The Serendipity Pyramid Finite Element

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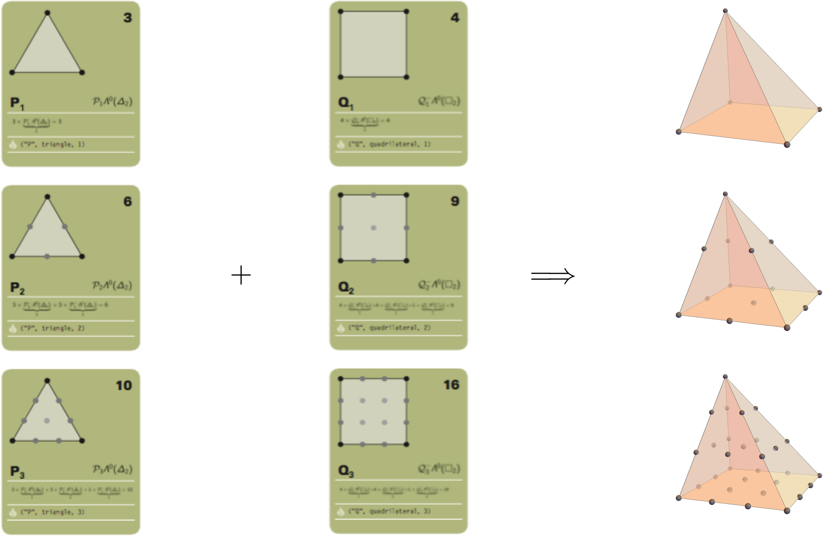
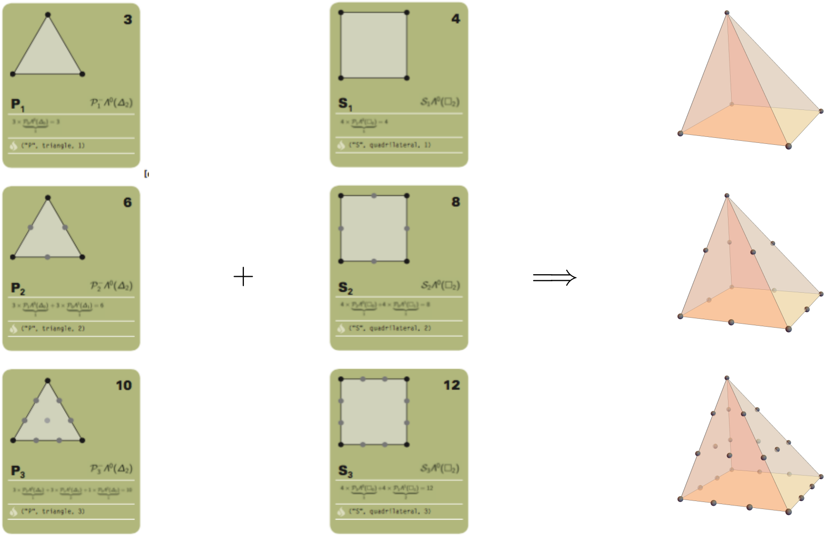
Outline

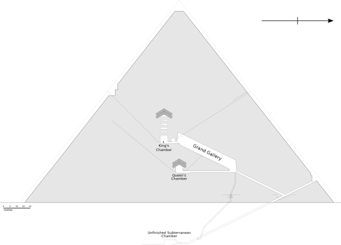
Regular pyramid



Serendipity pyramid



What lies inside the pyramids?

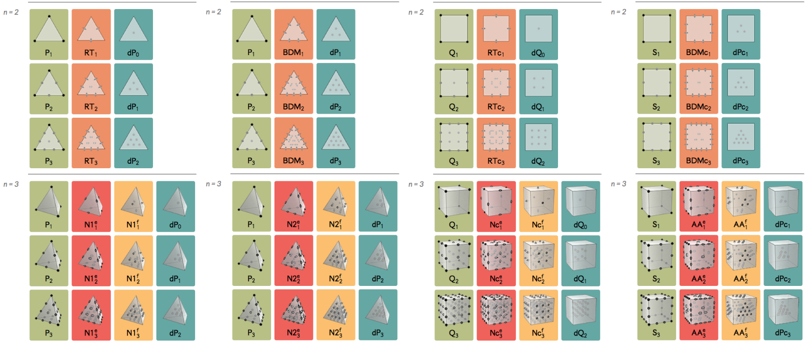


Classical pyramids have mysterious interiors.

Modern pyramids have tiny pyramids inside!

