Cardiac Electrophysiology Modeling with Serendipity Finite Elements

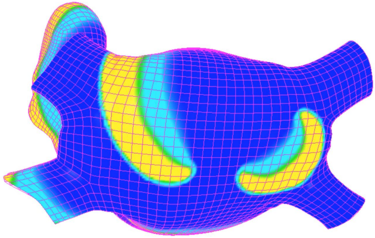
Andrew Gillette

Department of Mathematics
University of Arizona

http://math.arizona.edu/∼agillette/

Monodomain Equations for Electrophysiology

Numerical approximation of solutions to the **monodomain equations** allows the simulation of electrical propagation on patient-specific cardiac geometries.



Find $u, v : \Omega \to \mathbb{R}$ such that

$$\frac{ds}{dt} = F(s, v) \quad \text{in } \Omega$$

$$\frac{\lambda}{1+\lambda} \nabla \cdot (M_i \nabla v) = v_t + I(v, s) \quad \text{in } \Omega$$

$$-\frac{1}{1+\lambda} \nabla \cdot (M_i \nabla v) = \nabla \cdot (M_i \nabla u) \quad \text{in } \Omega$$

$$(M_i \nabla v) \cdot \hat{n} = 0 \quad \text{on } \partial\Omega$$

$$(M_i \nabla u) \cdot \hat{n} = 0 \quad \text{on } \partial\Omega$$

v = transmembrane potential

u = extracellular potential

s = state of ionic currents

$F(v, s)$ = voltage-dependent state update

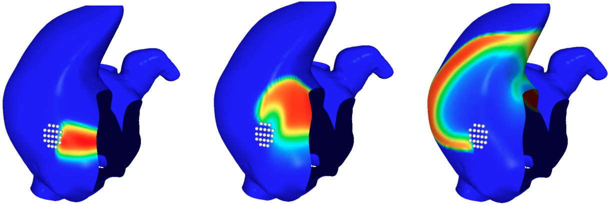
$I(v, s)$ = ionic current across membrane

M_i, M_e = intra-, extra-cellular conductivity tensors, satisfying $M_e = \lambda M_i$

Application Motivations and Challenges

Scientific and clinical motivations:

- Provide models of electrical activity at spatial and temporal resolutions beyond MRI and EKG data.
- Allow patient-specific modeling for ‘real-time’ simulations during surgery.
- Analyze ‘spiral waves’ that occur during atrial fibrillation:



[show videos]

Competing theories of the biophysical nature of this phenomenon have been proposed; computational simulations and mathematical analysis can provide evidence for or against treatment strategies (e.g. where to cauterize heart cells)

Application Motivations and Challenges

Computational challenges:

- Survey of various codes on a ‘benchmark problem’ revealed large discrepancies in the estimation of activation time, a basic measure of accuracy.
- In the application context,

$$\text{Péclet number} = \frac{k \cdot h^2}{6 \cdot D} = \frac{(\text{reaction rate}) \cdot (\text{mesh element length})^2}{6 \cdot (\text{diffusivity constant})} > 1$$

  which can lead to Gibbs type phenomena of spurious oscillations. This can be overcome either by using many mesh elements (small h values) or higher order methods, both of which can be computationally expensive.
- Using different functions for geometry and function approximation is expensive; unified methods are needed.

Serendipity finite element methods help address these challenges.

Table of Contents

Outline

1. Introduction to Serendipity FEM
2. The Cubic Tensor Product Case
3. The Cubic Serendipity Space Case
4. Future Directions

What is a serendipity finite element method?

Goal: Efficient, accurate approximation of the solution to a PDE over $\Omega \subset \mathbb{R}^n$.

Standard $O(h^r)$ **tensor product** finite element method in $\mathbb{R}^n$:

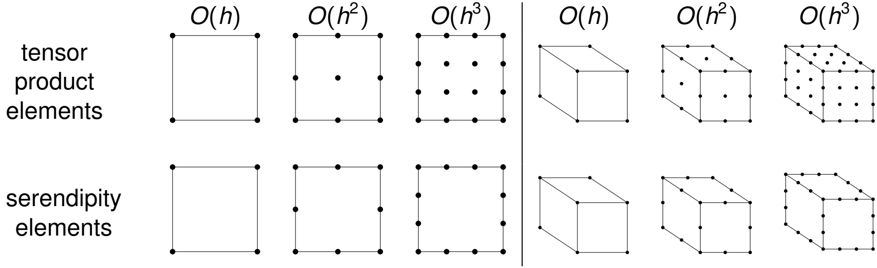
$\rightarrow$ Mesh Ω by n-dimensional cubes of side length h.

