & & \\ & & 0 & & & & & \ddots & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 4 \end{array}\right] \begin{bmatrix} \xi_{11} \\ \xi_{22} \\ \vdots \\ \xi_{nn} \\ \xi_{12} \\ \xi_{23} \\ \vdots \\ \xi_{n1} \end{bmatrix} = \mathbb{B}[\xi_{ij}].$$

Serendipity Theorem



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\} \xrightarrow{\mathbb{A}} \{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

Theorem [Rand, G, Bajaj]

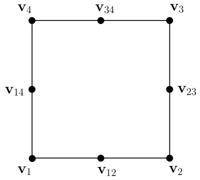
Given bounds on polygonal geometric quality:

- $||\mathbb{A}||$ is uniformly bounded,
- $||\mathbb{B}||$ is uniformly bounded, and
- span$\{\psi_{ij}\} \supset \mathcal{P}_2(\mathbb{R}^2)$ = quadratic polynomials in x and y

We obtain the quadratic *a priori* error estimate: $||u - u_h||_{H^1(\Omega)} \leq C\, h^{\mathbf{2}}\, |u|_{H^3(\Omega)}$

RAND, G, BAJAJ *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Math. Comp., 2011

Special case of a square



Bilinear functions are barycentric coordinates:

$$\lambda_1 = (1-x)(1-y)$$
$$\lambda_2 = x(1-y)$$
$$\lambda_3 = xy$$
$$\lambda_4 = (1-x)y$$

Compute $[\xi_{ij}] := \left[\ \mathbb{I}\ \middle|\ \mathbb{A}'\ \right][\lambda_a\lambda_b]$

$$\begin{bmatrix} \xi_{11} \\ \xi_{22} \\ \xi_{33} \\ \xi_{44} \\ \xi_{12} \\ \xi_{23} \\ \xi_{34} \\ \xi_{14} \end{bmatrix} = \begin{bmatrix} 1 & & 0 & -1 & 0 \\ 0 & & 0 & 0 & -1 \\ 0 & & 0 & -1 & 0 \\ 0 & \cdots & 0 & 0 & -1 \\ 0 & \cdots & 0 & 1/2 & 1/2 \\ 0 & & 0 & 1/2 & 1/2 \\ 0 & & 0 & 1/2 & 1/2 \\ 0 & & 1 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} \lambda_1\lambda_1 \\ \lambda_2\lambda_2 \\ \lambda_3\lambda_3 \\ \lambda_4\lambda_4 \\ \lambda_1\lambda_2 \\ \lambda_2\lambda_3 \\ \lambda_3\lambda_4 \\ \lambda_1\lambda_4 \end{bmatrix} = \begin{bmatrix} (1-x)(1-y)(1-x-y) \\ x(1-y)(x-y) \\ xy(-1+x+y) \\ (1-x)y(y-x) \\ (1-x)x(1-y) \\ x(1-y)y \\ (1-x)xy \\ (1-x)(1-y)y \end{bmatrix}$$

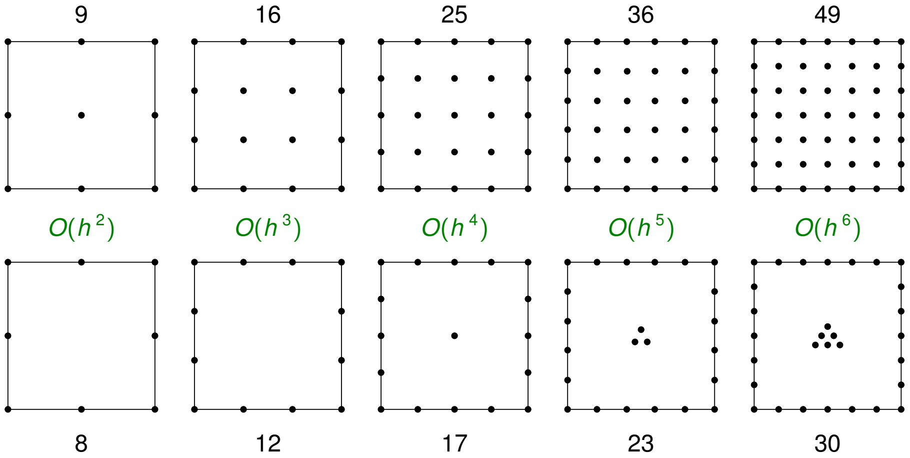
$$\text{span}\left\{\xi_{ii}, \xi_{i(i+1)}\right\} = \text{span}\left\{1, x, y, x^2, y^2, xy, x^2y, xy^2\right\} =: \mathcal{S}_2(I^2)$$

Hence, this provides a computational basis for the serendipity space $\mathcal{S}_2(I^2)$ defined in

ARNOLD, AWANOU *The serendipity family of finite elements*, Found. Comp. Math, 2011.

