A “Fifth Column” for Finite Elements

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joint work with

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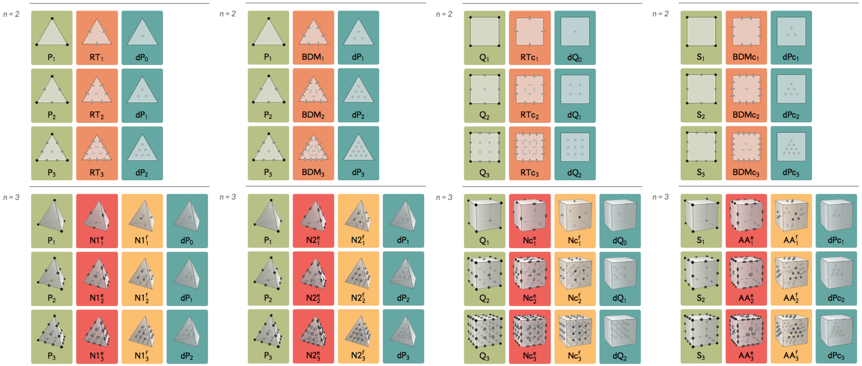


Table of Contents

Outline

1. Conforming finite elements in applications
2. The four columns of the Periodic Table of Finite Elements
3. The 5th column: Trimmed serendipity spaces
4. Benefits to applications and open questions

$H(\mathrm{div})$ / L^2 mixed form of Poisson problem

Derivation of a mixed method for the **Poisson** problem on a domain $\Omega \subset \mathbb{R}^3$:

2nd order equation

$$\Delta u + f = 0, \quad u \in H^2$$

1st order system

$$\begin{aligned} \mathrm{div}\ \sigma + f &= 0, \\ \sigma - \mathrm{grad}\ u &= 0, \end{aligned}$$

mixed weak form

$$\begin{aligned} (\mathrm{div}\ \sigma, \phi) + (f, \phi) &= 0, \quad \forall\ \phi \in L^2 = \Lambda^3(\Omega) \\ (\sigma, \omega) + (u, \mathrm{div}\ \omega) &= 0, \quad \underbrace{\forall\ \omega \in H(\mathrm{div})}_{\text{i.e. } \omega,\ \mathrm{div}\ \omega \in L^2(\Omega)} = \underbrace{\Lambda^2(\Omega)}_{\text{differential form notation}} \end{aligned}$$

mixed finite element method

$$\begin{aligned} (\mathrm{div}\ \sigma_h, \phi_h) + (f, \phi_h) &= 0, \quad \forall \phi_h \in \Lambda^3_h \subset L^2 \\ (\sigma_h, \omega_h) + (u_h, \mathrm{div}\ \omega_h) &= 0, \quad \underbrace{\forall \omega_h \in \Lambda^2_h}_{\text{finite element spaces to be chosen}} \underbrace{\subset H(\mathrm{div})}_{\text{enforcing containment} \Rightarrow \text{"conforming"}} \end{aligned}$$

A conforming mixed method for Darcy Flow

Movement of a fluid through porous media modeled via **Darcy flow**:

Given f and g, find pressure p and velocity $\mathbf{u}$ such that:

$$
\begin{aligned}
\mathbf{u} + K\nabla p &= 0 && \text{in } \Omega,\\
\operatorname{div}\ \mathbf{u} - f &= 0 && \text{in } \Omega,\\
p &= g && \text{on } \partial\Omega,
\end{aligned}
$$

where K is a symmetric, uniformly positive definite tensor for $\dfrac{\text{permeability}}{\text{viscosity}}$.