$\rightarrow$ Set up a linear system involving $(r+1)^n$ degrees of freedom (DoFs) per cube.

$\rightarrow$ For unknown continuous solution u and computed discrete approximation u_h:

$$\underbrace{||u - u_h||_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C\, h^r\, |u|_{H^{r+1}(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^{r+1}(\Omega).$$

A $O(h^r)$ **serendipity** FEM converges at the **same rate** with **fewer DoFs** per element:

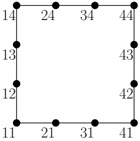


Example: For $O(h^3)$, $d = 3$, 50% fewer DoFs $\rightarrow$ $\approx$ 50% smaller linear system

What is a geometric decomposition?

A **geometric decomposition** for a finite element space is an explicit correspondence:

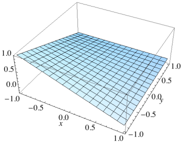
$$\{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, x^3y, xy^3\} \longleftrightarrow \{\vartheta_{11}, \vartheta_{14}, \vartheta_{41}, \vartheta_{44}, \vartheta_{12}, \vartheta_{13}, \vartheta_{42}, \vartheta_{43}, \vartheta_{21}, \vartheta_{31}, \vartheta_{24}, \vartheta_{34}\} \longleftrightarrow$$



monomials $\longleftrightarrow$ **basis functions** $\longleftrightarrow$ **domain points**

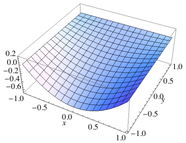
$\rightarrow$ Previously known basis functions employ Legendre polynomials

$\rightarrow$ These functions bear no symmetrical correspondence to the domain points



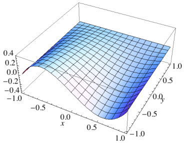
$\frac{1}{4}(x-1)(y-1)$

vertex



$-\frac{1}{4}\sqrt{\frac{3}{2}}\left(x^2-1\right)(y-1)$

edge (quadratic)



$-\frac{1}{4}\sqrt{\frac{5}{2}}x\left(x^2-1\right)(y-1)$

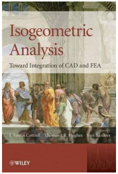
edge (cubic)

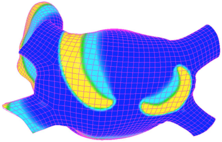
Szabó and I. Babuška *Finite element analysis*, Wiley Interscience, 1991.

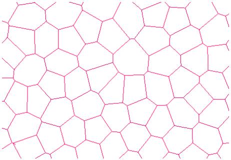
A 'canonical' geometric decomposition can aid in higher order generalizations.

Motivations and Related Topics

Goal: Construct geometric decompositions of serendipity spaces using linear combinations of standard tensor product functions. **Focus:** Cubic Hermites.







- **Isogeometric analysis:** Finding basis functions suitable for both domain description and PDE approximation avoids the expensive computational bottleneck of re-meshing.

  COTTRELL, HUGHES, BAZILEVS *Isogeometric Analysis: Toward Integration of CAD and FEA*, Wiley, 2009.

- **Modern mathematics:** Finite Element Exterior Calculus, Discrete Exterior Calculus, Virtual Element Methods. . .

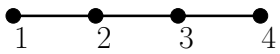
  ARNOLD, AWANOU *The serendipity family of finite elements*, Found. Comp. Math, 2011.
  DA VEIGA, BREZZI, CANGIANI, MANZINI, RUSSO *Basic Principles of Virtual Element Methods*, M3AS, 2013.

- **Flexible Domain Meshing:** Serendipity type elements for Voronoi meshes provide computational benefits without need of tensor product structure.

  RAND, G., BAJAJ *Quadratic Serendipity Finite Elements on Polygons Using Generalized Barycentric Coordinates*, Mathematics of Computation, in press.

