# (Trimmed) Serendipity Finite Element Methods in Theory and Practice

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*joint work with*
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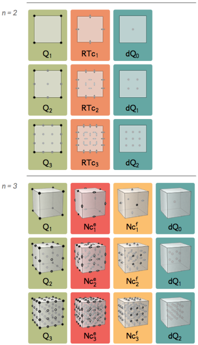


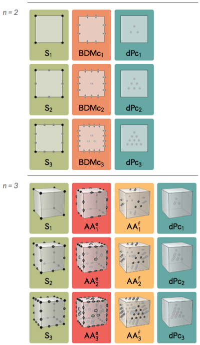


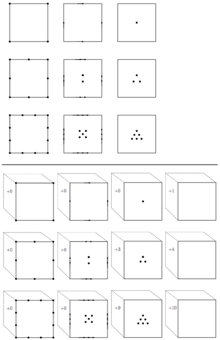

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3. Trimmed serendipity finite elements
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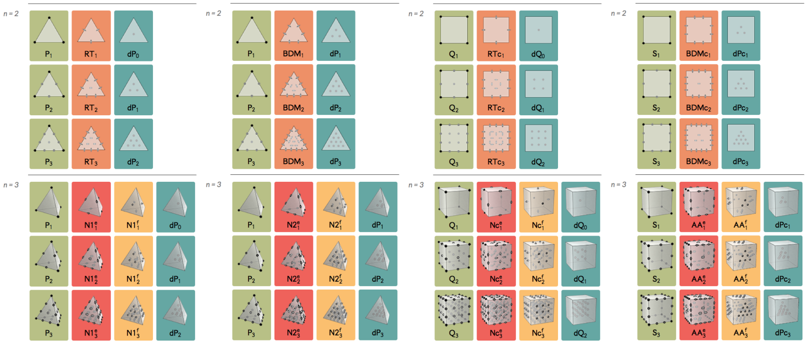
# The ‘Periodic Table of the Finite Elements’

ARNOLD, LOGG, “Periodic table of the finite elements,” *SIAM News*, 2014.




Classification of many common conforming finite element types.

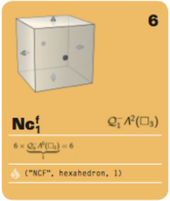
$n \rightarrow$ Domains in $\mathbb{R}^2$ (top half) and in $\mathbb{R}^3$ (bottom half)
$r \rightarrow$ Order $1, 2, 3$ of error decay (going down columns)
$k \rightarrow$ Conformity type $k = 0, \ldots, n$ (going across a row)

Geometry types: Simplices (left half) and cubes (right half).

# Classification of conforming methods

Conforming finite element method types can be broadly classified by three integers:

$n \rightarrow$ the spatial dimension of the domain
$r \rightarrow$ the order of error decay
$k \rightarrow$ the differential form order of the solution space




**Ex:** $\mathcal{Q}_1^-\Lambda^2(\square_3)$ is an element for

$n = 3 \rightarrow$ domains in $\mathbb{R}^3$
$r = 1 \rightarrow$ linear order of error decay
$k = 2 \rightarrow$ conformity in $\Lambda^2(\mathbb{R}^3) \rightsquigarrow H(\mathrm{div})$

$\mathcal{Q}_1^-\Lambda^2(\square_3)$ is part of the $\mathcal{Q}^-$ 'column' of elements,
is defined on geometry $\square_3$ (i.e. a cube),
has a **6** dimensional space of test functions,
and has an associated set of **6** degrees of freedom
that are unisolvent for the test function space.

# An abbreviated reading list (50 years of theory!)

RAVIART, THOMAS, "A mixed finite element method for 2nd order elliptic problems" *Lecture Notes in Mathematics*, 1977 $\leftarrow$ *3172 citations, including 150 from 2017!*

NÉDÉLEC, "Mixed finite elements in $\mathbb{R}^3$," *Numerische Mathematik*, 1980

BREZZI, DOUGLAS JR., MARINI, "Two families of mixed finite elements for second order elliptic problems," *Numerische Mathematik*, 1985

NÉDÉLEC, "A new family of mixed finite elements in $\mathbb{R}^3$," *Numerische Mathematik*, 1986

ARNOLD, FALK, WINTHER "Finite element exterior calculus, homological techniques, and applications," *Acta Numerica*, 2006

CHRISTIANSEN, "Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension," *Numerische Mathematik*, 2007

ARNOLD, FALK, WINTHER "Finite element exterior calculus: from Hodge theory to numerical stability," *Bulletin of the AMS*, 2010

ARNOLD, AWANOU "The serendipity family of finite elements ", *Found. Comp Math*, 2011

ARNOLD, AWANOU "Finite element differential forms on cubical meshes", *Math Comp.*, 2013

ARNOLD, BOFFI, BONIZZONI "Finite element differential forms on curvillinear meshes and their approximation properties," *Numerische Mathematik*, 2014

# $H(\text{div})$ / $L^2$ mixed form of Poisson problem

Derivation of a mixed method for the **Poisson** problem on a domain $\Omega \subset \mathbb{R}^3$:

Given $f : \Omega \to \mathbb{R}$, find a function $p \in H^2(\Omega)$ such that

$$\Delta p + f = 0, \qquad \text{in } \Omega, + \text{B.C.'s}$$

Writing this as a first order system: find $\mathbf{u} \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} \text{div } \mathbf{u} + f &= 0, && \text{in } \Omega, \\ \mathbf{u} - \text{grad } p &= 0, && \text{in } \Omega, \\ (\partial\Omega \text{ conditions}) &= 0 \end{aligned}$$

A **weak form** of these equations: find $\mathbf{u} \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} (\text{div } \mathbf{u}, w) + (f, w) &= 0, && \forall\ w \in L^2 = \Lambda^3(\Omega) \\ (\mathbf{u}, \mathbf{v}) + (p, \text{div } \mathbf{v}) &= 0, && \underbrace{\forall\ \mathbf{v} \in H(\text{div})}_{\text{i.e. } \mathbf{v},\ \text{div } \mathbf{v} \in L^2(\Omega)} = \underbrace{\Lambda^2(\Omega)}_{\text{differential form notation}} \end{aligned}$$

A conforming mixed **finite element** method: find $\mathbf{u}_h \in \Lambda_h^2$ and $p \in \Lambda_h^3$ such that

$$\begin{aligned} (\text{div } \mathbf{u}_h, w_h) + (f, w_h) &= 0 && \forall\ w_h \in \Lambda_h^3 \subset L^2(\Omega) \\ (\mathbf{u}_h, \mathbf{v}_h) + (p_h, \text{div } \mathbf{v}_h) &= [\partial\Omega \text{ terms}] && \forall\ \mathbf{v}_h \in \Lambda_h^2 \subset H(\text{div}) \\ (\partial\Omega \text{ conditions}) &= 0 \end{aligned}$$

# A conforming mixed method for Darcy Flow

Movement of a fluid through porous media modeled via **Darcy flow**:

Given $f$ and $g$, find pressure $p$ and velocity $\mathbf{u}$ such that:

$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 && \text{in } \Omega, \\ \operatorname{div}\, \mathbf{u} - f &= 0 && \text{in } \Omega, \\ p &= g && \text{on } \partial\Omega, \end{aligned}$$

where $K$ is a symmetric, uniformly positive definite tensor for $\dfrac{\text{permeability}}{\text{viscosity}}$.

