

Generalized barycentric coordinates for degenerate geometry in FEM

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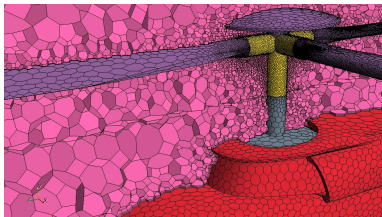
joint work with
Alexander Rand, CD-adapco

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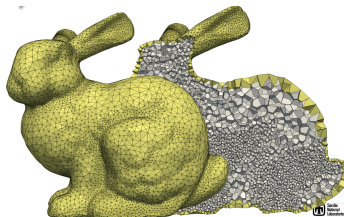
- 1 How are generalized barycentric coordinates used in FEM?
- 2 Deconstructing the 'a priori' estimate
- 3 Lessons from the triangle case
- 4 Experiments: GBCs on degenerate geometry
- 5 Comparison to theoretical results

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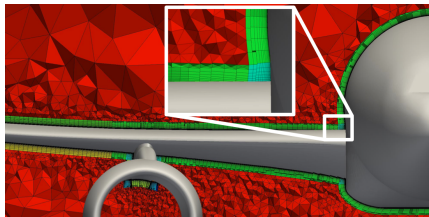
The growing world of polytopal meshing



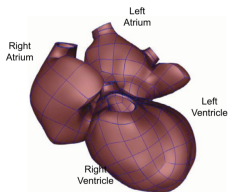
full polyhedral meshing
Star-CCM+ (CD-adapco)



surface-conforming meshing
VoroCrust (Sandia)



tet-hex-pyramid meshing
CADfix (ITI Transcendata)



NURBS meshing*
Continuity (UCSD)

*image from *Krishnamurthy et al*, CAGD 2016

Generalized barycentric coordinates applications

From the table of contents of the upcoming book:

*“Generalized Barycentric Coordinates in Computer Graphics
and Computational Mechanics”*

- Discrete Laplacians
- Mesh parameterization
- Shape deformation
- Self-supporting surfaces
- Extreme deformations
- BEM-based FEM
- Virtual element methods
- ... *and much more!*

Chapters in preparation by various authors; to appear in 2017.

In this talk, we focus on the use of GBCs in finite element methods, although our results apply to broader questions about interpolation quality.

The generalized barycentric coordinate approach

Let P be a convex polytope with vertex set V . We say that

$\lambda_{\mathbf{v}} : P \rightarrow \mathbb{R}$ are **generalized barycentric coordinates (GBCs)** on P

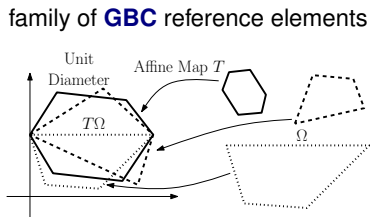
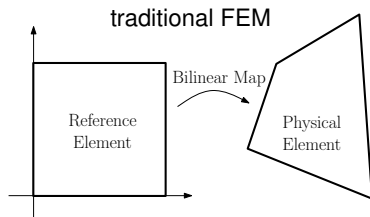
if they satisfy $\lambda_{\mathbf{v}} \geq 0$ on P and $L = \sum_{\mathbf{v} \in V} L(\mathbf{v}_{\mathbf{v}})\lambda_{\mathbf{v}}$, $\forall L : P \rightarrow \mathbb{R}$ linear.

Familiar properties are implied by this definition:

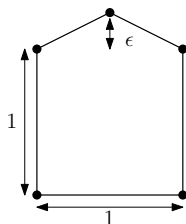
$$\underbrace{\sum_{\mathbf{v} \in V} \lambda_{\mathbf{v}} \equiv 1}_{\text{partition of unity}}$$

$$\underbrace{\sum_{\mathbf{v} \in V} \mathbf{v} \lambda_{\mathbf{v}}(\mathbf{x}) = \mathbf{x}}_{\text{linear precision}}$$

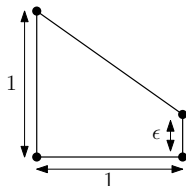
$$\underbrace{\lambda_{\mathbf{v}_i}(\mathbf{v}_j) = \delta_{ij}}_{\text{interpolation}}$$



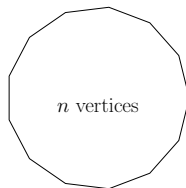
Some 'degenerate' geometry families



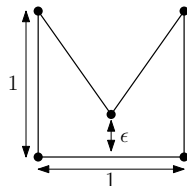
large angle



short edge



many vertices



vertex near
non-incident edge

Generalized barycentric coordinates help describe the relation between degenerate element geometry and interpolation error.