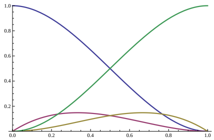
Outline

Cubic Hermite Geometric Decomposition: 1D

$\{1, x, x^2, x^3\}$	$\longleftrightarrow$ *approxim.* $\longleftrightarrow$	$\{\psi_1, \psi_2, \psi_3, \psi_4\}$	$\longleftrightarrow$ *geometry* $\longleftrightarrow$	
monomials		**basis functions**		**domain points**

Cubic Hermite Basis on [0, 1]

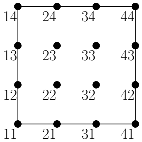
$$\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix} := \begin{bmatrix} 1 - 3x^2 + 2x^3 \\ x - 2x^2 + x^3 \\ x^2 - x^3 \\ 3x^2 - 2x^3 \end{bmatrix}$$



Approximation: $x^r = \sum_{i=1}^{4} \varepsilon_{r,i}\psi_i$, for $r = 0, 1, 2, 3$, where $[\varepsilon_{r,i}] = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -3 & 1 \end{pmatrix}$

Geometry: $u = u(0)\psi_1 + u'(0)\psi_2 - u'(1)\psi_3 + u(1)\psi_4, \qquad \forall u \in \underbrace{\mathcal{P}_3([0,1])}_{\text{cubic polynomials}}$

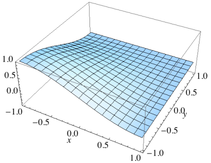
$$\underbrace{\left\{ \begin{array}{c} x^r y^s \\ 0 \leq r,s \leq 3 \end{array} \right\}}_{\mathcal{Q}_3([0,1]^2)} \longleftrightarrow \left\{ \begin{array}{c} \psi_i(x)\psi_j(y) \\ 1 \leq i,j \leq 4 \end{array} \right\} \longleftrightarrow$$



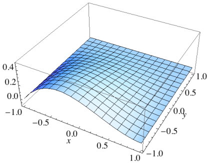
monomials $\longleftrightarrow$ **basis functions** $\longleftrightarrow$ **domain points**

Approximation: $x^r y^s = \sum_{i=1}^{4} \sum_{j=1}^{4} \varepsilon_{r,i}\varepsilon_{s,j}\psi_{ij}$, for $0 \leq r,s \leq 3$, $\varepsilon_{r,i}$ as in 1D.

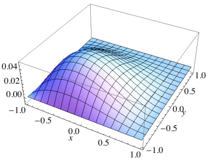
Geometry:



ψ_{11}



ψ_{21}

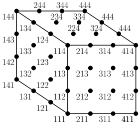


ψ_{22}

$$u = u|_{(0,0)}\psi_{11} + \partial_x u|_{(0,0)}\psi_{21} + \partial_y u|_{(0,0)}\psi_{12} + \partial_x\partial_y u|_{(0,0)}\psi_{22} + \cdots, \qquad \forall u \in \mathcal{Q}_3([0,1]^2)$$

Cubic Hermite Geometric Decomposition: 3D

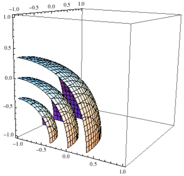
$$\underbrace{\left\{ \begin{array}{c} x^r y^s z^t \\ 0 \le r, s, t \le 3 \end{array} \right\}}_{\mathcal{Q}_3([0,1]^3)} \quad \longleftrightarrow \quad \begin{array}{c} \psi_i(x)\psi_j(y)\psi_k(z) \\ 1 \le i, j, k \le 4 \end{array} \quad \longleftrightarrow$$



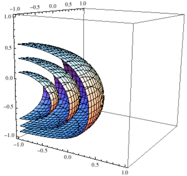
monomials $\longleftrightarrow$ **basis functions** $\longleftrightarrow$ **domain points**

Approximation: $x^r y^s z^t = \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{k=1}^{4} \varepsilon_{r,i}\varepsilon_{s,j}\varepsilon_{t,k}\psi_{ijk}$, for $0 \le r, s, t \le 3$, $\varepsilon_{r,i}$ as in 1D.