Outline

Tensor-product vs. serendipity degrees of freedom

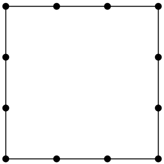


Elements in the same column exhibit the same rate of convergence:

$$\underbrace{||u-u_h||_{H^1(\Omega)}}_{\text{approximation error}} \le \underbrace{C\,h^{\mathbf{r}}\,|u|_{H^{\mathbf{r+1}}(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^{\mathbf{r+1}}(\Omega).$$

This 'serendipitous' observation led to the namesake of the elements.

Characterization of requisite monomials



total degree at most 3 (dim=10)

$$\{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, x^3y, xy^3, x^2y^2, x^3y^2, x^2y^3, x^3y^3\}$$

superlinear degree at most 3 (dim=12)

at most degree 3 in each variable (dim=16)

The **superlinear** degree of a polynomial ignores linearly-appearing variables.

Example: $\text{sldeg}(xy^3) = 3$, even though $\deg(xy^3) = 4$

Definition: $\text{sldeg}(x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}) := \left(\sum_{i=1}^{n} e_i\right) - \#\{e_i \ : \ e_i = 1\}$

Observe that the set $S_r = \left\{\alpha \in \mathbb{N}_0^n : \text{sldeg}(x^\alpha) \leq r\right\}$ has a partial ordering

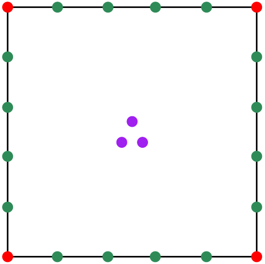
and is thus a **lower set**, meaning $\boxed{\alpha \in S_r,\ \mu \leq \alpha \implies \mu \in S_r}$

ARNOLD, AWANOU *The serendipity family of finite elements*, FoCM, 2011.

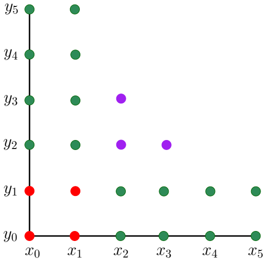
Partitioning and reordering the multi-indices

Theorem (Dyn and Floater, 2013) Fix a lower set $L \subset \mathbb{N}_0^n$ and points $z_\alpha \in \mathbb{R}^n$ for all $\alpha \in L$. For any sufficiently smooth n-variate real function f, there is a *unique polynomial p* in $\text{span}\{x^\alpha : \alpha \in L\}$ that *interpolates f at the points z_α*, with partial derivative interpolation for repeated z_α values.

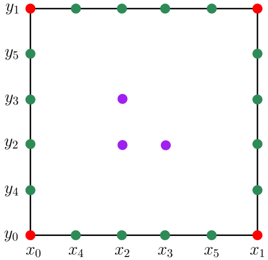
We apply the above theorem in the context of the lower set S_r:



The order 5 serendipity element, with degrees of freedom color-coded by dimensionality.



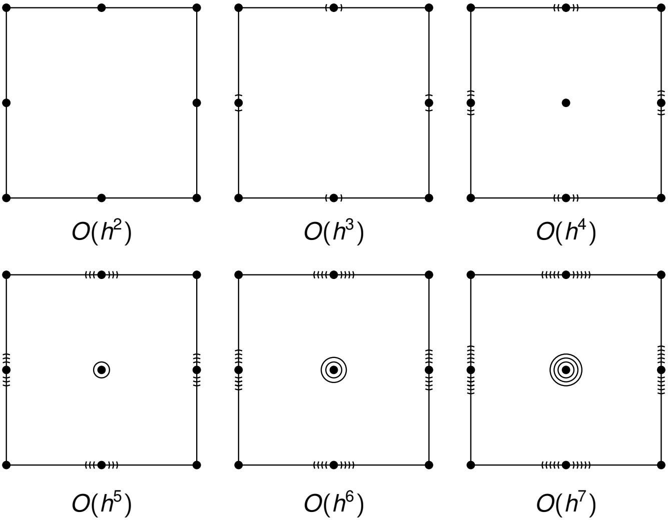
The lower set S_5, with equivalent color coding.