Outline

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Bounded interpolation estimate pairs

Define:

- ϕ , a type of generalized barycentric coordinates
- P , a polygon with vertices $\mathbf{v}_1, \dots, \mathbf{v}_n$
- $h = \text{diam}(P)$
- The interpolation procedure:

$$\mathcal{I}u = \sum_{i=1}^n u(\mathbf{v}_i)\phi_i$$

Suppose we can find a constant C_{BI} and class of polygons \mathfrak{p} such that

$$\left\| \sum_{i=1}^n u(\mathbf{v}_i)\phi_i \right\|_{H^1(P)} \leq C_{BI}h \|u\|_{H^2(P)} \quad u \in H^2(P), \quad P \in \mathfrak{p}$$

Then we say the pair (ϕ, \mathfrak{p}) has a **bounded interpolation** estimate.

The *a priori* error estimate

Fix a second order elliptic PDE.

Suppose (ϕ, \mathfrak{p}) has a bounded interpolation estimate:

$$\left\| \sum_{i=1}^n u(\mathbf{v}_i) \phi_i \right\|_{H^1(P)} \leq C_{BI} h \|u\|_{H^2(P)}$$

The ***a priori* error estimate** for a Galerkin FEM on a mesh of $P \in \mathfrak{p}$ is:

$$\underbrace{\|u - u_h\|_{H^1(\Omega)}}_{\text{finite element error}} \leq \underbrace{C_C}_{\text{Cea's Lemma}} \underbrace{\|u - \mathcal{I}u\|_{H^1(\Omega)}}_{\text{interpolation error}} \leq C_C C_{ie} h \underbrace{\|u\|_{H^2(\Omega)}}_{\text{2nd order oscillation}},$$

where

$$C_{ie} := \left(C_{BH}^2 (1 + C_{BI}^2) + C_{BI}^2 \right)^{1/2}$$

C_{BH} from Bramble-Hilbert Lemma

C_{BI} from bounded interpolation estimate

Our goal: Identify pairs (ϕ, \mathfrak{p}) with bounded interpolation estimates.

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Bounded interpolation pairs for triangles

Only choice for ϕ is regular barycentric coordinates.

Two classes of triangles to consider:

$t_{\min a}$ All angles are bounded away from zero: $\alpha_j > \alpha_* > 0$

$t_{\max a}$ All angles are bounded away from 180° : $\alpha_j < \alpha^* < 180^\circ$

We have

- $(\phi, t_{\min a}) \implies$ “minimum angle condition” \implies bounded interpolation
- $(\phi, t_{\max a}) \implies$ “maximum angle condition” \implies bounded interpolation

(BABUŠKA, AZIZ 1976 and JAMET 1976)



- Any class of triangles with bounded aspect ratio (ratio of longest edge to radius of smallest inscribed circle) can be described as a $t_{\min a}$ class.
- Any $t_{\min a}$ class also excludes large angles, so it can be described as a $t_{\max a}$ class.
- Raises the question: for a bounded interpolation estimate, is $t_{\max a}$ not only sufficient **but necessary**?

A maximum angle condition is not strictly necessary

Example from “The maximum angle condition is not necessary for convergence of the finite element method” HANNUKAINEN, KOROTOV, KŘÍŽEK 2012:

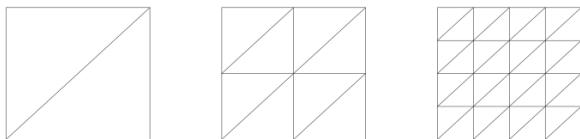


Fig. 5 Family \mathcal{F}_3 satisfying the minimum angle condition

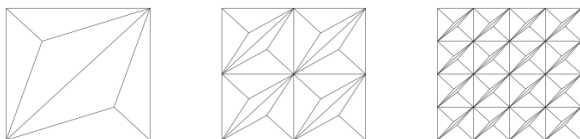


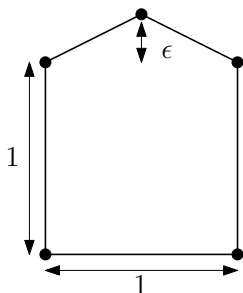
Fig. 6 Family \mathcal{F}_4 that does not satisfy the maximum angle condition

- Key point: $\mathcal{F}_4 \subset \mathcal{F}_3$ and \mathcal{F}_3 satisfies a maximum angle condition.
- Viewed another way: \mathcal{F}_4 is not the optimal triangulation of the lattice vertices.