Outline

The periodic table of finite elements *(prepared by Doug Arnold & Anders Logg)*:



- Square-based pyramids provide a link between tet and hex meshing regimes.
- A conforming pyramid element should match △ and □ elements on its faces.
- Serendipity elements offer substantial reduction in local basis size.
- Tensor-product-based pyramids are known; serendipity-based elements are not!

A history of pyramid elements

- First implementations in computational electromagnetics

  BEDROSIAN 1992, ZGAINSKI ET AL 1996, COULOMB ET AL 1997

- Generalizations to higher order elements and exact sequences

  BERGOT, COHEN, AND DURUFLÉ 2010

  NIGAM, PHILLIPS 2012 (two papers)

- Modern implementation and additional analysis

  FUENTES, KEITH, DEMKOWICZ, NAGARAJ 2015 (implementation)

  WITHERDEN, VINCENT 2015 (quadrature)

  CHAN, WARBURTON 2015 (trace inequalities)

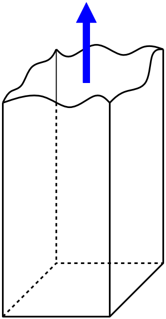
- Serentipity elements and pyramids

  ARNOLD, AWANOU 2011 (serendipity elements on cubes; superlinear degree)

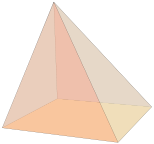
  LIU ET AL 2004 and 2011 (piecewise-defined quadratic serendipity pyramid)

Reference geometry and mappings

We follow the geometry conventions of Nigam and Phillips.



K_∞



$\hat{K}$

The **infinite pyramid** geometry is

$$K_\infty := \{\ (x,y,z) \in \mathbb{R}^3 \cup \infty\ :\ 0 \le x, y \le 1,\ 0 \le z \le \infty\ \}.$$

The **reference pyramid** geometry is

$$\hat{K} := \{\ (\hat{x},\hat{y},\hat{z}) \in \mathbb{R}^3\ :\ 0 \le \hat{x},\hat{y},\hat{z},\ \hat{x} \le 1-\hat{z},\ \hat{y} \le 1-\hat{z}\ \}.$$

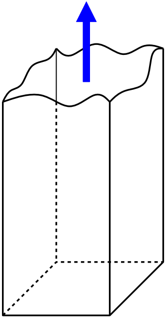
Define a bijective change of coordinates $\phi : K_\infty \to \hat{K}$ by

$$\phi(x,y,z) \quad = \begin{cases} \left(\dfrac{\hat{x}}{1+\hat{z}}, \dfrac{\hat{y}}{1+\hat{z}}, \dfrac{\hat{z}}{1+\hat{z}}\right), & 0 \le z < \infty \\ (0,0,1), & z = \infty \end{cases}$$

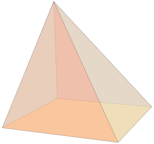
Given $u : \hat{K} \to \mathbb{R}$, the **pullback** of u to K^∞ by ϕ is

$$\phi^* u : K_\infty \to \mathbb{R} \quad \text{where} \quad (\phi^* u)(x,y,z) := u(\phi(x,y,z))$$

Pullback example



K_∞



$\hat{K}$

Consider the (possibly rational) function $u : \hat{K} \to \mathbb{R}$

$$u = \hat{x}^a \, \hat{y}^b \, (1 - \hat{z})^{c-a-b} \quad \text{with } c - a - b \geq 0.$$

The **pullback** of u to K^∞ by ϕ for $0 \leq z < \infty$ is

$$\begin{aligned}(\phi^* u)(x, y, z) &= u(\phi(x, y, z)) \\ &= u\left(\frac{\hat{x}}{1+\hat{z}}, \frac{\hat{y}}{1+\hat{z}}, \frac{\hat{z}}{1+\hat{z}}\right) \\ &= \frac{\hat{x}^a \, \hat{y}^b}{(1+\hat{z})^{a+b}} \left(1 - \frac{\hat{z}}{1+\hat{z}}\right)^{c-a-b} \\ &= \frac{\hat{x}^a \, \hat{y}^b}{(1+\hat{z})^c}\end{aligned}$$

and $(\phi^* u)(x, y, \infty) = u(0, 0, 1) = 0.$

Note that u pulls back to a 'nice' rational function that vanishes as $z \to \infty$

Outline

Shape functions on K_∞ and $\hat{K}$

The **superlinear degree** of a monomial is defined by:

$$\text{sldeg}\left(\prod_{i=1}^{n} x_i^{\alpha_i}\right) := \sum_{\alpha_i \neq 1} \alpha_i.$$