A **weak form** of these equations: find $\mathbf{u} \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

$$
\begin{aligned}
(K^{-1}\mathbf{u}, \mathbf{v}) - (p, \operatorname{div}\ \mathbf{v}) &= [\partial\Omega \text{ terms}] && \forall\ \mathbf{v} \in H(\text{div})\\
(\operatorname{div}\ \mathbf{u}, w) - (f, w) &= 0 && \forall\ w \in L^2(\Omega)\\
(\partial\Omega \text{ conditions}) &= 0
\end{aligned}
$$

A conforming mixed **finite element** method: find $\mathbf{u}_h \in \Lambda_h^2$ and $p \in \Lambda_h^3$ such that

$$
\begin{aligned}
(K^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div}\ \mathbf{v}_h) &= [\partial\Omega \text{ terms}] && \forall\ \mathbf{v}_h \in \Lambda_h^2 && \subset H(\text{div})\\
(\operatorname{div}\ \mathbf{u}_h, w_h) - (f, w_h) &= 0 && \forall\ w_h \in \Lambda_h^3 && \subset L^2(\Omega)\\
(\partial\Omega \text{ conditions}) &= 0
\end{aligned}
$$

Arbogast, Pencheva, Wheeler, Yotov "A Multiscale Mortar Mixed Finite Element Method"
Multiscale Modeling and Simulation (SIAM) 6:1, 2007.

H^1 / H(curl) mixed form of vector Poisson problem

Derivation of a mixed method for the **vector Poisson** problem on a domain $\Omega \subset \mathbb{R}^3$:

$$-\text{grad div } u + \text{curl curl } u = \text{f}, \qquad + \partial\Omega \textit{ conditions}$$

$$\begin{aligned} \sigma + \text{div } u &= 0, \qquad + \partial\Omega \textit{ conditions} \\ \text{grad } \sigma + \text{curl curl } u &= \text{f}, \end{aligned}$$

A **weak form** of these equations: find $\sigma \in H^1(\Omega)$ and $u \in H(\text{curl})$ such that

$$\begin{aligned} (\sigma, \tau) - (u, \text{grad } \tau) &= 0 \qquad \forall \tau \in H^1(\Omega) = \Lambda^0(\Omega) \\ (\text{grad } \sigma, v) + (\text{curl } u, \text{curl } v) &= (f, v), \qquad \underbrace{\forall v \in H(\text{curl})}_{\text{i.e. } \omega,\ \text{curl } \omega \in L^2(\Omega)} = \underbrace{\Lambda^1(\Omega)}_{\text{differential form notation}} \end{aligned}$$

A conforming mixed **finite element** method: find $\sigma_h \in \Lambda_h^0$ and $u_h \in \Lambda_h^1$ such that

$$\begin{aligned} (\sigma_h, \tau_h) - (u_h, \text{grad } \tau_h) &= 0, \qquad \forall \tau_h \in \Lambda_h^0 \subset H^1(\Omega) \\ (\text{grad } \sigma_h, v_h) + (\text{curl } u_h, \text{curl } v_h) &= (f, v_h), \qquad \forall v_h \in \Lambda_h^1 \subset H(\text{curl}) \end{aligned}$$

Applications to computational electromagnetism, image processing, visualization, . . .

An abbreviated reading list (50 years of theory!)

RAVIART, THOMAS, "A mixed finite element method for 2nd order elliptic problems" Lecture Notes in Mathematics, 1977. ← *2910 citations, more than 100 of which are from 2016!*

NÉDÉLEC, "Mixed finite elements in $\mathbb{R}^3$," Numerische Mathematik, 1980

BREZZI, DOUGLAS JR., MARINI, "Two families of mixed finite elements for second order elliptic problems," Numerische Mathematik, 1985.

NÉDÉLEC, "A new family of mixed finite elements in $\mathbb{R}^3$," Numerische Mathematik, 1986

ARNOLD, FALK, WINTHER "Finite element exterior calculus, homological techniques, and applications," *Acta Numerica*, 2006

CHRISTIANSEN, "Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension," Numerische Mathematik, 2007.

ARNOLD, FALK, WINTHER "Finite element exterior calculus: from hodge theory to numerical stability," *Bulletin of the AMS*, 2010.