The lower set S_5, with domain points z_α reordered.

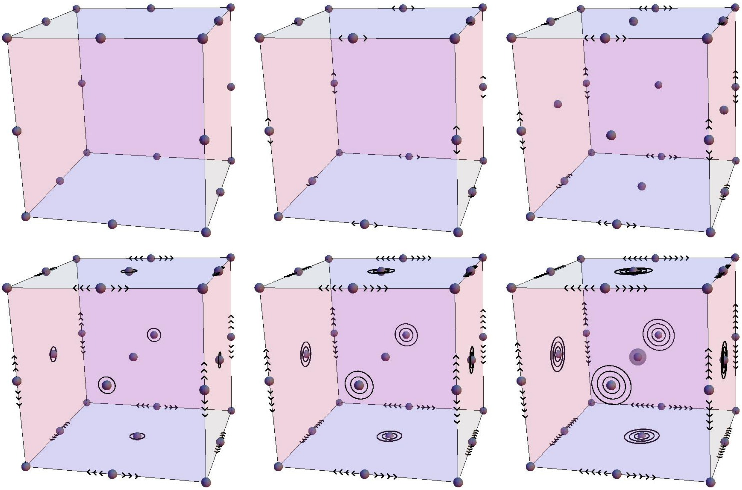
2D symmetric serendipity elements

We generate symmetric $O(h^r)$ serendipity elements on $[-1,1]^2$ by setting $x_j = y_j = 0$ for $2 \leq j \leq r$. This approach interpolates partial derivative information at the edge midpoints and square center.



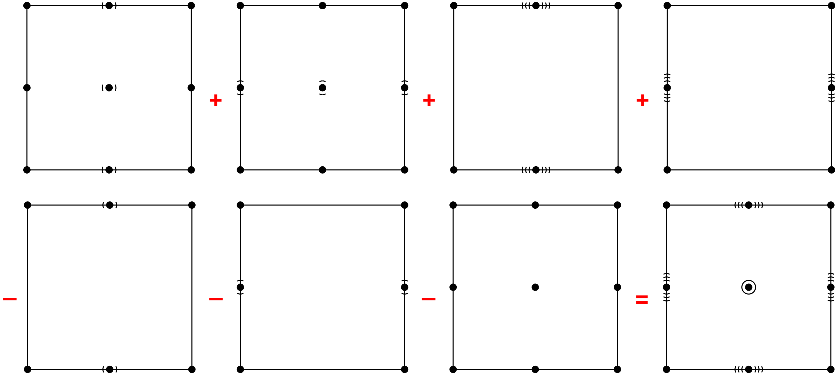
3D symmetric serendipity elements

The approach applies without modification to any dimension $n \geq 1$.



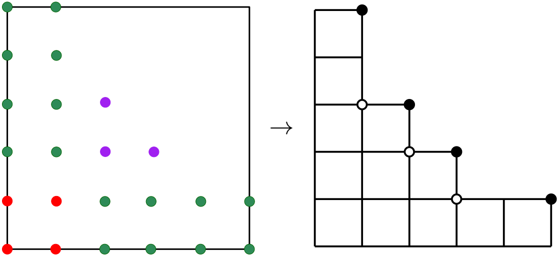
Linear combinations of tensor-products

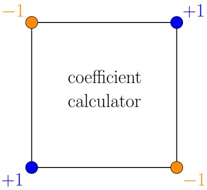
The Dyn-Floater theorem provides an explicit formula for computing the desired nodal interpolation scheme as a linear combination of standard tensor-product functions.



Tensor product structure

The Dyn-Floater interpolation scheme is expressed in terms of tensor product interpolation over ‘maximal blocks’ in the set using an inclusion-exclusion formula.