Define **shape functions on K_∞** as follows:

$$\mathcal{Q}_r^{[r,r]} := \bigoplus_{j=0}^{r} \left\{ \frac{x^a y^b}{(1+z)^j} \ : \ 0 \leq a, b \leq j \right\}$$

$$\mathcal{S}_r^{[r,r]} := \bigoplus_{j=0}^{r} \left\{ \frac{x^a y^b}{(1+z)^j} \ : \ 0 \leq a, b \leq j, \ \ \text{sldeg}(x^a y^b) \leq j \right\}$$

Define **shape functions on $\hat{K}$** as those whose pullback is a shape function on K_∞.

$$\phi\left(\mathcal{Q}_r^{[r,r]}\right) := \left\{ u : \hat{K} \to \mathbb{R} \ : \ \phi^* u \in \mathcal{Q}_r^{[r,r]} \right\}.$$

$$\phi\left(\mathcal{S}_r^{[r,r]}\right) := \left\{ u : \hat{K} \to \mathbb{R} \ : \ \phi^* u \in \mathcal{S}_r^{[r,r]} \right\}.$$

Counting the shape functions

$$\dim \mathcal{Q}_r^{[r,r]} = \dim \bigoplus_{j=0}^{r} \left\{ \frac{x^a y^b}{(1+z)^j} \ : \ 0 \le a, b \le j \right\}$$

$$= \sum_{j=0}^{r} \dim \mathcal{Q}_j^{-} \Lambda^0(I^2) \quad \leftarrow \text{2d tensor product, order } j$$

$$= \frac{1}{6}(2r^3 + 9r^2 + 13r + 6).$$

$$\dim \mathcal{S}_r^{[r,r]} = \dim \bigoplus_{j=0}^{r} \left\{ \frac{x^a y^b}{(1+z)^j} \ : \ 0 \le a, b \le j, \ \ \text{sldeg}(x^a y^b) \le j \right\}$$

$$= 1 + \sum_{j=1}^{r} \dim \mathcal{S}_j \Lambda^0(I^2) \quad \leftarrow \text{2d serendipity, order } j$$

$$= \frac{1}{6}(r^3 + 6r^2 + 23r)$$

Since ϕ is an isomorphism,

$$\dim \phi\left(\mathcal{S}_r^{[r,r]}\right) = \dim \mathcal{S}_r^{[r,r]} \qquad \text{and} \qquad \dim \phi\left(\mathcal{S}_r^{[r,r]}\right) = \dim \mathcal{S}_r^{[r,r]}.$$

The 'lowest order bubble function' on K_∞ and $\hat{K}$

Define the **lowest order bubble function** $\mathrm{b} : K_\infty \to \mathbb{R}$ by

$$\mathrm{b}(x, y, z) := \frac{x(1-x)y(1-y)z}{(1+z)^3}$$

b vanishes on ∂K_∞ i.e. on all of $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and as $z \to \infty$.

We can rewrite

$$\mathrm{b}(x, y, z) \quad = \quad \frac{x(1-x)y(1-y)}{(1+z)^2} - \frac{x(1-x)y(1-y)}{(1+z)^3}.$$

The term $\dfrac{x^2y^2}{(1+z)^3}$ appears after expanding numerators, so $\mathrm{b} \in \mathcal{Q}_3^{[3,3]}$.

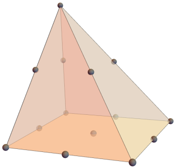
To allow x^2y^2 as a numerator in a serendipity shape function we need $\dfrac{\mathrm{b}(x, y, z)}{(1+z)^2}$,

or a higher exponent in the denominator since $\mathrm{sldeg}(x^2y^2) = 4$.

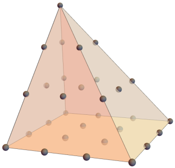
In all, we have $\mathrm{b} \in \mathcal{Q}_r^{[r,r]} \iff r \geq 3$ and $\mathrm{b} \in \mathcal{S}_r^{[r,r]} \iff r \geq 5$.

But where are these mysterious numbers 3 and 5 in the geometry?