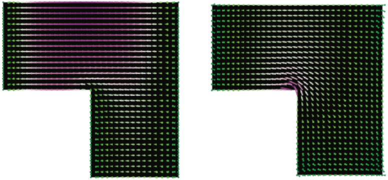
ARNOLD, AWANOU "The serendipity family of finite elements ", *Found. Comp Math*, 2011

ARNOLD, AWANOU "Finite element differential forms on cubical meshes", *Math Comp.*, 2013

ARNOLD, BOFFI, BONIZZONI "Finite element differential forms on curvilinear meshes and their approximation properties," Numerische Mathematik, 2014

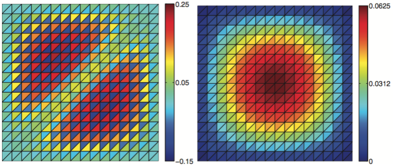
ARBOGAST, CORREA "Two families of H(div) mixed finite elements on quadrilaterals of minimal dimension," ICES Report, UT Austin, 2015

The importance of method selection



Vector Poisson problem

- Solutions by the standard non-mixed method (left) and by a mixed method (right).
- Only the second choice shows the correct behavior near the reentrant corner.



Poisson problem

- Solutions by two different choices for the finite element solution spaces in a mixed method.
- Only the second choice looks like the true solution is $x(1-x)y(1-y)$.

Examples and images borrowed from:

ARNOLD, FALK, WINTHER “Finite Element Exterior Calculus: From Hodge Theory to Numerical Stability,” *Bulletin of the AMS*, 47:2, 2010.

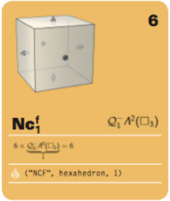
Outline

Unified notation via differential forms

Finite element method types can be broadly classified by three integers:

n	$\rightarrow$	the spatial dimension of the domain
r	$\rightarrow$	the order of error decay
k	$\rightarrow$	the differential form order of the solution space

An element type is defined in part by its **degrees of freedom**;
the more degrees of freedom, the greater the computational cost of the method.



Ex: $\mathcal{Q}_1^-\Lambda^2(\square_3)$ is an element for

$n = 3$	$\rightarrow$	domains in $\mathbb{R}^3$
$r = 1$	$\rightarrow$	linear order of error decay
$k = 2$	$\rightarrow$	conformity in $\Lambda^2(\mathbb{R}^3) \rightsquigarrow H(\text{div})$

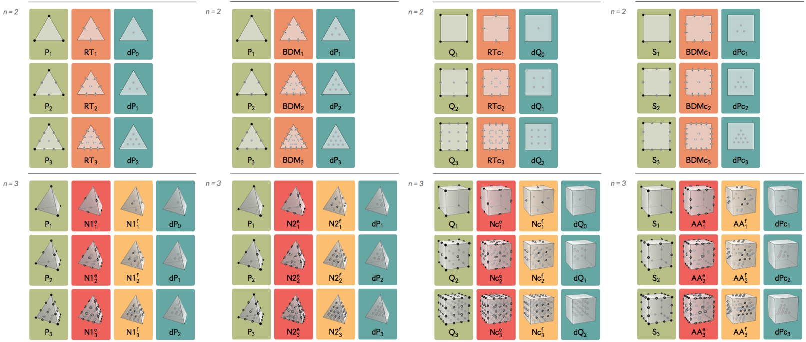
It has **6** degrees of freedom per element,

is defined on geometry $\square_3$ (i.e. a cube),

and is part of the $\mathcal{Q}^-$ 'column' of elements.

The ‘Periodic Table of the Finite Elements’

The periodic table of the finite elements *(prepared by Doug Arnold & Anders Logg)*:

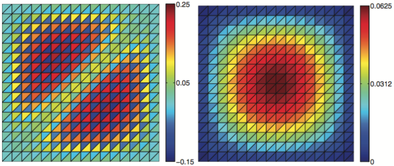


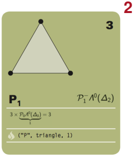
Classification of many common conforming finite element types.

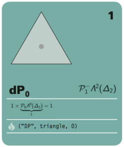
n	$\rightarrow$	Domains in $\mathbb{R}^2$ (top half) and in $\mathbb{R}^3$ (bottom half)
r	$\rightarrow$	Order $1, 2, 3$ of error decay (going down columns)
k	$\rightarrow$	Conformity type $k = 0, \ldots, n$ (going across a row)

Geometry types: Simplices (left half) and cubes (right half).