A **weak form** of these equations: find $\mathbf{u} \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} (K^{-1}\mathbf{u}, \mathbf{v}) - (p, \operatorname{div}\, \mathbf{v}) &= [\partial\Omega \text{ terms}] && \forall\, \mathbf{v} \in H(\text{div}) \\ (\operatorname{div}\, \mathbf{u}, w) - (f, w) &= 0 && \forall\, w \in L^2(\Omega) \\ (\partial\Omega \text{ conditions}) &= 0 \end{aligned}$$

A conforming mixed **finite element** method: find $\mathbf{u}_h \in \Lambda^2_h$ and $p \in \Lambda^3_h$ such that

$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div}\, \mathbf{v}_h) &= [\partial\Omega \text{ terms}] && \forall\, \mathbf{v}_h \in \Lambda^2_h && \subset H(\text{div}) \\ (\operatorname{div}\, \mathbf{u}_h, w_h) - (f, w_h) &= 0 && \forall\, w_h \in \Lambda^3_h && \subset L^2(\Omega) \\ (\partial\Omega \text{ conditions}) &= 0 \end{aligned}$$

Arbogast, Pencheva, Wheeler, Yotov "A Multiscale Mortar Mixed Finite Element Method" *Multiscale Modeling and Simulation* (SIAM) 6:1, 2007.

# Stable pairs of finite element spaces

$$\begin{aligned} (\mathbf{u}_h, \mathbf{v}_h) + (p_h, \mathrm{div}\ \mathbf{v}_h) &= [\partial\Omega \text{ terms}] & \forall\ \mathbf{v}_h \in \Lambda_h^2 &\subset H(\mathrm{div}) \\ (\mathrm{div}\ \mathbf{u}_h, w_h) + (f, w_h) &= 0 & \forall\ w_h \in \Lambda_h^3 &\subset L^2(\Omega) \end{aligned}$$

Given a selection for the finite element spaces $(\Lambda_h^2, \Lambda_h^3)$,

the method is said to be **stable** if the error in the computed solution $(\mathbf{u}_h, p_h)$ is within a constant multiple $C$ of the minimal *possible* error. That is:

$$||\mathbf{u} - \mathbf{u}_h||_{H(\mathrm{div})} + ||p - p_h||_{L^2} \leq C \left( \inf_{\mathbf{w}\in\Lambda_h^2} ||\mathbf{u} - \mathbf{w}||_{H(\mathrm{div})} + \inf_{q\in\Lambda_h^3} ||p - q||_{L^2} \right) \quad (*)$$

Brezzi's theorem establishes the following sufficient criteria for $(*)$:

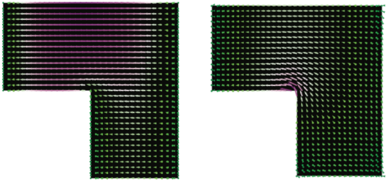
$$(\mathbf{w}, \mathbf{w}) \geq c\,||\mathbf{w}||^2_{H(\mathrm{div})}, \qquad \forall\ \mathbf{w} \in \mathbf{Z}_h := \left\{ \mathbf{w} \in \Lambda_h^2 \ :\ (\mathrm{div}\ \mathbf{w}, q) = 0,\ \ \forall\ q \in \Lambda_h^3 \right\},$$

$$\sup_{\mathbf{w}\in\Lambda_h^2} \frac{(\mathrm{div}\ \mathbf{w}, q)}{||\mathbf{w}||_{H(\mathrm{div})}} \geq c\,||q||_{L^2}, \qquad \forall\ q \in \Lambda_h^3.$$

If the pair $(\Lambda_h^2, \Lambda_h^3)$ satisfies these two criteria it is called a **stable pair**.

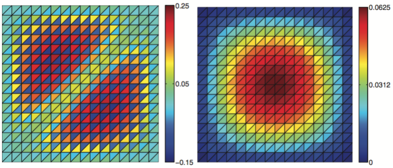
BREZZI, "On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers," RAIRO, 1974.

# The importance of method selection



Vector Poisson problem

- Solutions by the standard non-mixed method (left) and by a mixed method (right).
- Only the second choice shows the correct behavior near the reentrant corner.




Poisson problem

- Solutions by two different choices for the finite element solution spaces in a mixed method.
- Only the second choice looks like the true solution: $x(1-x)y(1-y)$.

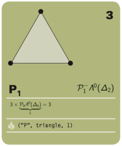
Examples and images borrowed from:

ARNOLD, FALK, WINTHER “Finite Element Exterior Calculus: From Hodge Theory to Numerical Stability,” *Bulletin of the AMS*, 47:2, 2010.

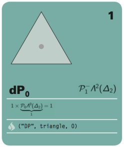
# Stable pairs of elements for mixed methods

Picking elements from the table for a mixed method for the Poisson problem:

2

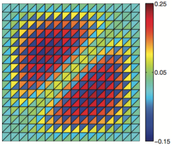



$\subset H^1 \times H^1$




$\subset L^2$

$\Rightarrow$




Unstable method

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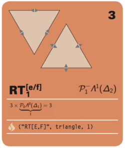
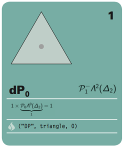
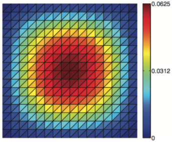



$\subset H(\text{div})$




$\subset L^2$

$\Rightarrow$




*Provably* stable method

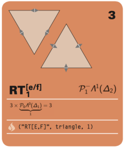
converges to

$u = x(1 - x)y(1 - y)$

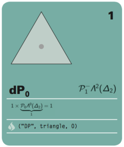
Example and images on right from:

ARNOLD, FALK, WINTHER "Finite Element Exterior Calculus. . ." *Bulletin of the AMS*, 47:2, 2010.