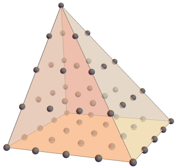
Visualizing the dimension



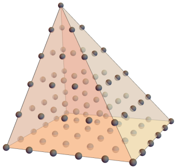
$\mathcal{Y}_2^- \Lambda^0$



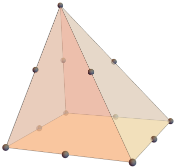
$\boxed{\mathcal{Y}_3^- \Lambda^0}$



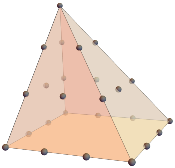
$\mathcal{Y}_4^- \Lambda^0$



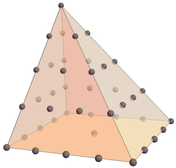
$\mathcal{Y}_5^- \Lambda^0$



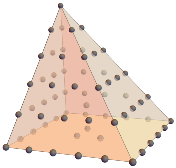
$\mathcal{Y}_2 \Lambda^0$



$\mathcal{Y}_3 \Lambda^0$



$\mathcal{Y}_4 \Lambda^0$

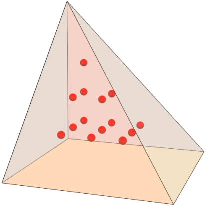


$\boxed{\mathcal{Y}_5 \Lambda^0}$

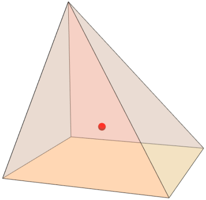
$\rightarrow$ 2D tensor product $\mathcal{Q}_r^-$ and serendipity elements $\mathcal{S}_r$ ‘stack’ to make pyramids

$\rightarrow$ First interior degrees of freedom appear at order 3 for $\mathcal{Q}_r^-$ and order 5 for $\mathcal{S}_r$

Visualizing the interior dimension



$\mathcal{Y}_5^- \Lambda^0$ interior



$\boxed{\mathcal{Y}_5 \Lambda^0}$ interior

$\rightarrow$ Note: $\mathcal{Q}_r^-$ (resp. $\mathcal{S}_r$) pyramids have tiny $\mathcal{Q}_{r-3}^-$ (resp. $\mathcal{S}_{r-5}$) pyramids inside!

Classical degrees of freedom

To each vertex $\mathbf{v}$,	$u \longmapsto u(\mathbf{v})$	
To each edge $\mathbf{e}$,	$u \longmapsto \int_{\mathbf{e}} (\mathrm{tr}_{\mathbf{e}}\, u) \wedge q,$	$q \in P_{\mathbf{e}}$
To each triangular face $\triangle$,	$u \longmapsto \int_{\triangle} (\mathrm{tr}_{\triangle}\, u) \wedge q,$	$q \in P_{\triangle}$
To the parallelogram face $\square$,	$u \longmapsto \int_{\square} (\mathrm{tr}_{\square}\, u) \wedge q,$	$q \in P_{\square}$
To the three-dimensional interior $\mathtt{int}$,	$u \longmapsto \int_{\mathtt{int}} (\mathrm{tr}_{\mathtt{int}}\, u) \wedge q,$	$q \in R_{\mathtt{int}}$

	$P_{\mathbf{v}}$	$P_{\mathbf{e}}$	$P_{\triangle}$	$P_{\square}$	$R_{\mathtt{int}}$
$\mathcal{Y}_r^- \Lambda^0$	$\mathbb{R}$	$\mathcal{P}_{r-2}\Lambda^1(\mathbf{e})$	$\mathcal{P}_{r-3}\Lambda^2(\triangle)$	$\mathcal{Q}_{r-1}^-\Lambda^2(\square)$	$\phi\left(\mathrm{b}\cdot\mathcal{Q}_{r-3}^{[r-3,r-3]}\right)\Lambda^3(\mathtt{int})$
$\mathcal{Y}_r \Lambda^0$	$\mathbb{R}$	$\mathcal{P}_{r-1}^-\Lambda^1(\mathbf{e})$	$\mathcal{P}_{r-2}^-\Lambda^2(\triangle)$	$\mathcal{P}_{r-4}\Lambda^2(\square)$	$\phi\left(\mathrm{b}\cdot\mathcal{S}_{r-5}^{[r-5,r-5]}\right)\Lambda^3(\mathtt{int})$

where $\phi\left(\mathrm{b}\cdot\mathcal{Q}_{r-3}^{[r-3,r-3]}\right)\Lambda^3(\mathtt{int}) := \mathrm{span}\left\{ u\, dV \;:\; \phi^* u = \mathrm{b} q \text{ with } q \in \mathcal{Q}_{r-3}^{[r-3,r-3]} \right\}$

