Efficient Delaunay Mesh Generation From Sampled Scalar Functions

Andrew Gillette

joint work with

Samrat Goswami and Chandrajit Bajaj

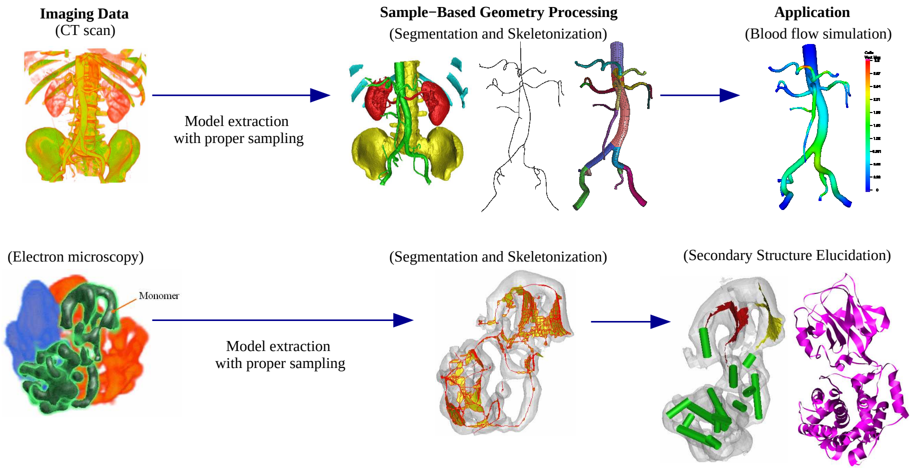
Department of Mathematics,
Institute of Computational Engineering and Sciences
University of Texas at Austin, Austin, Texas 78712, USA
http://www.math.utexas.edu/users/agillette

Outline

1. Motivation and Problem Statement
2. Prior work and background
3. Our Approach
4. Results
5. Future work

Outline

Motivation: Biological Modeling



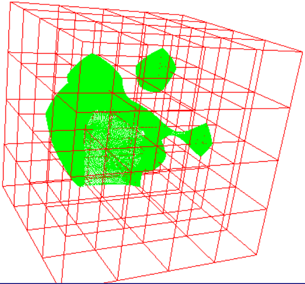
Biological modeling problems often require an enforced sampling density of image data in regions with complicated features.

Problem Statement

Input:

1. Rectilinear sampling of a bounded domain of an unknown scalar function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$.
2. An isovalue $v \in \mathbb{R}$.
3. A local interpolant (e.g. trilinear).
4. An accuracy bound $\epsilon \in \mathbb{R}$.

Inputs 1-3 define a surface Σ approximating the level set $F^{-1}(v)$.



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Inputs 1-3 define a surface Σ approximating the level set $F^{-1}(v)$.

Output a mesh M satisfying:

1. **Topological Guarantee:** M is homeomorphic to Σ.
2. **Geometrical Guarantee:** The Hausdorff distance from M to Σ within ϵ.
3. **Delaunay Conformity:** M is a subcomplex of the Delaunay triangulation of the vertex set of M.
4. **Adaptivity:** The user can decimate part of the volumetric data and still preserve properties I, II, and III.

Outline

Selected Prior Work 1

L. Chew *Guaranteed-quality mesh generation for curved surfaces* In Proc. SoCG, 1993.

- Provided a meshing algorithm for curved surfaces with provably good geometry.
- Could not guarantee topological correctness.

J.-D. Boissonnat and S. Oudot *Provably Good Sampling and Meshing of Surfaces* Graphical Models, 2005.

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P. Alliez, D. Cohen-Steiner, M. Yvinec and M. Desbrun *Variational tetrahedral meshing* SIGGRAPH, 2005.

- Employed Chew's algorithm to produce dense samples of Σ.
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H. Edelsbrunner and N. Shah *Triangulating Topological Spaces* Intl. Journal of Comput. Geom. and Appl, 1997.