# Method selection and cochain complexes

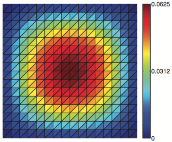



$\subset H(\mathrm{div})$




$\subset L^2$

$\Rightarrow$




*Provably* stable method

converges to
$\mathbf{u} = x(1-x)y(1-y)$

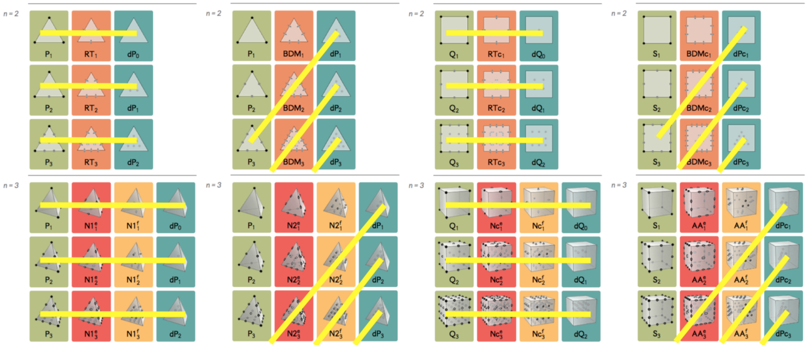
Stable pairs of elements for mixed Hodge-Laplacian problems are found by choosing consecutive spaces in compatible discretizations of the $L^2$ deRham Diagram.

$$H^1 \xrightarrow[\text{grad}]{\nabla} H(\mathrm{curl}) \xrightarrow[\text{curl}]{\nabla\times} H(\mathrm{div}) \xrightarrow[\text{div}]{\nabla\cdot} L^2$$

| | $H^1$ | $H(\mathrm{curl})$ | $H(\mathrm{div})$ | $L^2$ |
|---|---|---|---|---|
| vector Poisson | $\sigma$ | $\mu$ | | |
| Maxwell's eqn's | | $h$ | $b$ | |
| Darcy / Poisson | | | $\mathbf{u}$ | $p$ |

**Stable pairs are found from consecutive entries in a cochain complex.**

# Exact cochain complexes found in the table



Two kinds of families of cochain complexes on an tetrahedron in $\mathbb{R}^3$:

$$\mathcal{P}_r^-\Lambda^0 \to \mathcal{P}_r^-\Lambda^1 \to \mathcal{P}_r^-\Lambda^2 \to \mathcal{P}_r^-\Lambda^3$$ **'trimmed' polynomials**

$$\mathcal{P}_r\Lambda^0 \to \mathcal{P}_{r-1}\Lambda^1 \to \mathcal{P}_{r-2}\Lambda^2 \to \mathcal{P}_{r-3}\Lambda^3$$ **polynomials**

# Exact cochain complexes found in the table

On an $n$-simplex in $\mathbb{R}^n$:

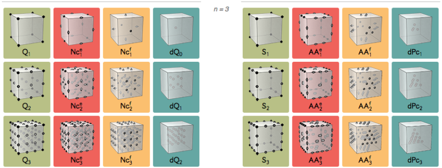
$$\mathcal{P}_r^-\Lambda^0 \to \mathcal{P}_r^-\Lambda^1 \to \cdots \to \mathcal{P}_r^-\Lambda^{n-1} \to \mathcal{P}_r^-\Lambda^n$$ **'trimmed' polynomials**

$$\mathcal{P}_r\Lambda^0 \to \mathcal{P}_{r-1}\Lambda^1 \to \cdots \to \mathcal{P}_{r-n+1}\Lambda^{n-1} \to \mathcal{P}_{r-n}\Lambda^n$$ **polynomials**

On an $n$-dimensional cube in $\mathbb{R}^n$:

$$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n$$ **tensor product**

$$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n$$ **serendipity**



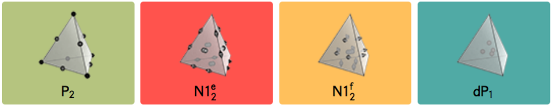

The 'minus' spaces proceed across rows of the PToFE (***r* is fixed**) while the 'regular' spaces proceed along diagonals (***r* decreases**)

Mysteriously, the degree of freedom count for mixed methods from the $\mathcal{P}_r^-$ spaces is smaller than those from the $\mathcal{P}_r$ spaces, while the opposite is true for the $\mathcal{Q}_r^-$ and $\mathcal{S}_r$ spaces.

# Outline

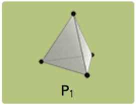
# Counting boundary and interior DoFs of $\mathcal{P}_r^- \Lambda^k$

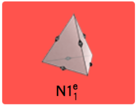

| | $\mathcal{P}_1^-\Lambda^0(\Delta_3)$ | $\mathcal{P}_1^-\Lambda^1(\Delta_3)$ | $\mathcal{P}_1^-\Lambda^2(\Delta_3)$ | $\mathcal{P}_1^-\Lambda^3(\Delta_3)$ |
|---|---|---|---|---|
| faces, edges, and, vertices | 4 | 6 | 4 | 0 |
| interior | 0 | 0 | 0 | 1 |
| total | 4 | 6 | 4 | 1 |

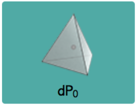



| | $\mathcal{P}_2^-\Lambda^0(\Delta_3)$ | $\mathcal{P}_2^-\Lambda^1(\Delta_3)$ | $\mathcal{P}_2^-\Lambda^2(\Delta_3)$ | $\mathcal{P}_2^-\Lambda^3(\Delta_3)$ |
|---|---|---|---|---|
| faces, edges, and, vertices | 10 | 20 | 12 | 0 |
| interior | 0 | 0 | 3 | 4 |
| total | 10 | 20 | 15 | 4 |

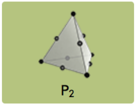
# Identifying an alternating sum pattern

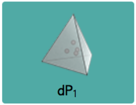






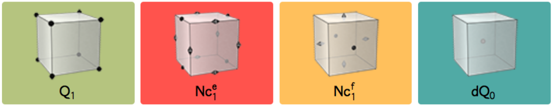

| | $\mathcal{P}_1^-\Lambda^0(\Delta_3)$ | $\mathcal{P}_1^-\Lambda^1(\Delta_3)$ | $\mathcal{P}_1^-\Lambda^2(\Delta_3)$ | $\mathcal{P}_1^-\Lambda^3(\Delta_3)$ | $\pm$ sum |
|---|---|---|---|---|---|
| boundary | 4 | 6 | 4 | 0 | 2 |
| interior | 0 | 0 | 0 | 1 | -1 |
| total | 4 | 6 | 4 | 1 | 1 |



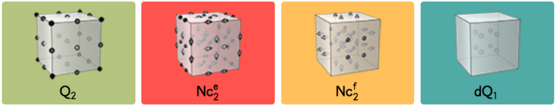



| | $\mathcal{P}_2^-\Lambda^0(\Delta_3)$ | $\mathcal{P}_2^-\Lambda^1(\Delta_3)$ | $\mathcal{P}_2^-\Lambda^2(\Delta_3)$ | $\mathcal{P}_2^-\Lambda^3(\Delta_3)$ | $\pm$ sum |
|---|---|---|---|---|---|
| boundary | 10 | 20 | 12 | 0 | 2 |
| interior | 0 | 0 | 3 | 4 | -1 |
| total | 10 | 20 | 15 | 4 | 1 |