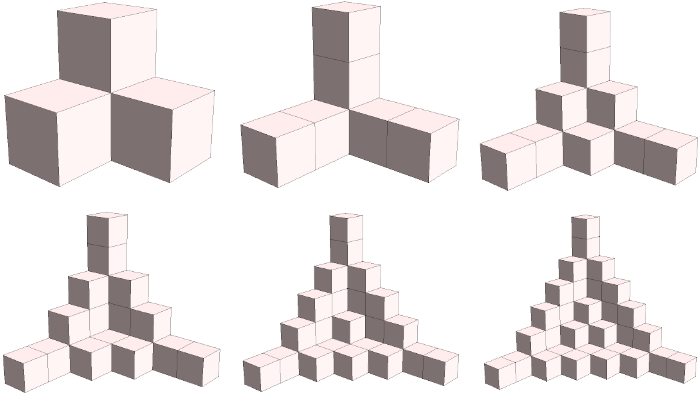
Put differently, the linear combination is the sum over *all* blocks within the lower set with coefficients determined as follows:

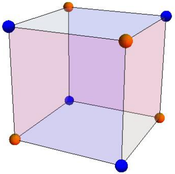
→ Place the coefficient calculator at the extremal block corner.
→ Add up all values appearing in the lower set.
→ The coefficient for the block is the value of the sum.

Hence: black dots → +1; white dots → -1; others → 0.

3D coefficient computation

Lower sets corresponding to S_2 through S_7 in 3 variables.





Decomposition into a linear combination of tensor product interpolants works the same as in 2D, using the 3D coefficient calculator at left. (Blue $\rightarrow$ +1; Orange $\rightarrow$ -1).

FLOATER, G, *Nodal bases for the serendipity family of finite elements*, Submitted, 2014. Available as arXiv:1404.6275

Outline

Matlab code for Wachspress coordinates on polygons

Input: The vertices $v_1, \ldots, v_n$ of a polygon and a point x

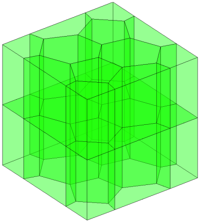
Output: Wachspress functions λ_i and their gradients $\nabla\lambda_i$

```
function [phi dphi] = wachspress2d(v,x)
n = size(v,1);
w = zeros(n,1);
R = zeros(n,2);
phi = zeros(n,1);
dphi = zeros(n,2);

un = getNormals(v); % computes the outward unit normal to each edge

p = zeros(n,2);
for i = 1:n
   h = dot(v(i,:)  - x,un(i,:));
   p(i,:)  = un(i,:)  / h;
end

for i = 1:n
   im1 = mod(i-2,n) + 1;
   w(i) = det([p(im1,:);p(i,:)]);
   R(i,:)  = p(im1,:)  + p(i,:);
end

wsum = sum(w);
phi = w/wsum;

phiR = phi' * R;
for k = 1:2
   dphi(:,k) = phi .* (R(:,k) - phiR(:,k));
end
```

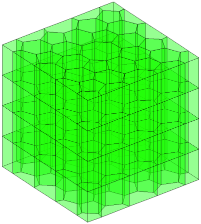
Matlab code for polygons *and polyhedra* (simple or non-simple) included in appendix of

FLOATER, G, SUKUMAR *Gradient bounds for Wachspress coordinates on polytopes*, SIAM J. Numerical Analysis, 2014.

Numerical results



h= 0.7071



h= 0.3955

→ We fix a sequence of polyhedral meshes where h denotes the maximum diameter of a mesh element.

→ $\exists\ \gamma > 0$ such that if any element from any mesh in the sequence is scaled to have diameter 1, the computed value of h_* will be $\geq \gamma$.

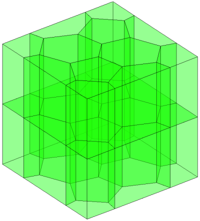
→ We solve the weak form of the Poisson problem:

$$\int_\Omega \nabla u \cdot \nabla w \, d\mathbf{x} = \int_\Omega f \; w \, d\mathbf{x}, \quad \forall w \in H_0^1(\Omega),$$

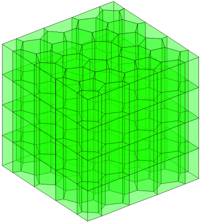
where $f(\mathbf{x})$ is defined so that the exact solution is $u(\mathbf{x}) = xyz(1-x)(1-y)(1-z)$.