- Established the **closed ball property**, a criterion that ensures a mesh is Delaunay-conforming.
- An important theoretical result, not an application paper.

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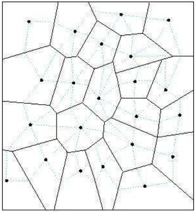
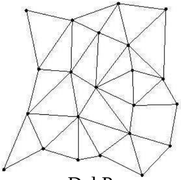
The Closed Ball Property

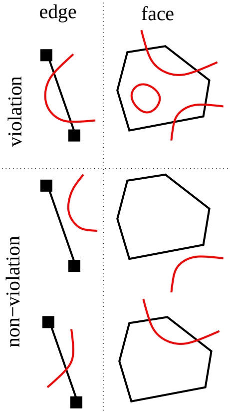

Vor P


Del P

- Vor P = Voronoi diagram of pointset P.
- Del P = Delaunay diagram of pointset P.

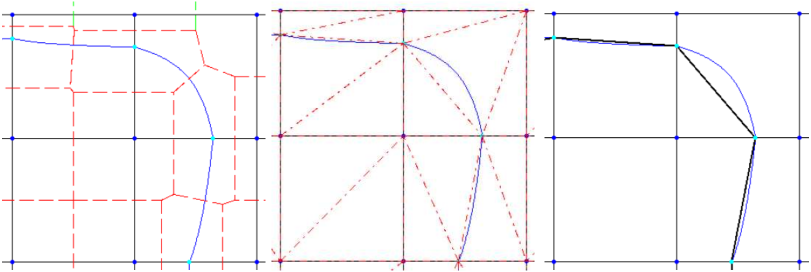
Closed ball property [ES,1997]

A Voronoi object V of dimension k satisfies the closed ball property iff $V \cap \Sigma = \emptyset$ or $V \cap \Sigma$ is homeomorphic to a closed ball of dimension $k - 1$.



Delaunay Conformity

$\mathrm{Del}\, P|_{\Sigma}$ = the set of Delaunay objects of $\mathrm{Del}\, P$ whose dual Voronoi objects have non-zero intersection with Σ.



Theorem [ES,1997]

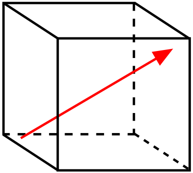
If Σ intersects each Voronoi object of $\mathrm{Vor}\, P$ transversally and $\mathrm{Vor}\, P$ satisfies the closed ball property, then $\mathrm{Del}\, P|_{\Sigma}$ is homeomorphic to Σ.

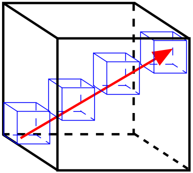
Ray-Surface Intersection Code

To construct a Delaunay-conforming surface, current methods require the use of a Ray-Surface Intersection Code (RSIC).

RSIC Input:	An arbitrary ray $\vec{r}$ and surface Σ.
RSIC Output:	Locations where $\vec{r}$ intersects Σ.

Problem: For a scalar volume of n^3 vertices, a ray may pass through n separate function domains, since Σ is defined piecewise.





Goal: Confine RSIC computation to neighborhoods of constant size.

Outline

Algorithm

Idea: Build a "protective layer" of points near Σ to bound Ray-Surface Intersection Code calculations.

DELSURFMESH(Σ)

1. Compute the point set E sampling Σ.
2. Compute the protective layer G of grid points.
3. Compute the Voronoi and Delaunay diagrams of $E \cup G$.
4. Insert new sample points (N) repeatedly until $\mathrm{Vor}\,(E \cup G \cup N)$ satisfies the closed ball property.
5. Output the Restricted Delaunay triangulation $\mathrm{Del}\,(E \cup G \cup N)|_{\Sigma}$.