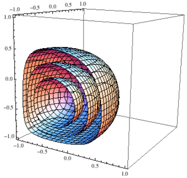
Geometry: Contours of level sets of the basis functions:



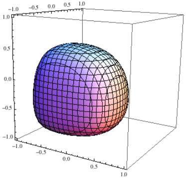
ψ_{111}



ψ_{112}



ψ_{212}



ψ_{222}

Two families of finite elements on cubical meshes

$\mathcal{Q}_r^-\Lambda^k([0,1]^n) \longrightarrow$ tensor product spaces ($\leq$ degree r in each variable)

early work: RAVIART, THOMAS 1976, NEDELEC 1980
more recently: ARNOLD, BOFFI, BONIZZONI arXiv:1212.6559, 2012

$\mathcal{S}_r\Lambda^k([0,1]^n) \longrightarrow$ serendipity finite element spaces (superlinear degree r)

early work: STRANG, FIX *An analysis of the finite element method* 1973
more recently: ARNOLD, AWANOU FoCM 11:3, 2011, and arXiv:1204.2595, 2012.

The **superlinear** degree of a polynomial ignores linearly-appearing variables.

$$n=2: \quad \{\overbrace{\underbrace{1,x,y,x^2,y^2,xy,x^3,y^3,x^2y,xy^2,x^3y,xy^3}_{\mathcal{S}_3\Lambda^0([0,1]^2)\ \text{(dim=12)}},x^2y^2,x^3y^2,x^2y^3,x^3y^3}^{\mathcal{Q}_3\Lambda^0([0,1]^2)\ \text{(dim=16)}}\}$$

$$n=3: \quad \{\overbrace{\underbrace{1,\ldots,xyz,\ \ x^3y,x^3z,y^3z,\ldots,x^3yz,xy^3z,xyz^3}_{\mathcal{S}_3\Lambda^0([0,1]^3)\ \text{(dim=32)}},x^3y^2,\ldots,x^3y^3z^3}^{\mathcal{Q}_3\Lambda^0([0,1]^3)\ \text{(dim=64)}}\}$$

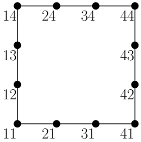
$\mathcal{Q}_r^-\Lambda^k$ and $\mathcal{S}_r\Lambda^k$ and have the **same** key mathematical properties needed for FEEC
(degree, inclusion, trace, subcomplex, unisolvence, commuting projections)
but for fixed $k \geq 0$, $r, n \geq 2$ the serendipity spaces have **fewer** degrees of freedom

Outline

Cubic Hermite Serendipity Geom. Decomp: 2D

Theorem [G, 2012]: A Hermite-like geometric decomposition of $\mathcal{S}_3([0,1]^2)$ exists.

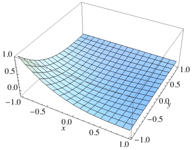
$$\underbrace{\left\{ \begin{matrix} x^r y^s \\ \text{sldeg} \leq 3 \end{matrix} \right\}}_{\mathcal{S}_3([0,1]^2)} \longleftrightarrow \left\{ \begin{matrix} \vartheta_{\ell m} \\ \text{(to be defined)} \end{matrix} \right\} \longleftrightarrow$$



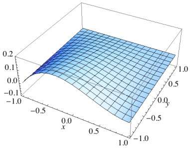
monomials $\longleftrightarrow$ **basis functions** $\longleftrightarrow$ **domain points**

Approximation: $x^r y^s = \sum_{\ell m} \varepsilon_{r,i} \varepsilon_{s,j} \vartheta_{\ell m}$, for *superlineardegree*$(x^r y^s) \leq 3$

Geometry:



ϑ_{11}



ϑ_{21}

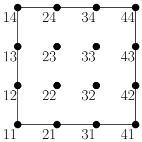
$$\begin{aligned} u \quad &= u|_{(0,0)} \vartheta_{11} \\ &+ \partial_x u|_{(0,0)} \vartheta_{21} \\ &+ \partial_y u|_{(0,0)} \vartheta_{12} \\ &+ \cdots \end{aligned}$$

$\forall u \in \mathcal{S}_3([0,1]^2)$

Proof Overview:

1 Fix index sets and basis orderings based on domain points:

$$\begin{aligned} V &= \text{vertices } (11, 14, \ldots) \\ E &= \text{edges } (12, 13, \ldots) \\ D &= \text{interior } (22, 23, \ldots) \end{aligned} \quad \longleftrightarrow$$



$$[\vartheta_{\ell m}] := [\, \vartheta_{11}, \vartheta_{14}, \vartheta_{41}, \vartheta_{44}, \quad \vartheta_{12}, \vartheta_{13}, \vartheta_{42}, \vartheta_{43}, \vartheta_{21}, \vartheta_{31}, \vartheta_{24}, \vartheta_{34} \,],$$

$$[\psi_{ij}] := [\, \underbrace{\psi_{11}, \psi_{14}, \psi_{41}, \psi_{44},}_{\text{indices in } V} \quad \underbrace{\psi_{12}, \psi_{13}, \psi_{42}, \psi_{43}, \psi_{21}, \psi_{31}, \psi_{24}, \psi_{34},}_{\text{indices in } E} \quad \underbrace{\psi_{22}, \psi_{23}, \psi_{32}, \psi_{33}}_{\text{indices in } D} \,]$$

2 Define a 12×16 matrix $\mathbb{H}$ with entries $h_{ij}^{\ell m}$ so that $\ell m \in V \cup E$, $ij \in V \cup E \cup D$.

3 Define the serendipity basis functions $\vartheta_{\ell m}$ via

$$\boxed{[\vartheta_{\ell m}] := \mathbb{H}[\psi_{ij}]}$$

and show that the approximation and geometry properties hold.

Cubic Hermite Serendipity Geom. Decomp: 2D

Proof Details:

2 Define a 12×16 matrix $\mathbb{H}$ with entries $h_{ij}^{\ell m}$ so that $\ell m \in V \cup E$, $ij \in V \cup E \cup D$.

$$\mathbb{H} := \left[\begin{array}{c|cccc} & -1 & 1 & 1 & -1 \\ & 1 & -1 & -1 & 1 \\ & 1 & -1 & -1 & 1 \\ & -1 & 1 & 1 & -1 \\ & -1 & 0 & 1 & 0 \\ \mathbb{I} & 0 & -1 & 0 & 1 \\ \text{(12x12 identity matrix)} & 1 & 0 & -1 & 0 \\ & 0 & 1 & 0 & -1 \\ & -1 & 1 & 0 & 0 \\ & 0 & 0 & -1 & 1 \\ & 1 & -1 & 0 & 0 \\ & 0 & 0 & 1 & -1 \end{array}\right]$$

3 Define $[\vartheta_{\ell m}] := \mathbb{H}[\psi_{ij}]$. The geometry property holds since for $\ell m \in V \cup E$,

$$\vartheta_{\ell m} = \underbrace{\psi_{\ell m}}_{\text{bicubic Hermite}} + \underbrace{\sum_{ij \in D} h_{ij}^{\ell m} \psi_{ij}}_{\text{zero on boundary}} \implies \vartheta_{\ell m} \equiv \psi_{\ell m} \text{ on edges}$$

Proof Details:

To prove that the approximation property holds, observe:

$$[\vartheta_{\ell m}] := \mathbb{H}[\psi_{ij}] \quad \text{implies} \quad \sum_{ij} h_{ij}^{\ell m}\psi_{ij} = \vartheta_{\ell m}$$

For all (r, s) pairs such that $\text{sldeg}(x^r y^s) \leq 3$, the matrix entries in column ij satisfy

$$\varepsilon_{r,i}\varepsilon_{s,j} = \sum_{\ell m \in V \cup E} \varepsilon_{r,\ell}\varepsilon_{s,m}h_{ij}^{\ell m}$$

Substitute these into the Hermite 2D approximation property:

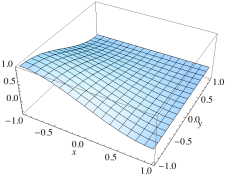
$$x^r y^s = \sum_{ij \in V \cup E \cup D} \varepsilon_{r,i}\varepsilon_{s,j}\psi_{ij} = \sum_{ij}\sum_{\ell m} \varepsilon_{r,\ell}\varepsilon_{s,m}h_{ij}^{\ell m}\psi_{ij}$$

$$= \sum_{\ell m} \varepsilon_{r,\ell}\varepsilon_{s,m}\sum_{ij} h_{ij}^{\ell m}\psi_{ij} = \sum_{\ell m} \varepsilon_{r,\ell}\varepsilon_{s,m}\vartheta_{\ell m}$$