# Counting DoFs of $\mathcal{Q}_r^- \Lambda^k$




| | $\mathcal{Q}_1^- \Lambda^0(\square_3)$ | $\mathcal{Q}_1^- \Lambda^1(\square_3)$ | $\mathcal{Q}_1^- \Lambda^2(\square_3)$ | $\mathcal{Q}_1^- \Lambda^3(\square_3)$ | $\pm$ sum |
|---|---|---|---|---|---|
| boundary | 8 | 12 | 6 | 0 | 2 |
| interior | 0 | 0 | 0 | 1 | -1 |
| total | 8 | 12 | 6 | 1 | 1 |




| | $\mathcal{Q}_2^- \Lambda^0(\square_3)$ | $\mathcal{Q}_2^- \Lambda^1(\square_3)$ | $\mathcal{Q}_2^- \Lambda^2(\square_3)$ | $\mathcal{Q}_2^- \Lambda^3(\square_3)$ | $\pm$ sum |
|---|---|---|---|---|---|
| boundary | 26 | 48 | 24 | 0 | 2 |
| interior | 1 | 6 | 12 | 8 | -1 |
| total | 27 | 54 | 36 | 8 | 1 |

# Predicting DoFs of $\mathcal{S}_r^-\Lambda^k$

**How big would a "minimal dimension" cochain complex on cubes be?**

Expect to recover $\mathcal{Q}_1^-\Lambda^k$ in lowest order case:

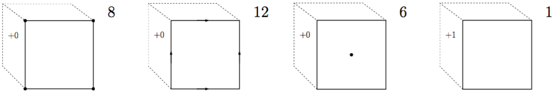
| | $\mathcal{S}_1^-\Lambda^0(\square_3)$ | $\mathcal{S}_1^-\Lambda^1(\square_3)$ | $\mathcal{S}_1^-\Lambda^2(\square_3)$ | $\mathcal{S}_1^-\Lambda^3(\square_3)$ | $\pm$ sum |
|---|---|---|---|---|---|
| boundary | 8 | 12 | 6 | 0 | 2 |
| interior | 0 | 0 | 0 | 1 | -1 |
| total | 8 | 12 | 6 | 1 | 1 |

For $r > 1$, we must have a constant multiple of DoFs per edge or face, and we have expected dimensions (by other reasoning) for $\mathcal{S}_2^-\Lambda^0$ and $\mathcal{S}_2^-\Lambda^3$:

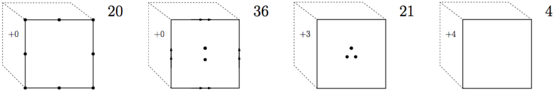
| | $\mathcal{S}_2^-\Lambda^0(\square_3)$ | $\mathcal{S}_2^-\Lambda^1(\square_3)$ | $\mathcal{S}_2^-\Lambda^2(\square_3)$ | $\mathcal{S}_2^-\Lambda^3(\square_3)$ | $\pm$ sum |
|---|---|---|---|---|---|
| boundary | 20 | $12e_1 + 6f_1$ | $6f_2$ | 0 | 2 |
| interior | 0 | $i_1$ | $i_2$ | 4 | -1 |
| total | 20 | $12e_1 + 6f_1 + i_1$ | $6f_2 + i_2$ | 4 | 1 |

Also expect $e_1 = 2$ since this would augment the DoFs per edge by 1 from $r = 1$ case.

# Actual DoFs of $\mathcal{S}_r^-\Lambda^k$ $(r = 1, 2)$

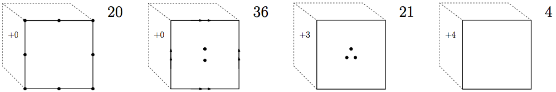



| | $\mathcal{S}_1^-\Lambda^0(\square_3)$ | $\mathcal{S}_1^-\Lambda^1(\square_3)$ | $\mathcal{S}_1^-\Lambda^2(\square_3)$ | $\mathcal{S}_1^-\Lambda^3(\square_3)$ | $\pm$ sum |
|---|---|---|---|---|---|
| boundary | 8 | 12 | 6 | 0 | 2 |
| interior | 0 | 0 | 0 | 1 | -1 |
| total | 8 | 12 | 6 | 1 | 1 |

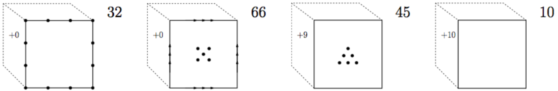



| | $\mathcal{S}_2^-\Lambda^0(\square_3)$ | $\mathcal{S}_2^-\Lambda^1(\square_3)$ | $\mathcal{S}_2^-\Lambda^2(\square_3)$ | $\mathcal{S}_2^-\Lambda^3(\square_3)$ | $\pm$ sum |
|---|---|---|---|---|---|
| boundary | 20 | 36 | 18 | 0 | 2 |
| interior | 0 | 0 | 3 | 4 | -1 |
| total | 20 | 36 | 21 | 4 | 1 |

# Actual DoFs of $\mathcal{S}_r^-\Lambda^k$ $(r = 2, 3)$




| | $\mathcal{S}_2^-\Lambda^0(\square_3)$ | $\mathcal{S}_2^-\Lambda^1(\square_3)$ | $\mathcal{S}_2^-\Lambda^2(\square_3)$ | $\mathcal{S}_2^-\Lambda^3(\square_3)$ | $\pm$ sum |
|---|---|---|---|---|---|
| boundary | 20 | 36 | 18 | 0 | 2 |
| interior | 0 | 0 | 3 | 4 | -1 |
| total | 20 | 36 | 21 | 4 | 1 |

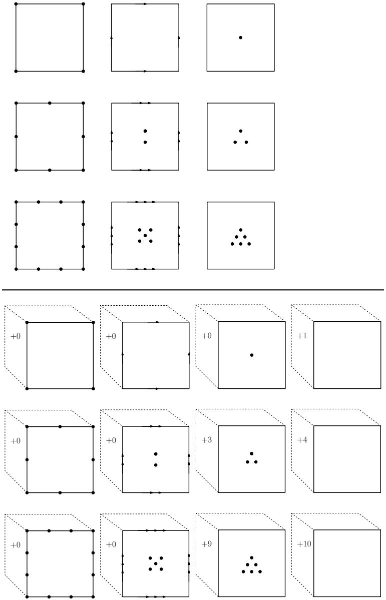



| | $\mathcal{S}_3^-\Lambda^0(\square_3)$ | $\mathcal{S}_3^-\Lambda^1(\square_3)$ | $\mathcal{S}_3^-\Lambda^2(\square_3)$ | $\mathcal{S}_3^-\Lambda^3(\square_3)$ | $\pm$ sum |
|---|---|---|---|---|---|
| boundary | 32 | 66 | 36 | 0 | 2 |
| interior | 0 | 0 | 9 | 10 | -1 |
| total | 32 | 66 | 45 | 10 | 1 |

# Outline

1. The "Periodic Table of the Finite Elements"
2. How to find new finite elements by counting
3. Trimmed serendipity finite elements
4. Computational bases for serendipity-type spaces
5. Extension to generic quads and hexes

# The 5th column: Trimmed serendipity spaces




A new column for the PToFE:
the **trimmed serendipity** elements.