Stable choices for mixed methods

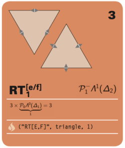


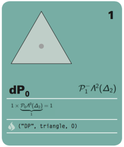




$\subset H^1 \times H^1$ $\subset L^2$

Unstable method, as shown





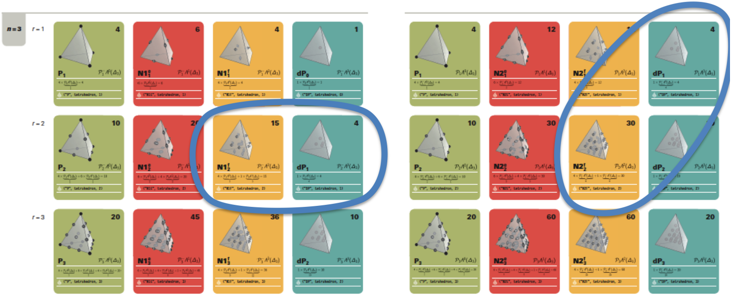
$\subset H(\text{div})$ $\subset L^2$

Provably stable method

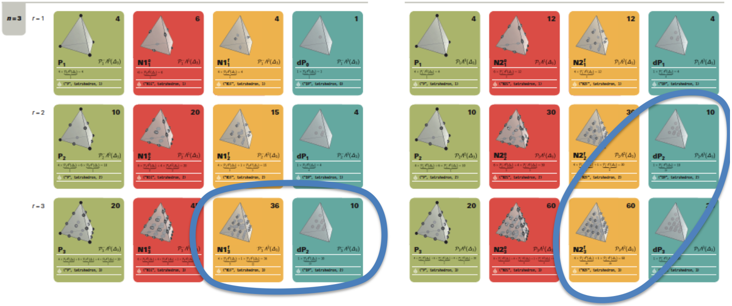
→ The term **stable pair** of elements for mixed methods has a precise mathematical meaning (see e.g. Arnold, Falk, Winther paper).

→ The Periodic Table of Finite Elements lets us 'read off' stable pairs visually.

Stable pairs for simplicial meshes



Stable pairs for simplicial meshes



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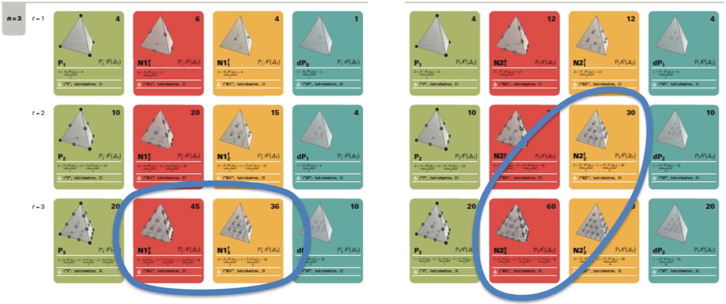
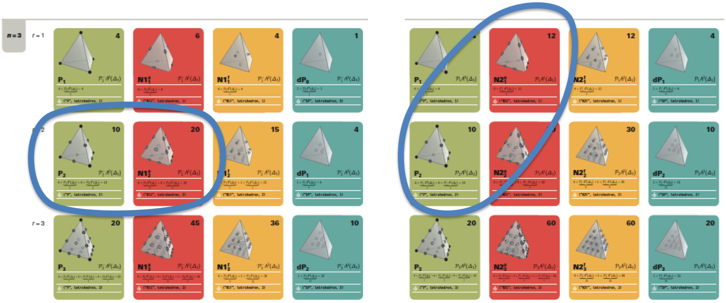
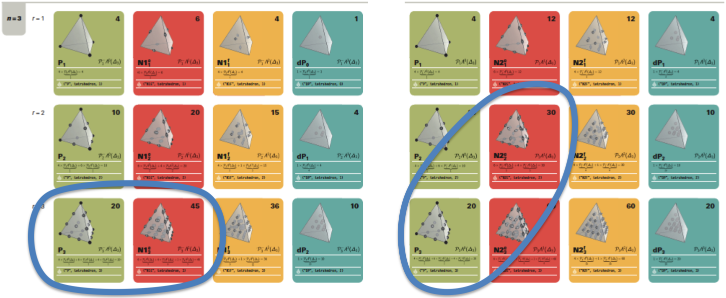