Algorithm

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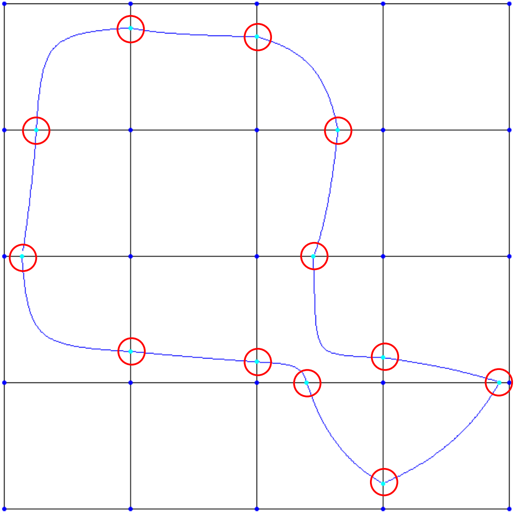
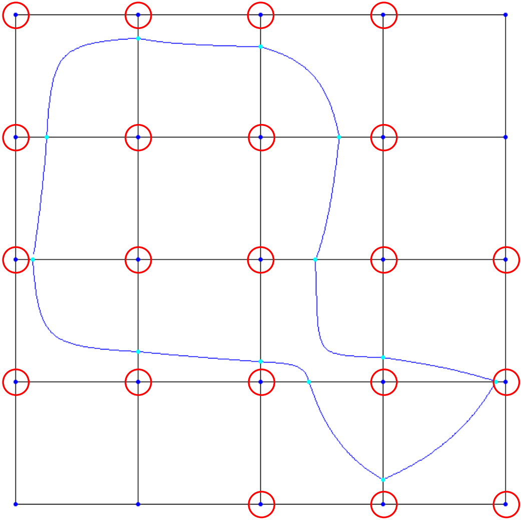


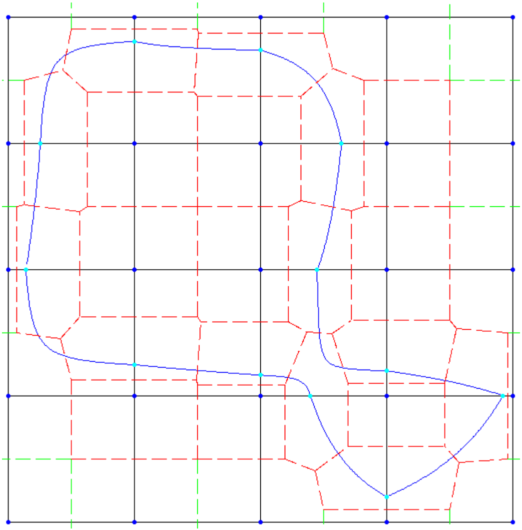
Algorithm

2. Compute the protective layer G of grid points.



Algorithm

3. Compute the **Voronoi** and Delaunay diagrams of $E \cup G$.



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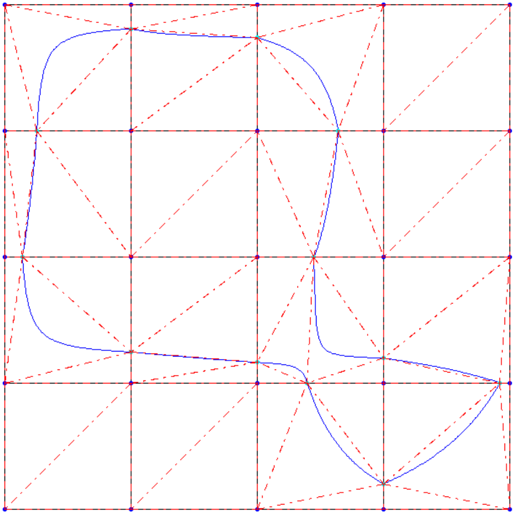
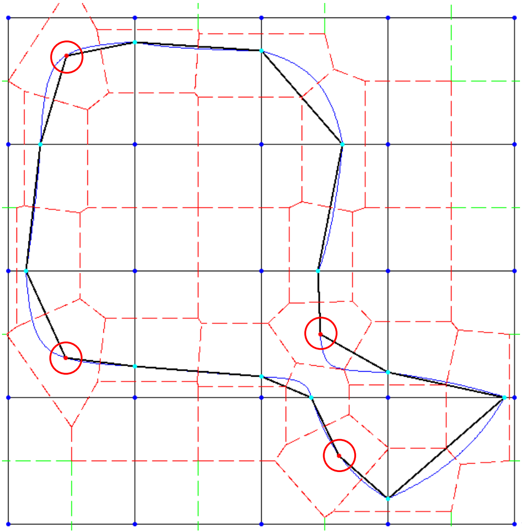