$\boxed{\mathcal{S}_r^- \Lambda^k(\square_n)}$ denotes

approximation order $r$,
subset of $k$-form space $\Lambda^k(\Omega)$,
use on meshes of $n$-dim'l cubes.

Defined for any $n \geq 1$, $0 \leq k \leq n$, $r \geq 1$

Identical or analogous properties to all the other colummns in the table.

The advantage of the $\mathcal{S}_r^- \Lambda^k$ spaces is that they have fewer degrees of freedom for mixed methods than their tensor product and serendipity counterparts.

# The polynomial space of $\mathcal{S}_r^-\Lambda^k$

$\mathcal{S}_r^-\Lambda^k(\square_n)$ is a space of differential $k$-forms whose coefficients are polynomials in $\mathbb{R}^n$.

$$\mathcal{S}_r^-\Lambda^k = \mathcal{P}_r^-\Lambda^k \oplus \mathcal{J}_r\Lambda^k \oplus d\mathcal{J}_r\Lambda^{k-1}$$

Polynomial coefficients in each summand:

$\mathcal{P}_r^-\Lambda^k$ : anything up to degree $r-1$ and some degree $r$

$\mathcal{J}_r\Lambda^k$ : certain polynomials whose degree is between $r+1$ and $r+n-k-1$

$d\mathcal{J}_r\Lambda^{k-1}$ : certain polynomials whose degree is between $r$ and $r+n-k-2$

The "regular" serendipity space has an analogous decomposition:

$$\mathcal{S}_r\Lambda^k = \mathcal{P}_r\Lambda^k \oplus \mathcal{J}_r\Lambda^k \oplus d\mathcal{J}_{r+1}\Lambda^{k-1}$$

$$- - - - - - - - - - - - - - -$$

This decomposition provides a direct sum into some precise but elaborate subspaces:

$$\mathcal{J}_r\Lambda^k(\mathbb{R}^n) := \textstyle\sum_{l\geq 1} \kappa\mathcal{H}_{r+l-1,l}\Lambda^{k+1}(\mathbb{R}^n),$$

where $$\mathcal{H}_{r,l}\Lambda^k(\mathbb{R}^n) := \left\{\omega \in \mathcal{H}_r\Lambda^k(\mathbb{R}^n) \mid \text{ldeg}\,\omega \geq l\right\},$$

where $$\text{ldeg}(x^\alpha dx_\sigma) := \#\{i \in \sigma^* \;:\; \alpha_i = 1\}.$$

# The degrees of freedom of $\mathcal{S}_r^- \Lambda^k$

The **degrees of freedom** associated to a $d$-dimensional sub-face $f$ of an $n$-dimensional cube $\square_n$ are (for any $k \leq d \leq \min\{n, \lfloor r/2 \rfloor + k\}$):

$$u \longmapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r-2(d-k)-1}\Lambda^{d-k}(f) \;\oplus\; d\mathcal{H}_{r-2(d-k)+1}\Lambda^{d-k-1}(f),$$

These degrees of freedom are **unisolvent** for $\mathcal{S}_r^- \Lambda^k(\square_n)$.

– – – – – – – – – – – – –

The direct sum decomposition of the indexing space gives one way to count the dimension precisely:

$$\underbrace{\mathcal{P}_{r-2(d-k)-1}\Lambda^{d-k}(f)}_{\text{indexing space for } \mathcal{S}_{r-1}\Lambda^k(f)} \;\oplus\; \underbrace{d\mathcal{H}_{r-2(d-k)+1}\Lambda^{d-k-1}(f)}_{\text{subspace of } \mathcal{H}_{r-2(d-k)}\Lambda^{d-k}(f)}$$

# Dimension count and comparison

Formula for counting degrees of freedom of $\mathcal{S}_r^- \Lambda^k(\square_n)$:

$$\sum_{d=k}^{\min\{n,\lfloor r/2\rfloor+k\}} 2^{n-d}\binom{n}{d}\left(\binom{r-d+2k-1}{r-d+k-1}\binom{r-d+k-1}{d-k}+\binom{r-d+2k}{k}\binom{r-d+k-1}{d-k-1}\right)$$

| | k | r=1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| n=2 | 0 | 4 | 8 | 12 | 17 | 23 | 30 | 38 |
| | 1 | 4 | 10 | 17 | 26 | 37 | 50 | 65 |
| | 2 | 1 | 3 | 6 | 10 | 15 | 21 | 28 |
| n=3 | 0 | 8 | 20 | 32 | 50 | 74 | 105 | 144 |
| | 1 | 12 | 36 | 66 | 111 | 173 | 255 | 360 |
| | 2 | 6 | 21 | 45 | 82 | 135 | 207 | 301 |
| | 3 | 1 | 4 | 10 | 20 | 35 | 56 | 84 |
| n=4 | 0 | 16 | 48 | 80 | 136 | 216 | 328 | 480 |
| | 1 | 32 | 112 | 216 | 392 | 656 | 1036 | 1563 |
| | 2 | 24 | 96 | 216 | 422 | 746 | 1227 | 1910 |
| | 3 | 8 | 36 | 94 | 200 | 375 | 644 | 1036 |
| | 4 | 1 | 5 | 15 | 35 | 70 | 126 | 210 |

# Key properties of the trimmed serendipity spaces

$$\begin{array}{llllll} \mathcal{Q}_r^-\Lambda^0 \to & \mathcal{Q}_r^-\Lambda^1 \to & \cdots \to & \mathcal{Q}_r^-\Lambda^{n-1} & \to \mathcal{Q}_r^-\Lambda^n & \textbf{tensor product} \\ \mathcal{S}_r\Lambda^0 \to & \mathcal{S}_{r-1}\Lambda^1 \to & \cdots \to & \mathcal{S}_{r-n+1}\Lambda^{n-1} & \to \mathcal{S}_{r-n}\Lambda^n & \textbf{serendipity} \\ \mathcal{S}_r^-\Lambda^0 \to & \mathcal{S}_r^-\Lambda^1 \to & \cdots \to & \mathcal{S}_r^-\Lambda^{n-1} & \to \mathcal{S}_r^-\Lambda^n & \textbf{trimmed serendipity} \end{array}$$