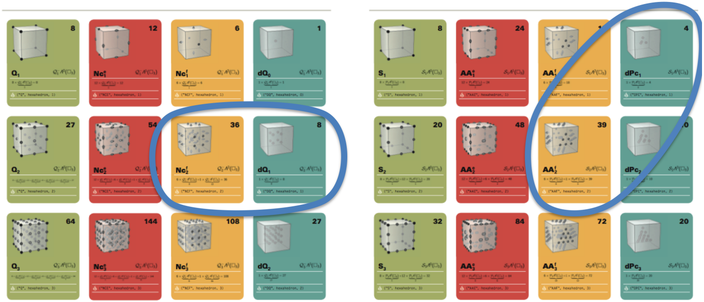
Stable pairs for simplicial meshes



Stable pairs for simplicial meshes



Stable pairs for cubical mehes



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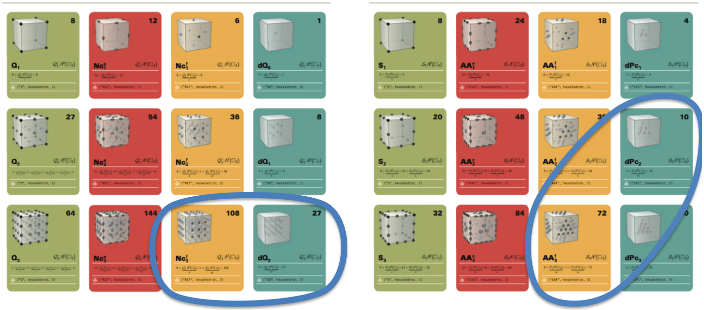

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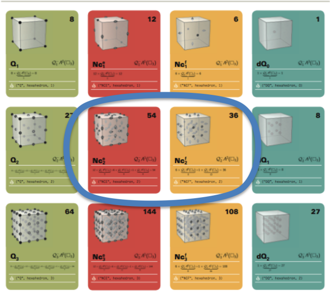

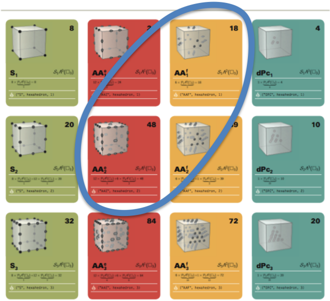

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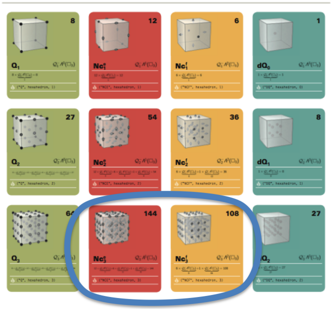

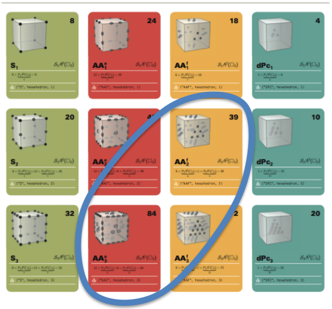

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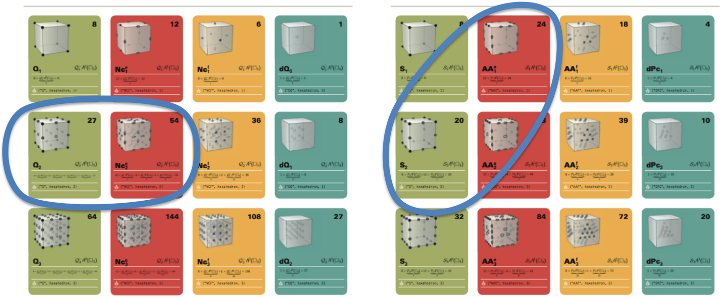
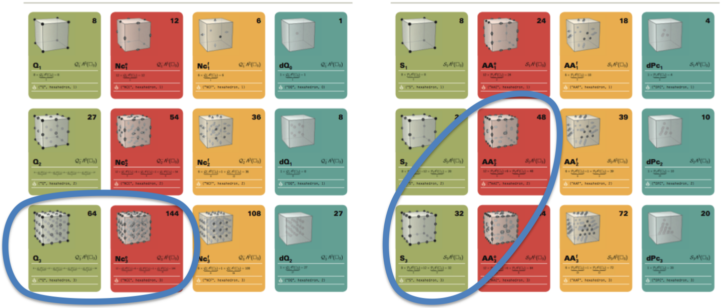