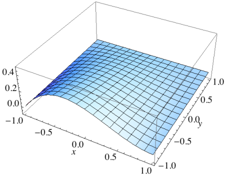
Hence $[\vartheta_{\ell m}]$ is a basis for $\mathcal{S}_2([0,1]^2)$, completing the geometric decomposition. □

Hermite Style Serendipity Functions (2D)

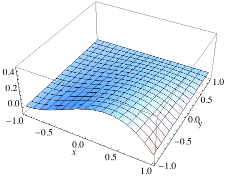
$$
[\vartheta_{\ell m}] = \begin{bmatrix} \vartheta_{11} \\ \vartheta_{14} \\ \vartheta_{41} \\ \vartheta_{44} \\ \vartheta_{12} \\ \vartheta_{13} \\ \vartheta_{42} \\ \vartheta_{43} \\ \vartheta_{21} \\ \vartheta_{31} \\ \vartheta_{24} \\ \vartheta_{34} \end{bmatrix} = \begin{bmatrix} -(x-1)(y-1)(-2+x+x^2+y+y^2) \\ (x-1)(y+1)(-2+x+x^2-y+y^2) \\ (x+1)(y-1)(-2-x+x^2+y+y^2) \\ -(x+1)(y+1)(-2-x+x^2-y+y^2) \\ -(x-1)(y-1)^2(y+1) \\ (x-1)(y-1)(y+1)^2 \\ (x+1)(y-1)^2(y+1) \\ -(x+1)(y-1)(y+1)^2 \\ -(x-1)^2(x+1)(y-1) \\ (x-1)(x+1)^2(y-1) \\ (x-1)^2(x+1)(y+1) \\ -(x-1)(x+1)^2(y+1) \end{bmatrix} \cdot \frac{1}{8}
$$



ϑ_{11}



ϑ_{21}

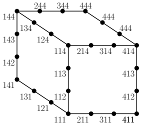


ϑ_{31}

Cubic Hermite Serendipity Geom. Decomp: 3D

Theorem [G, 2012]: A Hermite-like geometric decomposition of $\mathcal{S}_3([0,1]^3)$ exists.

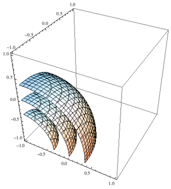
$$\underbrace{\left\{ \begin{array}{c} x^r y^s z^t \\ \text{sldeg} \leq 3 \end{array} \right\}}_{\mathcal{S}_3([0,1]^3)} \longleftrightarrow \left\{ \begin{array}{c} \vartheta_{\ell mn} \\ \text{(to be defined)} \end{array} \right\} \longleftrightarrow$$



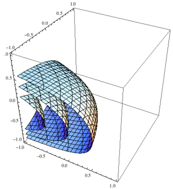
monomials $\longleftrightarrow$ **basis functions** $\longleftrightarrow$ **domain points**

Approximation: $x^r y^s z^t = \sum_{\ell mn} \varepsilon_{r,i} \varepsilon_{s,j} \varepsilon_{t,k} \vartheta_{\ell mn}$, for *superlineardegree*$(x^r y^s z^t) \leq 3$

Geometry:



ϑ_{111}



ϑ_{112}

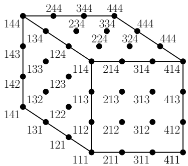
$$\begin{aligned} u &= u|_{(0,0,0)} \vartheta_{111} \\ &+ \partial_x u|_{(0,0,0)} \vartheta_{211} \\ &+ \partial_y u|_{(0,0,0)} \vartheta_{121} \\ &+ \partial_z u|_{(0,0,0)} \vartheta_{112} \\ &+ \cdots \end{aligned}$$
$$\forall u \in \mathcal{S}_3([0,1]^3)$$

Cubic Hermite Serendipity Geom. Decomp: 3D

Proof Overview:

1 Fix index sets and basis orderings based on domain points:

$$\begin{aligned} V &= \text{vertices } (111, \ldots) \\ E &= \text{edges } (112, \ldots) \\ F &= \text{face interior } (122, \ldots) \\ M &= \text{volume interior } (222, \ldots) \end{aligned} \quad \longleftrightarrow$$



$$[\vartheta_{\ell mn}] := [\ \vartheta_{111}, \ldots, \vartheta_{444},\ \ \vartheta_{112}, \ldots, \vartheta_{443}\ ],$$

$$[\psi_{ijk}] := [\ \underbrace{\psi_{111}, \ldots, \psi_{444},}_{\text{indices in } V}\ \underbrace{\psi_{112}, \ldots, \psi_{443},}_{\text{indices in } E}\ \underbrace{\psi_{122}, \ldots, \psi_{433},}_{\text{indices in } F}\ \underbrace{\psi_{222}, \ldots, \psi_{333}}_{\text{indices in } M}\ ]$$

2 Define a 32×64 matrix $\mathbb{W}$ with entries $h_{ijk}^{\ell mn}$ (where $\ell mn \in V \cup E$)

3 Define the serendipity basis functions $\vartheta_{\ell mn}$ via

$$\boxed{[\vartheta_{\ell mn}] := \mathbb{W}[\psi_{ijk}]}$$

and show that the approximation and geometry properties hold.

Proof Details:

2 Define a 32×64 matrix $\mathbb{W}$ with entries $h_{ijk}^{\ell mn}$ so that $\ell mn \in V \cup E$

$$\mathbb{W} := \left[\begin{array}{c|c} \mathbb{I} & \text{specific full rank} \\ \text{(32x32 identity matrix)} & \text{32x32 matrix} \\ & \text{with entries -1, 0, or 1} \end{array} \right]$$

3 Define $[\vartheta_{\ell mn}] := \mathbb{W}[\psi_{ijk}]$.

$\longrightarrow$ Confirm directly that $[\vartheta_{\ell mn}]$ restricts to $[\vartheta_{\ell m}]$ on faces.

$\longrightarrow$ Similar proof technique confirms geometry and approximation properties.

$$[\vartheta_{\ell mn}] = \begin{bmatrix} \vartheta_{111} \\ \vartheta_{114} \\ \vdots \\ \vartheta_{442} \\ \vartheta_{443} \end{bmatrix} = \begin{bmatrix} (x-1)(y-1)(z-1)(-2+x+x^2+y+y^2+z+z^2) \\ -(x-1)(y-1)(z+1)(-2+x+x^2+y+y^2-z+z^2) \\ \vdots \\ (x+1)(y+1)(z-1)^2(z+1) \\ -(x+1)(y+1)(z-1)(z+1)^2 \end{bmatrix} \cdot \frac{1}{16}$$

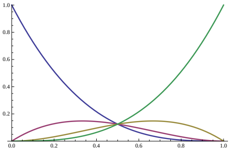
Complete list and more details in my paper:

GILLETTE *Hermite and Bernstein Style Basis Functions for Cubic Serendipity Spaces on Squares and Cubes*, arXiv:1208.5973, 2012

Cubic Bernstein Serendipity Geom. Decomp: 2D, 3D

Cubic Bernstein Basis on $[0,1]$

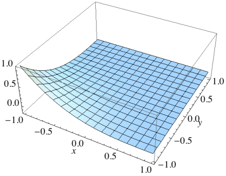
$$\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix} := \begin{bmatrix}(1-x)^3\\ (1-x)^2x\\ (1-x)x^2\\ x^3\end{bmatrix}$$



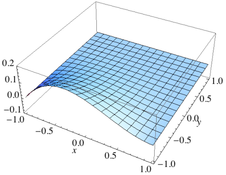
Theorem [G, 2012]: Bernstein-like geometric decompositions of $\mathcal{S}_3([0,1]^2)$ and $\mathcal{S}_3([0,1]^3)$ exist.

$$[\xi_{\ell m}] = \begin{bmatrix}\xi_{11}\\ \xi_{14}\\ \vdots\\ \xi_{24}\\ \xi_{34}\end{bmatrix} = \begin{bmatrix}(x-1)(y-1)(-2-2x+x^2-2y+y^2)\\ -(x-1)(y+1)(-2-2x+x^2+2y+y^2)\\ \vdots\\ (x-1)^2(x+1)(y+1)\\ -(x-1)(x+1)^2(y+1)\end{bmatrix} \cdot \frac{1}{16}$$