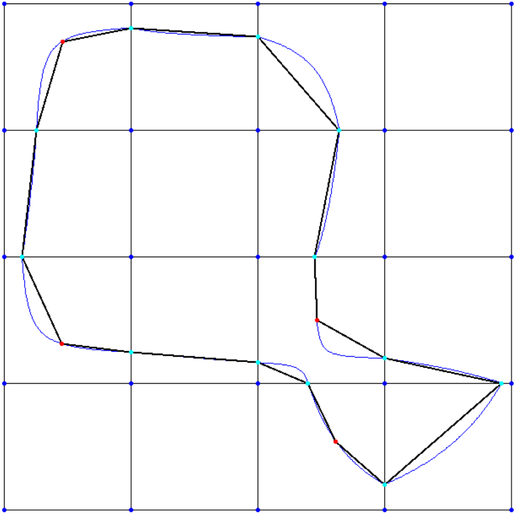
Algorithm

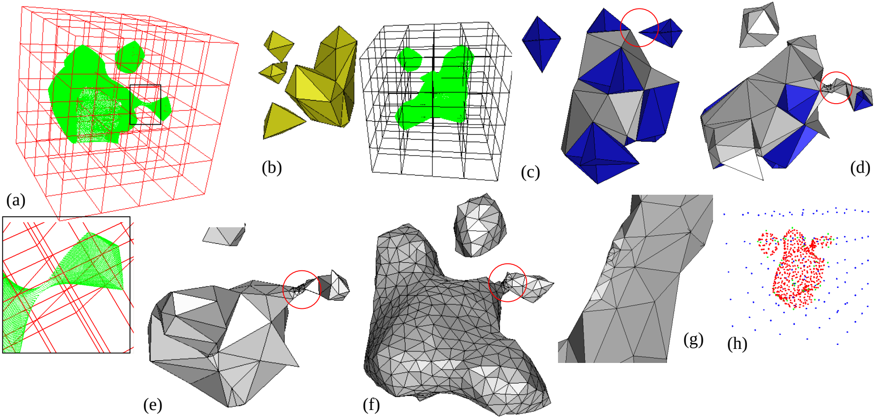
4. Insert new sample points (N) repeatedly until $\mathrm{Vor}\,(E \cup G \cup N)$ satisfies the closed ball property.



Algorithm

5. Output the Restricted Delaunay triangulation $\mathrm{Del}\,(E \cup G \cup N)|_{\Sigma}$.



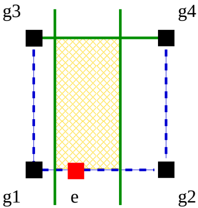


Steps of the algorithm on a toy data set. Compare to Marching Cubes output in (b).

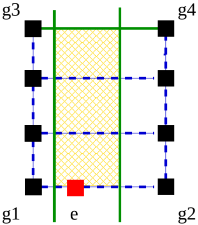
Efficiency Results

Theorem:

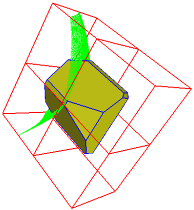
There is an $O(1)$ bound on the number of voxels that a Voronoi cell of an E or N point may intersect. In the case of uniform rectilinear gridding with voxels that are cubes, this bound is four voxels.