Stable pairs for cubical mehes



Two patterns of choices

The stable pairs just shown come from one of four **sequences** of spaces:

On simplicial meshes in $\mathbb{R}^n$:

$$\mathcal{P}_r^-\Lambda^0 \to \mathcal{P}_r^-\Lambda^1 \to \cdots \to \mathcal{P}_r^-\Lambda^{n-1} \to \mathcal{P}_r^-\Lambda^n$$ **'trimmed' polynomials**

$$\mathcal{P}_r\Lambda^0 \to \mathcal{P}_{r-1}\Lambda^1 \to \cdots \to \mathcal{P}_{r-n+1}\Lambda^{n-1} \to \mathcal{P}_{r-n}\Lambda^n$$ **polynomials**

On cubical meshes in $\mathbb{R}^n$:

$$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n$$ **tensor product**

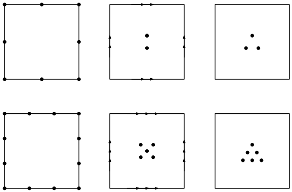
$$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n$$ **serendipity**

- The 'minus' spaces proceed across rows of the PToFE (r stays fixed)
- The 'regular' spaces proceed on SW-NE diagonals of the PToFE (r decreases)
- Mysteriously, the degree of freedom count for mixed methods from the $\mathcal{P}_r^-$ spaces is smaller than those from the $\mathcal{P}_r$ spaces, while the opposite is true for the $\mathcal{Q}_r^-$ and $\mathcal{S}_r$ spaces.

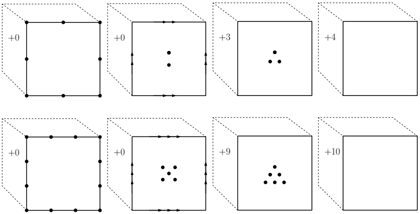
Outline

The 5th column: Trimmed serendipity spaces

$n = 2$



$n = 3$



A new column for the PToFE:
the **trimmed serendipity** elements.

$\boxed{\mathcal{S}_r^- \Lambda^k(\square_n)}$ denotes
- approximation order r,
- subset of k-form space $\Lambda^k(\Omega)$,
- use on meshes of n-dim'l cubes.

Defined for any $n \geq 1$, $0 \leq k \leq n$, $r \geq 1$

Identical or analogous properties to all the other colummns in the table.

The advantage of the $\mathcal{S}_r^- \Lambda^k$ spaces is that they have **minimal dimension**, in a certain sense to be described, and thus allow potential computational benefits.

Key properties of the trimmed serendipity spaces

$$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n$$ **tensor product**

$$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n$$ **serendipity**

$$\mathcal{S}_r^-\Lambda^0 \to \mathcal{S}_r^-\Lambda^1 \to \cdots \to \mathcal{S}_r^-\Lambda^{n-1} \to \mathcal{S}_r^-\Lambda^n$$ **trimmed serendipity**

Subcomplex: $d\mathcal{S}_r^-\Lambda^k \subset \mathcal{S}_r^-\Lambda^{k+1}$

Exactness: The above sequence is exact.
i.e. the image of incoming map = kernel of outgoing map

Inclusion: $\mathcal{S}_r\Lambda^k \subset \mathcal{S}_{r+1}^-\Lambda^k \subset \mathcal{S}_{r+1}\Lambda^k$

Trace: $\mathrm{tr}_f\mathcal{S}_r^-\Lambda^k(\mathbb{R}^n) \subset \mathcal{S}_r^-\Lambda^k(f)$, for any $(n-1)$-hyperplane f in $\mathbb{R}^n$

Special cases:
$\mathcal{S}_r^-\Lambda^0 = \mathcal{S}_r\Lambda^0$
$\mathcal{S}_r^-\Lambda^n = \mathcal{S}_{r-1}\Lambda^n$
$\mathcal{S}_r^-\Lambda^k + d\mathcal{S}_{r+1}\Lambda^{k-1} = \mathcal{S}_r\Lambda^k$.