**Subcomplex:** $d\mathcal{S}_r^-\Lambda^k \subset \mathcal{S}_r^-\Lambda^{k+1}$

**Exactness:** The above sequence is exact.
i.e. the image of incoming map = kernel of outgoing map

**Inclusion:** $\mathcal{S}_r\Lambda^k \subset \mathcal{S}_{r+1}^-\Lambda^k \subset \mathcal{S}_{r+1}\Lambda^k$

**Trace:** $\mathrm{tr}_f\mathcal{S}_r^-\Lambda^k(\mathbb{R}^n) \subset \mathcal{S}_r^-\Lambda^k(f)$, for any $(n-1)$-hyperplane $f$ in $\mathbb{R}^n$

**Special cases:**
$$\mathcal{S}_r^-\Lambda^0 = \mathcal{S}_r\Lambda^0$$
$$\mathcal{S}_r^-\Lambda^n = \mathcal{S}_{r-1}\Lambda^n$$
$$\mathcal{S}_r^-\Lambda^k + d\mathcal{S}_{r+1}\Lambda^{k-1} = \mathcal{S}_r\Lambda^k.$$

Replace '$\mathcal{S}$' by '$\mathcal{P}$' $\rightsquigarrow$ key properties about the first two columns for $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$!

# Mixed Method dimension comparison 1

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \operatorname{div}\ \mathbf{u} - f &= 0 \end{aligned}$$

We compare degree of freedom counts among the three families for use on meshes of affinely-mapped squares or cubes, when a conforming method with (at least) order $r$ decay in the approximation of $p$, $\mathbf{u}$, and div $\mathbf{u}$ is desired.

**Total** # of degrees of freedom on a square ($n = 2$):

| $r$ | $\lvert\mathcal{Q}_r^-\Lambda^1\rvert + \lvert\mathcal{Q}_r^-\Lambda^2\rvert$ | $\lvert\mathcal{S}_r\Lambda^1\rvert + \lvert\mathcal{S}_{r-1}\Lambda^2\rvert$ | $\lvert\mathcal{S}_r^-\Lambda^1\rvert + \lvert\mathcal{S}_r^-\Lambda^2\rvert$ |
|---|---|---|---|
| 1 | 4+1 = 5 | 8+1 = 9 | 4+1 = 5 |
| 2 | 12+4 = 16 | 14+3 = 17 | 10+3 = 13 |
| 3 | 24+9 = 33 | 22+6 = 28 | 17+6 = 23 |

**Total** # of degrees of freedom on a cube ($n = 3$):

| $r$ | $\lvert\mathcal{Q}_r^-\Lambda^2\rvert + \lvert\mathcal{Q}_r^-\Lambda^3\rvert$ | $\lvert\mathcal{S}_r\Lambda^2\rvert + \lvert\mathcal{S}_{r-1}\Lambda^3\rvert$ | $\lvert\mathcal{S}_r^-\Lambda^2\rvert + \lvert\mathcal{S}_r^-\Lambda^3\rvert$ |
|---|---|---|---|
| 1 | 6+1 = 7 | 18+1 = 19 | 6+1 = 7 |
| 2 | 36+8 = 44 | 39+4 = 43 | 21+4 = 25 |
| 3 | 108+27 = 135 | 72+10 = 82 | 45+10 = 55 |

# Mixed Method dimension comparison 2

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$$

The number of interior degrees of freedom is reduced from tensor product, to serendipity, to trimmed serendipity:

\# of **interior** degrees of freedom on a square ($n = 2$):

| $r$ | $\lvert\mathcal{Q}_r^-\Lambda_{\mathbf{0}}^1\rvert + \lvert\mathcal{Q}_r^-\Lambda_{\mathbf{0}}^2\rvert$ | $\lvert\mathcal{S}_r\Lambda_{\mathbf{0}}^1\rvert + \lvert\mathcal{S}_{r-1}\Lambda_{\mathbf{0}}^2\rvert$ | $\lvert\mathcal{S}_r^-\Lambda_{\mathbf{0}}^1\rvert + \lvert\mathcal{S}_r^-\Lambda_{\mathbf{0}}^2\rvert$ |
|---|---|---|---|
| 1 | 0+1 = 1 | 0+1 = 1 | 0+1 = 1 |
| 2 | 4+4 = 8 | 2+3 = 5 | 2+3 = 5 |
| 3 | 12+9 = 21 | 6+6 = 12 | 5+6 = 11 |

\# of **interior** degrees of freedom on a cube ($n = 3$):

| $r$ | $\lvert\mathcal{Q}_r^-\Lambda_{\mathbf{0}}^2\rvert + \lvert\mathcal{Q}_r^-\Lambda_{\mathbf{0}}^3\rvert$ | $\lvert\mathcal{S}_r\Lambda_{\mathbf{0}}^2\rvert + \lvert\mathcal{S}_{r-1}\Lambda_{\mathbf{0}}^3\rvert$ | $\lvert\mathcal{S}_r^-\Lambda_{\mathbf{0}}^2\rvert + \lvert\mathcal{S}_r^-\Lambda_{\mathbf{0}}^3\rvert$ |
|---|---|---|---|
| 1 | 0+1 = 1 | 0+1 = 1 | 0+1 = 1 |
| 2 | 12+8 = 20 | 3+4 = 7 | 3+4 = 7 |
| 3 | 54+27 = 81 | 12+10 = 22 | 9+10 = 19 |

# Mixed Method dimension comparison 3

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$$

Assuming interior degrees of freedom could be dealt with efficiently (e.g. by static condensation), trimmed serendipity elements *still* have the fewest DoFs:

# of **interface** (edge) degrees of freedom on a square ($n = 2$):

| $r$ | $\lvert\mathcal{Q}_r^-\Lambda^1(\partial\square_2)\rvert$ | $\lvert\mathcal{S}_r\Lambda^1(\partial\square_2)\rvert$ | $\lvert\mathcal{S}_r^-\Lambda^1(\partial\square_2)\rvert$ |
|---|---|---|---|
| 1 | 4 | 8 | 4 |
| 2 | 8 | 12 | 8 |
| 3 | 12 | 16 | 12 |

# of **interface** (edge+face) degrees of freedom on a cube ($n = 3$):

| $r$ | $\lvert\mathcal{Q}_r^-\Lambda^2(\partial\square_3)\rvert$ | $\lvert\mathcal{S}_r\Lambda^2(\partial\square_3)\rvert$ | $\lvert\mathcal{S}_r^-\Lambda^2(\partial\square_3)\rvert$ |
|---|---|---|---|
| 1 | 6 | 18 | 6 |
| 2 | 24 | 36 | 18 |
| 3 | 54 | 60 | 36 |