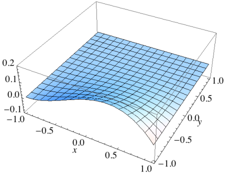
$$[\xi_{\ell mn}] = \begin{bmatrix}\xi_{111}\\ \xi_{114}\\ \vdots\\ \xi_{442}\\ \xi_{443}\end{bmatrix} = \begin{bmatrix}-(x-1)(y-1)(z-1)(-5-2x+x^2-2y+y^2-2z+z^2)\\ (x-1)(y-1)(z+1)(-5-2x+x^2-2y+y^2+2z+z^2)\\ \vdots\\ (x+1)(y+1)(z-1)^2(z+1)\\ -(x+1)(y+1)(z-1)(z+1)^2\end{bmatrix} \cdot \frac{1}{32}$$



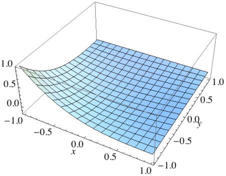
β_{11}



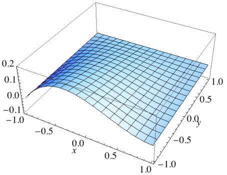
β_{21}



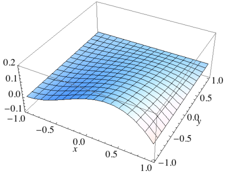
β_{31}



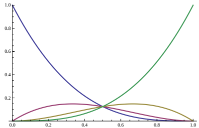
ξ_{11}



ξ_{21}



ξ_{31}



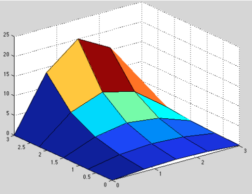
Bicubic Bernstein functions (top) and Bernstein-style serendipity functions (bottom).

$\rightarrow$ Note boundary agreement with Bernstein functions.

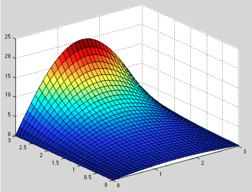
Computational Evidence

Serendipity cubic Hermite-like method implemented in Matlab to solve Poisson's equation with exact solution $u(x,y) = \sin(x)\, e^y$

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	2.89e-1		1.44e+0	
4	1.32e-2	4.4	1.62e-1	3.1
8	6.90e-4	4.3	1.82e-2	3.2
16	3.83e-5	4.2	2.09e-3	3.1
32	2.22e-6	4.1	2.48e-4	3.1



$n = 2$



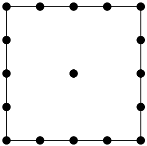
$n = 32$

Confirms the expected **cubic** order *a priori* error estimate

$$\underbrace{||u - u_h||_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C\, h^{3}\, |u|_{H^4(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^{4}(\Omega),$$

Outline

Serendipity spaces for large r



Let $p(x,y) := (1+x)(1-x)(1-y)(1+y)$.

Observe $p \in \mathcal{P}_4 \subset \mathcal{S}_4$, but $p \equiv 0$ on $\partial([0,1]^2)$.

When $r > 3$, there are interior domain points for the serendipity spaces as characterized in

Arnold, Awanou *The serendipity family of finite elements*, Found. Comp. Math, 2011.

	1	2	3	4	5	6	7	$r \geq 2n$
$n = 2$								
dim $\mathcal{Q}_r$	4	9	16	25	36	49	64	$r^2 + 2r + 1$
dim $\mathcal{S}_r$	4	8	12	17	23	30	38	$\frac{1}{2}(r^2 + 3r + 6)$
$n = 3$								
dim $\mathcal{Q}_r$	8	27	64	125	216	343	512	$r^3 + 3r^2 + 3r + 1$
dim $\mathcal{S}_r$	8	20	32	50	74	105	144	$\frac{1}{6}(r^3 + 6r^2 + 29r + 24)$

- Thus, when $r \geq 2n$, the expected computational savings by serendipity methods is **50% in 2D** and **83% in 3D**.
- I am currently pursuing related theoretical results with Snorre Christiansen and Michael Floater (both from U. Oslo).

Acknowledgments

UC San Diego:

Matt Gonzalez
Kevin Vincent
Poorya Mirkhosravi

Andrew McCulloch
Michael Holst

National Biomedical Computation Resource

Slides and pre-prints: `http://math.arizona.edu/~agillette/`