

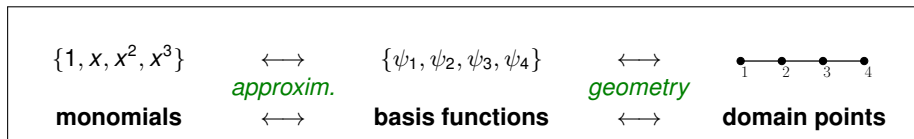
Hermite Polynomials and Geometric Decompositions

Andrew Gillette

Department of Mathematics
University of California, San Diego

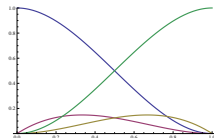
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Cubic Hermite Geometric Decomposition: 1D



Cubic Hermite Basis
on $[0, 1]$

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} := \begin{bmatrix} 1 - 3x^2 + 2x^3 \\ x - 2x^2 + x^3 \\ x^2 - x^3 \\ 3x^2 - 2x^3 \end{bmatrix}$$



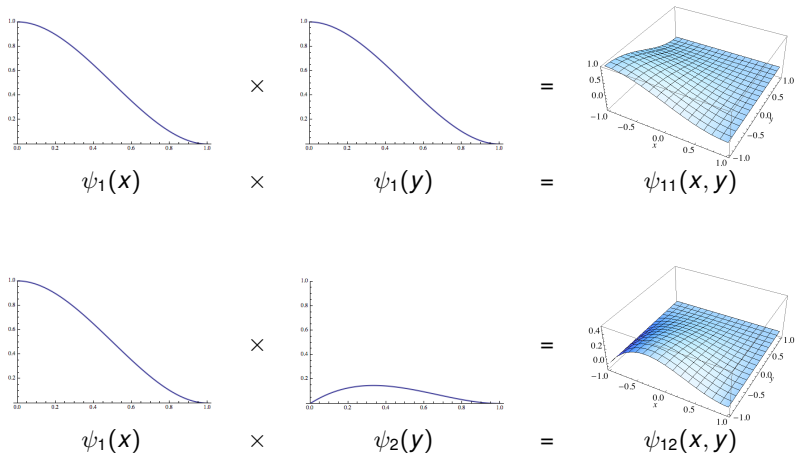
Approximation: $x^r = \sum_{i=1}^4 \varepsilon_{r,i} \psi_i$, for $r = 0, 1, 2, 3$, where $[\varepsilon_{r,i}] = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -3 & 1 \end{pmatrix}$

Geometry: If $a(x)$ is a cubic polynomial then:

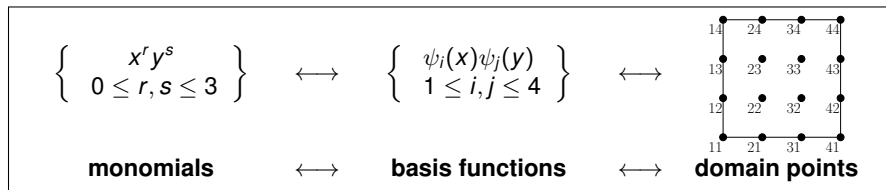
$$a(x) = \underbrace{a(0)}_{\text{value}} \psi_1 + \underbrace{a'(0)}_{\text{derivative}} \psi_2 - \underbrace{a'(1)}_{\text{derivative}} \psi_3 + \underbrace{a(1)}_{\text{value}} \psi_4$$

Tensor Product Polynomials

We can use our 1D Hermite functions to make 2D Hermite functions:

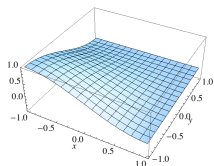


Cubic Hermite Geometric Decomposition: 2D

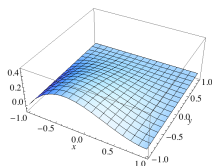


Approximation: $x^r y^s = \sum_{i=1}^4 \sum_{j=1}^4 \varepsilon_{r,i} \varepsilon_{s,j} \psi_{ij}$, for $0 \leq r, s \leq 3$, $\varepsilon_{r,i}$ as in 1D.