# Outline

# Building a computational basis


| $dx$ | $dy$ | $dz$ |
|---|---|---|
| $-yz$ | $xz$ | $0$ |
| $0$ | $-xz$ | $xy$ |
| $yz$ | $xz$ | $xy$ |
| $2xy$ | $x^2$ | $0$ |
| $2xz$ | $0$ | $x^2$ |
| $y^2$ | $2xy$ | $0$ |
| $0$ | $2yz$ | $y^2$ |
| $z^2$ | $0$ | $2xz$ |
| $0$ | $z^2$ | $2yz$ |
| $2xyz$ | $x^2z$ | $x^2y$ |
| $y^2z$ | $2xyz$ | $xy^2$ |
| $yz^2$ | $xz^2$ | $2xyz$ |

**Goal:** find a computational basis for $\mathcal{S}_1\Lambda^1(\square_3)$:

- Must be $H(\text{curl})$-conforming
- Must have 24 functions, 2 associated to each edge of cube
- Must recover constant and linear approx. on each edge
- The approximation space contains:

  (1) Any polynomial coefficient of at most linear order:
  $\{1, x, y, z\} \times \{dx, dy, dz\} \;\rightarrow$ 12 forms
  (2) Certain forms with quadratic or cubic order coefficients shown in table at left $\rightarrow$ 12 forms

- For constants, use "obvious" functions:

$$\{(y \pm 1)(z \pm 1)dx, \;\; (x \pm 1)(z \pm 1)dy, \;\; (x \pm 1)(y \pm 1)dz\}$$

e.g. $(y+1)(z+1)dx$ evaluates to zero on every edge ***except*** $\{y = 1, z = 1\}$ where it is $\equiv 4 \rightarrow$ constant approx.

Also, $(y+1)(z+1)dx$ can be written as a linear combo, by using the first three forms at left to get the $yz\,dx$ term

# Building a computational basis


| $dx$ | $dy$ | $dz$ |
|---|---|---|
| $-yz$ | $xz$ | $0$ |
| $0$ | $-xz$ | $xy$ |
| $yz$ | $xz$ | $xy$ |
| $2xy$ | $x^2$ | $0$ |
| $2xz$ | $0$ | $x^2$ |
| $y^2$ | $2xy$ | $0$ |
| $0$ | $2yz$ | $y^2$ |
| $z^2$ | $0$ | $2xz$ |
| $0$ | $z^2$ | $2yz$ |
| $2xyz$ | $x^2z$ | $x^2y$ |
| $y^2z$ | $2xyz$ | $xy^2$ |
| $yz^2$ | $xz^2$ | $2xyz$ |

- For constant approx on edges, we used:

$$\{(y\pm 1)(z\pm 1)dx,\ \ (x\pm 1)(z\pm 1)dy,\ \ (x\pm 1)(y\pm 1)dz\}$$

- Guess for linear approx on edges:

$$\{x(y\pm 1)(z\pm 1)dx,\ \ y(x\pm 1)(z\pm 1)dy,\ \ z(x\pm 1)(y\pm 1)dz\}$$

e.g. $x(y+1)(z+1)dx$ evaluates to $4x$ on $\{y=1, z=1\}$.

- Unfortunately: $\boxed{x(y+1)(z+1)dx \notin \mathcal{S}_1\Lambda(\square_3)!}$

Why? $x(y+1)(z+1)dx = (xyz + xy + xz + x)dx$

but *xyz dx only appears with other cubic order coefficients!*

- Remedy: add $dy$ and $dz$ terms that vanish on all edges.

# Building a computational basis


| $dx$ | $dy$ | $dz$ |
|---|---|---|
| $-yz$ | $xz$ | $0$ |
| $0$ | $-xz$ | $xy$ |
| $yz$ | $xz$ | $xy$ |
| $2xy$ | $x^2$ | $0$ |
| $2xz$ | $0$ | $x^2$ |
| $y^2$ | $2xy$ | $0$ |
| $0$ | $2yz$ | $y^2$ |
| $z^2$ | $0$ | $2xz$ |
| $0$ | $z^2$ | $2yz$ |
| $2xyz$ | $x^2z$ | $x^2y$ |
| $y^2z$ | $2xyz$ | $xy^2$ |
| $yz^2$ | $xz^2$ | $2xyz$ |

Computational basis element associated to $\{y = 1, z = 1\}$:

$$\boxed{2x(y+1)(z+1)\,dx + (z+1)(x^2-1)\,dy + (y+1)(x^2-1)\,dz}$$

✓ Evaluates to $4x$ on $\{y = 1, z = 1\}$ (linear approx.)

✓ Evaluates to 0 on all other edges

✓ Belongs to the space $\mathcal{S}_1\Lambda(\square_3)$:

$$\begin{array}{rcrcrl}
2xyz\,dx & + & x^2z\,dy & + & x^2y\,dz & \\
2xy\,dx & + & x^2\,dy & + & 0\,dz & \\
2xz\,dx & + & 0\,dy & + & x^2\,dz & \\
2x\,dx & + & (-z-1)dy & + & (-y-1)dz & \leftarrow \text{linear order} \\
\hline
\end{array}$$

$\hookrightarrow$ summation and factoring yields the desired form)

There are 11 other such functions, one per edge. We have:

$$\begin{array}{ccccc}
\mathcal{S}_1\Lambda(\square_3) & = & \underbrace{E_0\Lambda^1(\square_3)}_{\text{“obvious” basis for constant approx}} & \oplus & \underbrace{\tilde{E}_1\Lambda^1(\square_3)}_{\text{modified basis for linear approx}} \\
\dim\ 24 & = & 12 & + & 12
\end{array}$$