Polynomial spaces and degrees of freedom

$\mathcal{S}_r^- \Lambda^k(\square_n)$ is a space of differential k-forms whose coefficients are polynomials in $\mathbb{R}^n$.

$$\mathcal{S}_r^- \Lambda^k = \mathcal{P}_r^- \Lambda^k \oplus \mathcal{J}_r \Lambda^k \oplus d\mathcal{J}_r \Lambda^{k-1}$$

Polynomial coefficients in each summand:

$\mathcal{P}_r^- \Lambda^k$: anything up to degree $r-1$ and some degree r

$\mathcal{J}_r \Lambda^k$: certain polynomials whose degree is between $r+1$ and $r+n-k-1$

$d\mathcal{J}_r \Lambda^{k-1}$: certain polynomials whose degree is between r and $r+n-k-2$

The 'regular' serendtipity space has an analogous decomposition:

$$\mathcal{S}_r \Lambda^k = \mathcal{P}_r \Lambda^k \oplus \mathcal{J}_r \Lambda^k \oplus d\mathcal{J}_{r+1} \Lambda^{k-1}$$

– – – – – – – – – – – – –

The **degrees of freedom** associated to a d-dimensional sub-face f of an n-dimensional cube $\square_n$ are (for any $k \leq d \leq \min\{n, \lfloor r/2 \rfloor + k\}$):

$$u \longmapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \underbrace{\mathcal{P}_{r-2(d-k)-1} \Lambda^{d-k}(f)}_{\text{indexing space for } \mathcal{S}_{r-1}\Lambda^k(f)} \oplus d\mathcal{H}_{r-2(d-k)+1} \Lambda^{d-k-1}(f),$$

Dimension count and comparison

Formula for counting degrees of freedom of $\mathcal{S}_r^- \Lambda^k(\square_n)$:

$$\sum_{d=k}^{\min\{n,\lfloor r/2\rfloor+k\}} 2^{n-d}\binom{n}{d}\left(\binom{r-d+2k-1}{r-d+k-1}\binom{r-d+k-1}{d-k}+\binom{r-d+2k}{k}\binom{r-d+k-1}{d-k-1}\right)$$

	k	r=1	2	3	4	5	6	7
n=2	0	4	8	12	17	23	30	38
	1	4	10	17	26	37	50	65
	2	1	3	6	10	15	21	28
n=3	0	8	20	32	50	74	105	144
	1	12	36	66	111	173	255	360
	2	6	21	45	82	135	207	301
	3	1	4	10	20	35	56	84
n=4	0	16	48	80	136	216	328	480
	1	32	112	216	392	656	1036	1563
	2	24	96	216	422	746	1227	1910
	3	8	36	94	200	375	644	1036
	4	1	5	15	35	70	126	210

Outline

New $H(\mathrm{curl})$ and $H(\mathrm{div})$ elements in $\mathbb{R}^3$

The degrees of freedom for $\mathcal{S}_r^-\Lambda^1(\square_3)$ (i.e. $H(\mathrm{curl})$-conforming in $\mathbb{R}^3$):

$$u \longmapsto \int_e u \cdot \vec{t}\, p, \qquad p \in \mathcal{P}_{r-1}(e), \quad e \text{ an edge of } \square_3 \text{ with unit tangent } \vec{t},$$

$$u \longmapsto \int_f (u \times \hat{n}) \cdot \vec{p}, \qquad \vec{p} \in [\mathcal{P}_{r-3}(f)]^2 \oplus \mathrm{grad}\, \mathcal{H}_{r-1}\Lambda^0(f),$$

$$f \text{ a face of } \square_3 \text{ with unit normal } \hat{n},$$

$$u \longmapsto \int_{\square_3} u \cdot \vec{p}, \qquad \vec{p} \in [\mathcal{P}_{r-5}(\square_3)]^3 \oplus \mathrm{curl}\, \mathcal{H}_{r-3}\Lambda^1(\square_3).$$