Counting the degrees of freedom

$$\begin{aligned}
\dim \mathcal{Y}_r^- \Lambda^0 &= 5 + 8\left|\mathcal{P}_{r-2}\Lambda^1(\mathbf{e})\right| + 4\left|\mathcal{P}_{r-3}\Lambda^2(\triangle)\right| + \left|\mathcal{Q}_{r-1}^-\Lambda^2(\square)\right| + \left|\mathcal{Q}_{r-3}^{[r-3,r-3]}\right| \\
&= 5 + 8(r-1) + 2(r-2)(r-1) + (r-1)^2 + \frac{(2r-3)(r-2)(r-1)}{6} \\
&= \frac{1}{6}(2r^3 + 9r^2 + 13r + 6) \\
&= \dim \mathcal{Q}_r^{[r,r]}
\end{aligned}$$

$$\begin{aligned}
\dim \mathcal{Y}_r \Lambda^0 &= 5 + 8\left|\mathcal{P}_{r-1}^-\Lambda^1(\mathbf{e})\right| + 4\left|\mathcal{P}_{r-2}^-\Lambda^2(\triangle)\right| + \left|\mathcal{P}_{r-4}\Lambda^2(\square)\right| + \left|\mathcal{S}_{r-5}^{[r-5,r-5]}\right| \\
&= 5 + 8(r-1) + 2(r-2)(r-1) + \frac{(r-3)(r-2)}{2} + \frac{(r-4)(r-3)(r-2)}{6} \\
&= \frac{1}{6}(r^3 + 6r^2 + 23r) \\
&= \dim \mathcal{S}_r^{[r,r]}
\end{aligned}$$

Unisolvence and polynomial reproduction

Theorem [G., 2015]

The degrees of freedom for $\mathcal{Y}_r^- \Lambda^0$ are **unisolvent** for $\phi\left(\mathcal{Q}_r^{[r,r]}\right)$.

The degrees of freedom for $\mathcal{Y}_r \Lambda^0$ are **unisolvent** for $\phi\left(\mathcal{S}_r^{[r,r]}\right)$.

Proof idea: Appeal to known elements for unisolvence on boundary;
use definition of shape functions for interior.

Theorem [G., 2015]

The shape functions of order r on $\hat{K}$ **reproduce polynomials** of degree $\leq r$, i.e.

$$\mathcal{P}_r(\mathbb{R}^3) \subset \phi\left(\mathcal{Q}_r^{[r,r]}\right) \quad \text{and} \quad \mathcal{P}_r(\mathbb{R}^3) \subset \phi\left(\mathcal{S}_r^{[r,r]}\right)$$

Proof idea: Write $p \in \mathcal{P}_r$ in powers of x, y, $1 - z$;
show pullback is in shape function space.

Outline

Comparing dimension counts

$r \rightarrow$	1	2	3	4	5	6	7	key reference
$\dim \mathcal{Y}_r \Lambda^0$	5	13	25	42	65	95	133	Gillette
$\dim \mathcal{Y}_r^- \Lambda^0$ $\dim \hat{P}_r$ $\dim \mathcal{R}_r^{(0)}$	5	14	30	55	91	140	204	Gillette Bergot, Cohen, Duruflé Nigam, Phillips (M2AN)
$\dim \mathcal{U}^{(0),r}$	5	15	37	77	141	235	365	Nigam, Phillips (IMA J.) Fuentes et al.

→ Bergot, Cohen, Duruflé. *Journal of Scientific Computing*, 2010.

→ Fuentes, Keith, Demkowicz, Nagaraj. *Computers & Mathematics with Applications*, 2015.

→ Gillette. `arXiv:1512.07269`, 2015

→ Nigam, Phillips. *IMA Journal of Numerical Analysis*, 2012.

→ Nigam, Phillips. *Mathematical Modelling and Numerical Analysis*, 2012.

Minimality

What is the ***minimum*** number of degrees of freedom required to make an order r pyramid element?

$\rightarrow$ On a triangular face: $\mathcal{P}_r\Lambda^0(\triangle)$ *(No choice, really)*

$\rightarrow$ On the quadrilateral base: $\mathcal{S}_r\Lambda^0(\square)$ *(See paper with Christiansen below)*

$\rightarrow$ On the interior: $\mathcal{P}_{r-5}\Lambda^3(\mathbb{R}^3)$

(Loosely, b/c there are 5 linearly independent constraints to make a bubble function.)

Virtual element methods associate at least $\binom{r-2}{3}$ degrees of freedom to the interior of a pyramid and $\dim \mathcal{P}_{r-5}\Lambda^3(\mathbb{R}^3) = \binom{r-2}{3}$.

So, our construction appears to be minimal!

CHRISTIANSEN, G. "Constructions of some minimal finite element systems."
Mathematical Modelling and Numerical Analysis, 2016.

BEIRAO DA VEIGA, BREZZI, MARINI, RUSSO "Serendipity nodal VEM spaces"
arXiv:1510.08477, 2015.

Acknowledgments

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Slides and Pre-prints

`http://math.arizona.edu/~agillette/`