# A complete table of computational bases

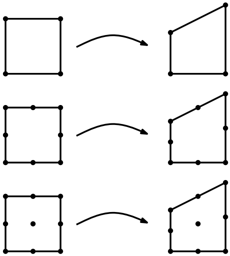
| $n=3$ | $k=0$ | $k=1$ | $k=2$ | $k=3$ |
|---|---|---|---|---|
| $\mathcal{S}_r\Lambda^k$ | $V\Lambda^0(\square_3)$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| | $\bigoplus_{i=0}^{r-2} E_i\Lambda^0(\square_3)$ | $\bigoplus_{i=0}^{r-1} E_i\Lambda^1(\square_3) \oplus \tilde{E}_r\Lambda^1(\square_3)$ | $\emptyset$ | $\emptyset$ |
| | $\bigoplus_{i=4}^{r} F_i\Lambda^0(\square_3)$ | $\bigoplus_{i=2}^{r-1} F_i\Lambda^1(\square_3) \oplus \hat{F}_r\Lambda^1(\square_3)$ | $\bigoplus_{i=0}^{r-1} F_i\Lambda^2(\square_3) \oplus \tilde{F}_r\Lambda^2(\square_3)$ | $\emptyset$ |
| | $\bigoplus_{i=6}^{r} I_i\Lambda^0(\square_3)$ | $\bigoplus_{i=4}^{r} I_i\Lambda^1(\square_3)$ | $\bigoplus_{i=2}^{r} I_i\Lambda^2(\square_3)$ | $\bigoplus_{i=2}^{r} I_i\Lambda^3(\square_3)$ |
| $\mathcal{S}_r^-\Lambda^k$ | $V\Lambda^0(\square_3)$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| | $\bigoplus_{i=0}^{r-2} E_i\Lambda^0(\square_3)$ | $\bigoplus_{i=0}^{r-1} E_i\Lambda^1(\square_3)$ | $\emptyset$ | $\emptyset$ |
| | $\bigoplus_{i=4}^{r} F_i\Lambda^0(\square_3)$ | $\bigoplus_{i=2}^{r-1} F_i\Lambda^1(\square_3) \oplus \tilde{F}_r\Lambda^1(\square_3)$ | $\bigoplus_{i=0}^{r-1} F_i\Lambda^2(\square_3)$ | $\emptyset$ |
| | $\bigoplus_{i=6}^{r} I_i\Lambda^0(\square_3)$ | $\bigoplus_{i=4}^{r-1} I_i\Lambda^1(\square_3) \oplus \tilde{I}_r\Lambda^1(\square_3)$ | $\bigoplus_{i=2}^{r-1} I_i\Lambda^2(\square_3) \oplus \tilde{I}_r\Lambda^2(\square_3)$ | $\bigoplus_{i=2}^{r-1} I_i\Lambda^3(\square_3)$ |

# Outline

# Serendipity elements struggle with reference mapping

Quadratic serendipity elements, mapped non-affinely, are only expected to converge at the rate of *linear* elements.

ARNOLD, BOFFI, FALK, “Approximation by Quadrilateral Finite Elements,” *Math. Comp.*, 2002

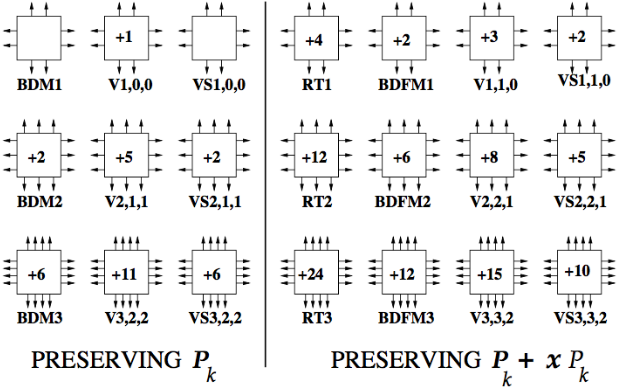


| | $\|\|u - u_h\|\|_{L^2}$ | $\|\|\nabla(u - u_h)\|\|_{L^2}$ |
|---|---|---|
| linear | $O(h^2)$ | $O(h)$ |
| quadratic serendipity | $O(h^2)$ | $O(h)$ |
| quadratic tensor prod. | $O(h^3)$ | $O(h^2)$ |

Extensions to vector-valued and higher dimensions:

ARNOLD, BOFFI, FALK, “Quadrilateral $H$(div) Finite Elements,” *SINUM*, 2005.
ARNOLD, BOFFI, BONIZZONI, “Finite element differential forms on curvilinear cubic meshes,” *Numer. Math.*, 2014

# The virtual element technique



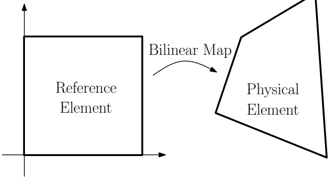

$\rightarrow$ Analogues of conforming finite element spaces on squares can be treated as virtual elements.

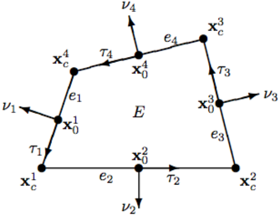
$\rightarrow$ Explicit basis functions are not needed to implement the method.

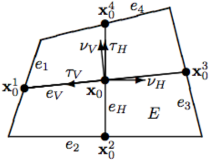
$\rightarrow$ Related polygonal element methods (HHO, HDG, WG. . .) may offer similar approaches.

BEIRÃO DA VEIGA, BREZZI, MARINI, RUSSO "Serendipity face and edge VEM spaces" *Rendiconti Lincei-Matematica e Applicazioni*, 2017.

# The Arbogast-Correa technique



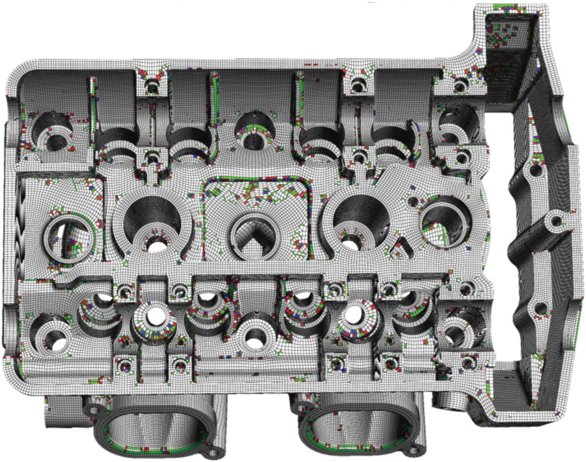




A finite element space on a general quadrilateral is built in two parts:

- Apply Piola mapping to functions associated to boundary of reference element.
- Define functions on the physical element corresponding to interior degrees of freedom in a way that ensures relevant polynomial approximation properties.

ARBOGAST, CORREA "Two families of $H(\mathrm{div})$ mixed finite elements on quadrilaterals of minimal dimension," *SIAM J. Numerical Analysis*, 2016

# Recent advances in hex-dominant meshing



→ A hex-dominant mesh with ≈1.3 million cells, including ≈1 million hexahedra.

→ Re-meshed from a mesh of ≈ 10 million tetrahedra.

SOKOLOV ET AL. “Hexahedral-Dominant Meshing,” *ACM Trans. Graphics*, 2016

# Open source finite element software

FEniCS primarily supports simplicial elements

`deal.ii` primarily supports quad/hex elements

ALNÆS ET AL. "The FEniCS Project Version 1.5" *Archive of Numerical Software* 2015
BANGERTH ET AL. "The `deal.ii` Library, Version 8.4," *Journal of Num. Math.*, 2016

*Neither package supports (trimmed) serendipity elements yet. . .*
*. . . but that is likely to change in the near future!*

# Acknowledgments

**Thanks for the invitation to speak!**

Related Publications

Christiansen, G. "Constructions of some minimal finite element systems." ESAIM: M2AN, 50:3, pp. 833–850, 2016.

G., Kloefkorn "Trimmed Serendipity Finite Element Differential Forms." Mathematics of Computation, to appear. See `arXiv:1607.00571`

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Slides and Pre-prints

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