2 pixel bound
(square pixels)



6 pixel bound
(rectangular pixels)



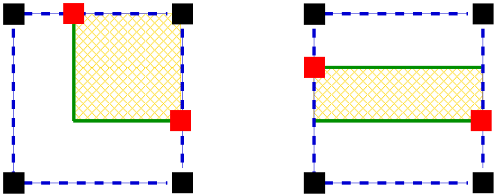
4 voxel bound
(cubic voxels)

Efficiency Results

Theorem:

Let *VF* be a Voronoi facet formed between two grid points and *VE* a Voronoi edge formed among three grid points at any point in the algorithm. Then $VF \cap \Sigma = \emptyset$ and $VE \cap \Sigma = \emptyset$.

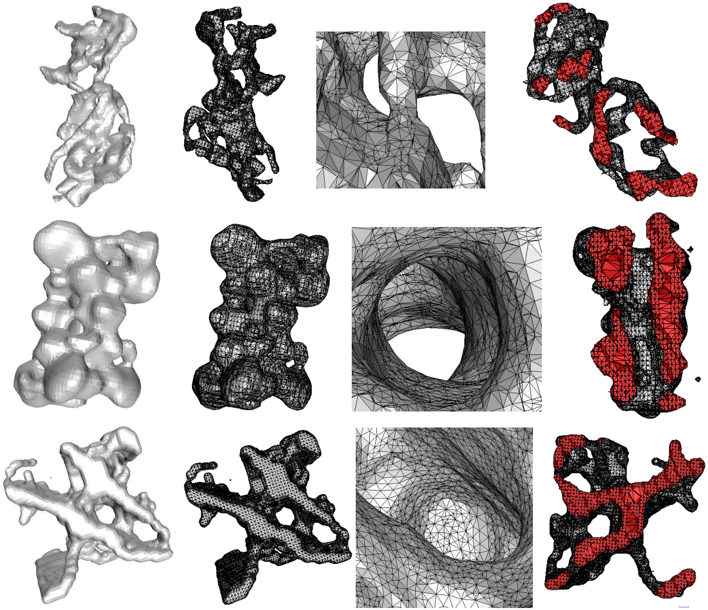
Loosely: Σ avoids Voronoi objects formed only by grid points



(2D case: Σ must lie inside the shaded yellow region)

Outline

Results



1CID

1MAG

BONE

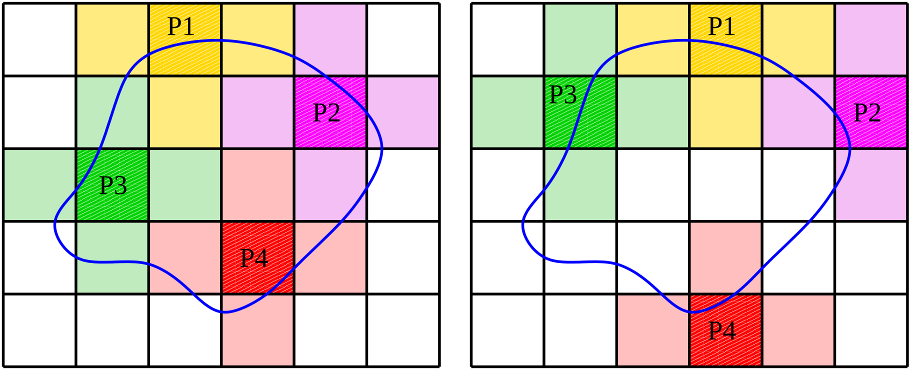
Results

Dataset	Size	ϵ	Init. Triang	Enforce CBP	Recover Geom.
1CID	128^3	10^{-5}	$1.0s$	$45s$	$35s$
1MAG	128^3	10^{-5}	$1.5s$	$20s$	$28s$
BONE	64^3	10^{-4}	$5.0s$	$65s$	$89s$

Observe: The complexity depends on the number of voxels that Σ passes through, not the size of the dataset.

Outline

Parallelization



Since the enforcement of Delaunay conformity can be done locally, our algorithm is amenable to parallel processing for mesh generation.

Acknowledgements

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