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Large angle experiment



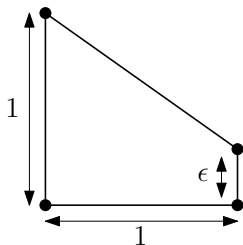
On a **single** element with a large angle:

- Set boundary values on domain shown from $u(x, y) = 0.77223(x - 0.331)^2 + 1.1123(y + 0.177344)^2$
- Compute harmonic (**hm**), Wachspress (**wa**), mean value (**mv**), discrete harmonic (**dh**), moving least squares (**ls-***), and maximum entropy (**me-***) coordinates as $\epsilon \rightarrow 0$
- Report $|u_h^{\text{hm}}|_{H^1}$ and $|u_h^{\text{hm}} - u_h^*|_{H^1}$

ϵ	hm	wa	mv	dh	ls-1	ls-2	ls-3	me-t	me-u
0.1600	1.3e0	2.6e-1	6.0e-2	2.3e-1	1.8e-1	2.2e-2	5.4e-2	7.9e-2	5.1e-1
0.0400	1.3e0	1.3e0	1.1e-1	1.5e0	3.6e-1	5.4e-2	1.2e-1	1.6e-1	2.2e0
0.0100	1.3e0	3.1e0	1.3e-1	3.9e0	4.4e-1	6.5e-2	1.4e-1	1.9e-1	5.1e0
0.0025	1.3e0	6.4e0	1.3e-1	8.3e0	4.5e-1	6.7e-2	1.4e-1	2.0e-1	9.3e0
0.0000	1.3e0	-	1.3e-1	-	4.5e-1	6.8e-2	1.5e-1	2.0e-1	-

Only *some* coordinates lose interpolation quality as the geometry degenerates.
(in this case, **wa**, **dh**, and **me-u**).

Short edge experiment

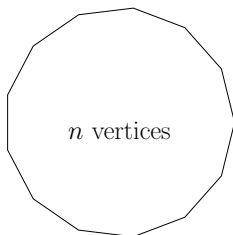


On a quadrilateral with a very short edge:

ϵ	hm	wa	mv	dh	ls-1	ls-2	ls-3	me-t	me-u
0.1600	7.5e-1	1.1e-1	4.0e-2	2.3e-1	2.2e-1	3.4e-2	6.5e-2	2.5e-1	7.3e-1
0.0400	8.5e-1	3.9e-2	1.2e-2	8.3e-2	9.0e-2	1.3e-2	2.9e-2	9.6e-2	4.6e-1
0.0100	8.9e-1	1.2e-2	3.2e-3	1.6e-1	2.9e-2	4.3e-3	1.0e-2	3.0e-2	2.5e-1
0.0025	9.1e-1	3.3e-3	6.7e-4	3.7e-3	7.9e-3	1.2e-3	3.2e-3	8.9e-3	9.9e-2

In this case, *none* of the coordinates lose interpolation quality as the geometry degenerates!

Large vertex count experiment

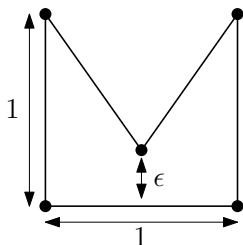


On a regular polygon with n vertices:

n	hm	wa	mv	dh	ls-1	ls-2	ls-3	me-t	me-u
4	1.5e0	0.0e0	4.5e-2	0.0e0	1.1e-1	8.3e-3	1.2e-2	0.0e0	0.0e0
8	1.9e0	5.0e-2	1.0e-2	5.0e-2	1.3e-1	9.9e-3	2.3e-2	7.5e-2	1.3e-1
16	1.9e0	7.9e-2	4.2e-2	7.9e-2	1.7e-1	5.5e-3	4.0e-2	8.2e-2	1.7e-1
32	2.0e0	8.8e-2	4.3e-2	8.8e-2	1.7e-1	2.3e-3	4.3e-2	6.8e-2	1.8e-1
64	2.0e0	9.1e-2	4.3e-2	9.1e-2	1.7e-1	9.7e-4	4.3e-2	5.2e-2	-
128	2.0e0	9.2e-2	4.3e-2	9.2e-2	1.8e-1	5.8e-4	4.4e-2	3.9e-2	-
256	2.0e0	9.2e-2	4.3e-2	9.2e-2	1.8e-1	5.0e-4	4.4e-2	2.9e-2	-

Again, *none* of the coordinates lose interpolation quality as the geometry degenerates. . . but computing them becomes expensive.