→ Using Wachspress coordinates $\lambda_\mathbf{v}$, the local stiffness matrix has entries of the form $\displaystyle\int_P \nabla\lambda_\mathbf{v} \cdot \nabla\lambda_\mathbf{w} \; d\mathbf{x}$, which we integrate by tetrahedralizing P and using a second-order accurate quadrature rule (4 points per tetrahedron).

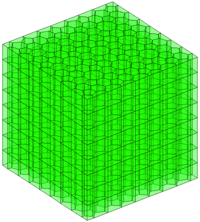
Numerical results



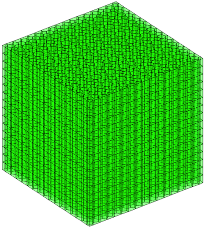
h= 0.7071



h= 0.3955



h= 0.1977

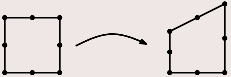


h= 0.0989

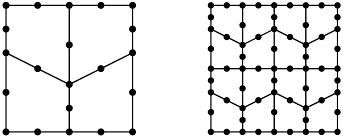
As expected, we observe optimal convergence convergence rates: quadratic in L^2 norm and linear in H^1 semi-norm.

Mesh	# of nodes	h	$\frac{\|\|u-u^h\|\|_{0,P}}{\|\|u\|\|_{0,P}}$	Rate	$\frac{\|u-u^h\|_{1,P}}{\|u\|_{1,P}}$	Rate
a	78	0.7071	2.0×10^{-1}	–	4.1×10^{-1}	–
b	380	0.3955	5.4×10^{-2}	2.28	2.1×10^{-1}	1.14
c	2340	0.1977	1.4×10^{-2}	1.96	1.1×10^{-1}	0.97
d	16388	0.0989	3.5×10^{-3}	1.99	5.4×10^{-2}	0.99
e	122628	0.0494	8.8×10^{-4}	**2.00**	2.7×10^{-2}	**0.99**

Numerical evidence for non-affine image of a square

Instead of mapping  ,
use quadratic serendipity **GBC** interpolation with mean value coordinates:

$$u_h = I_q u := \sum_{i=1}^{n} u(\mathbf{v}_i)\psi_{ii} + u\left(\frac{\mathbf{v}_i + \mathbf{v}_{i+1}}{2}\right)\psi_{i(i+1)}$$



$n = 2$ $n = 4$

Non-affine bilinear mapping

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	5.0e-2		6.2e-1	
4	6.7e-3	2.9	1.8e-1	1.8
8	9.7e-4	2.8	5.9e-2	1.6
16	1.6e-4	2.6	2.3e-2	1.4
32	3.3e-5	2.3	1.0e-2	1.2
64	7.4e-6	2.1	4.96e-3	1.1

ARNOLD, BOFFI, FALK, Math. Comp., 2002

Quadratic serendipity **GBC** method

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	2.34e-3		2.22e-2	
4	3.03e-4	2.95	6.10e-3	1.87
8	3.87e-5	2.97	1.59e-3	1.94
16	4.88e-6	2.99	4.04e-4	1.97
32	6.13e-7	3.00	1.02e-4	1.99
64	7.67e-8	3.00	2.56e-5	1.99
128	9.59e-9	3.00	6.40e-6	2.00
256	1.20e-9	3.00	1.64e-6	1.96

Future Directions

- Expand serendipity results from cubes to polyhedra.
- Incorporate elements into finite element software packages.
- Analyze speed vs. accuracy trade-offs.

	1	2	3	4	5	6	7	$r \geq 2n$
$n = 2$								
$\dim \mathcal{Q}_r$	4	9	16	25	36	49	64	$r^2 + 2r + 1$
$\dim \mathcal{S}_r$	4	8	12	17	23	30	38	$\frac{1}{2}(r^2 + 3r + 6)$
$n = 3$								
$\dim \mathcal{Q}_r$	8	27	64	125	216	343	512	$r^3 + 3r^2 + 3r + 1$
$\dim \mathcal{S}_r$	8	20	32	50	74	105	144	$\frac{1}{6}(r^3 + 6r^2 + 29r + 24)$

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Michael Floater	University of Oslo
N. Sukumar	UC Davis

Thanks for the invitation to speak!

Slides and pre-prints: `http://math.arizona.edu/~agillette/`

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Mini-symposium at the 13th US National Congress on Computational Mechanics
San Diego, July 2015
Accepting abstract submissions: deadline in Feburary 2015.