The degrees of freedom for $\mathcal{S}_r^-\Lambda^2(\square_3)$ (i.e. $H(\mathrm{div})$-conforming in $\mathbb{R}^3$):

$$u \longmapsto \int_f u \cdot \hat{n}\, p, \qquad p \in \mathcal{P}_{r-1}(f), \; f \text{ a face of } \square_3 \text{ with unit normal } \hat{n},$$

$$u \longmapsto \int_{\square_3} u \cdot p, \qquad \vec{p} \in [\mathcal{P}_{r-3}(\square_3)]^3 \oplus \mathrm{grad}\, \mathcal{H}_{r-1}\Lambda^0(\square_3).$$

Dimension Comparison

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$$

We compare degree of freedom counts among the three families for use on meshes of affinely-mapped squares or cubes, when a conforming method with (at least) order r decay in the approximation of p, $\mathbf{u}$, and div $\mathbf{u}$ is desired.

Meshes of squares ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1\rvert + \lvert\mathcal{Q}_r^-\Lambda^2\rvert$	$\lvert\mathcal{S}_r\Lambda^1\rvert + \lvert\mathcal{S}_{r-1}\Lambda^2\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1\rvert + \lvert\mathcal{S}_r^-\Lambda^2\rvert$
1	4+1 = 5	8+1 = 9	4+1 = 5
2	12+4 = 16	14+3 = 17	10+3 = 13
3	24+9 = 33	22+6 = 28	17+6 = 23

Meshes of cubes ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2\rvert + \lvert\mathcal{Q}_r^-\Lambda^3\rvert$	$\lvert\mathcal{S}_{r+1}\Lambda^2\rvert + \lvert\mathcal{S}_r\Lambda^3\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2\rvert + \lvert\mathcal{S}_r^-\Lambda^3\rvert$
1	6+1 = 7	18+1 = 19	6+1 = 7
2	36+8 = 44	39+4 = 43	21+4 = 25
3	108+27 = 135	72+10 = 82	45+10 = 55

Minimality

Theorem [G, Kloefkorn]

$\mathcal{S}_r^- \Lambda^\bullet(\square_n)$ is a minimal compatible finite element system containing $\mathcal{P}_{r-1}\Lambda^\bullet(\square_n)$.

The theorem implies that of all possible families of finite element spaces with the error decay properties of $\mathcal{P}_{r-1}\Lambda^\bullet(\square_n)$, the $\mathcal{S}_r^- \Lambda^k(\square_n)$ family has minimal dimension.

References:

CHRISTIANSEN, G. "Constructions of some minimal finite element systems." Mathematical Modelling and Numerical Analysis, 2016.

G., KLOEFKORN "Trimmed Serendipity Finite Element Differential Forms" `arXiv:1607.00571`, 2016.

Related contemporary work:

ARBOGAST, CORREA "Two families of $H(\text{div})$ mixed finite elements on quadrilaterals of minimal dimension," ICES Report, UT Austin, 2015

BEIRAO DA VEIGA, BREZZI, MARINI, RUSSO "Serendipity nodal VEM spaces" `arXiv:1510.08477`, 2015.

COCKBURN, FU "A systematic construction of finite element commuting exact sequences" `arXiv:1605.00132`, 2016.

Open questions

- Unisolvence in general case.

  We have proved all cases of immediate relevance to applications:

  $1 \leq n \leq 4$, $0 \leq k \leq n$, $1 \leq r \leq 10$.
- Construction of basis functions for implementation
- Extension to non-affine maps of cubes
- Further comparison to other approaches

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Collaborators on this work

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Upcoming Events

Advances in Quadrilateral and Hexahedral Finite Elements

Poster session at SIAM CSE in Atlanta, February 2017.

Slides and Pre-prints

`http://math.arizona.edu/~agillette/`