Non-convex experiment



On a non-convex polygon:

ϵ	hm*	mv	ls-1	ls-2	ls-3	me-t
0.1600	2.4e0	8.2e-1	2.3e-0	4.2e-1	8.9e-1	1.4e-0
0.0400	2.2e0	7.4e-1	2.3e-0	3.0e-1	8.7e-1	1.4e-0
0.0100	2.4e0	7.6e-1	2.4e-0	2.6e-1	8.6e-1	1.6e-0
0.0025	2.6e0	8.2e-1	2.6e-0	2.4e-1	1.1e0	1.7e-0

Again, *none* of the coordinates lose interpolation quality as the geometry degenerates. . . but some are not defined!

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Classes of polygons and polytopes

On polygons:

p_{\min} All interior angles are bounded away from zero: $\alpha_j > \alpha_* > 0$.

p_{\max} All interior angles are bounded away from 180° : $\alpha_j < \alpha^* < 180^\circ$.

On d -polytopes:

p_{cvx} P is convex.

p_{ar} The aspect ratio is bounded: $\frac{\text{diam}(P)}{\text{radius of max. insc. } d\text{-ball.}} \leq \gamma^*$.

p_{nv} The number of vertices is bounded: $n < n^*$.

p_{edge} The ratio of diameter to the shortest edge is bounded: $\frac{\text{diam}(P)}{|e_i|} < e^*$.

$p_{\text{v-f}}$ The value of h_* is bounded away from zero: $h_*(P) \geq h_* > 0$ where
 $h_*(P) := \min \text{dist from a vertex to hyper-plane of a non-incident face}$

p_{simp} Every vertex is incident to exactly d other vertices

Theoretical results

The following pairs have bounded interpolation estimates. . .

. . . on polygons:

Wachspress: $(\phi_{\mathbf{wa}}, \mathfrak{p}_{\mathit{cvx}} \cap \mathfrak{p}_{\mathit{ar}} \cap \mathfrak{p}_{\mathit{edge}} \cap \mathfrak{p}_{\mathit{maxa}})$

Sibson: $(\phi_{\mathbf{sib}}, \mathfrak{p}_{\mathit{cvx}} \cap \mathfrak{p}_{\mathit{ar}} \cap \mathfrak{p}_{\mathit{edge}})$

Mean Value: $(\phi_{\mathbf{mv}}, \mathfrak{p}_{\mathit{cvx}} \cap \mathfrak{p}_{\mathit{ar}} \cap \mathfrak{p}_{\mathit{edge}})$

Triangulation: $(\phi_{\mathbf{tri}}, \mathfrak{p}_{\mathit{maxa}})$

Harmonic: $(\phi_{\mathbf{hm}}, \mathfrak{p}_{\mathit{cvx}} \cap \mathfrak{p}_{\mathit{ar}})$

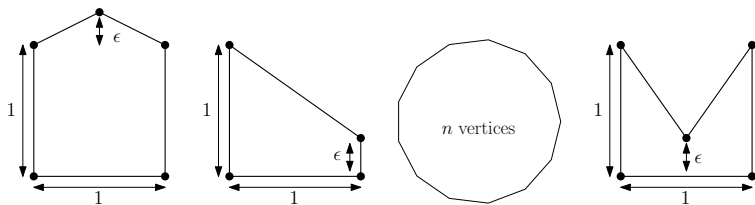
(see G., RAND, BAJAJ, 2012, RAND, G., BAJAJ, 2013, G., RAND, 2016)

. . . on d -polytopes:

Wachspress: $(\phi_{\mathbf{wa}}, \mathfrak{p}_{\mathit{cvx}} \cap \mathfrak{p}_{\mathit{v-f}} \cap \mathfrak{p}_{\mathit{simp}})$

(see FLOATER, G., SUKUMAR, 2014.)

Experiments reveal more work to do



p_{cvx}	✓	✓	✓	✗
p_{ar}	✓	✓	✓	✓
p_{nv}	✓	✓	✗	✓
p_{edge}	✓	✗	✗	✓
p_{mina}	✓	✓	✓	✓
p_{maxa}	✗	✓	✗	✓
p_{v-f}	✗	✗	✗	✗
<hr/>				
wa	✗	✓	✓	n/a
mv	✓	✓	✓	✓
dh	✗	✓	✓	n/a
ls-*	✓	✓	✓	✓
me-t	✓	✓	✓	n/a
me-u	✗	✓	n/a	n/a

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Collaborators on this work

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Michael Floater	U. Oslo	math
Alexander Rand	CD-adapco	industry
N. Sukumar	UC Davis	civil engineering

Slides and pre-prints

<http://math.arizona.edu/~agillette/>

Forthcoming book chapter:

“Shape Quality for Generalized Barycentric Interpolation”