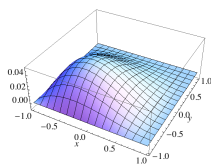
Geometry:



ψ_{11}



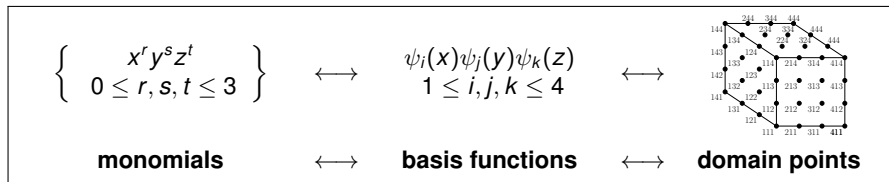
ψ_{21}



ψ_{22}

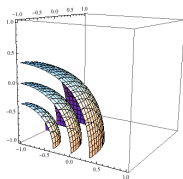
$$a(x, y) = \underbrace{a|_{(0,0)}}_{\text{value}} \psi_{11} + \underbrace{\partial_x a|_{(0,0)}}_{\text{1st deriv.}} \psi_{21} + \underbrace{\partial_y a|_{(0,0)}}_{\text{1st deriv.}} \psi_{12} + \underbrace{\partial_x \partial_y a|_{(0,0)}}_{\text{2nd deriv.}} \psi_{22} + \dots$$

Cubic Hermite Geometric Decomposition: 3D

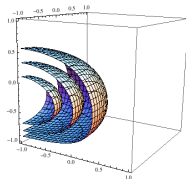


Approximation: $x^r y^s z^t = \sum_{i=1}^4 \sum_{j=1}^4 \sum_{k=1}^4 \varepsilon_{r,i} \varepsilon_{s,j} \varepsilon_{t,k} \psi_{ijk}$, for $0 \leq r, s, t \leq 3$, $\varepsilon_{r,i}$ as in 1D.

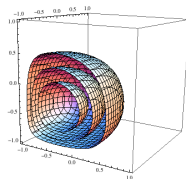
Geometry: Contours of level sets of the basis functions:



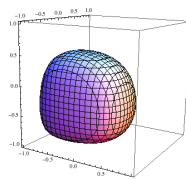
ψ_{111}



ψ_{112}



ψ_{212}



ψ_{222}

Which monomials do we really need?

$$\text{degree}(x^r y^s) = r + s$$

$$\text{superlinear degree}(x^r y^s) = r + s - \{\# \text{ of linearly appearing variables}\}$$

	degree	superlinear degree
xy^2	3	2
x^3y	4	3
xy^3	4	3
x^2y^2	4	4
x^3y^2	5	5

regular cubics (dim=16)

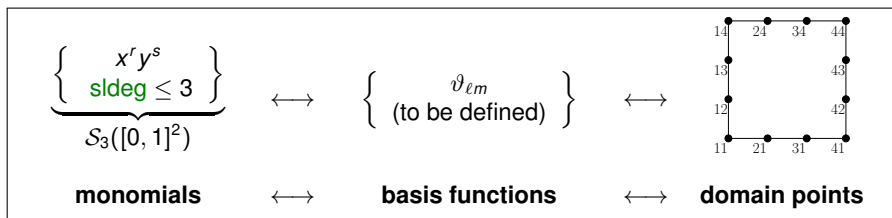
$$\{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, x^3y, xy^3, x^2y^2, x^3y^2, x^2y^3, x^3y^3\}$$

superlinear cubics (dim=12)

Recent mathematical research shows that degree r polynomials and **superlinear** degree r polynomials have the **same** approximation power for the models we are studying.

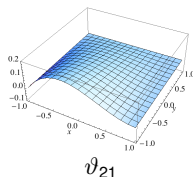
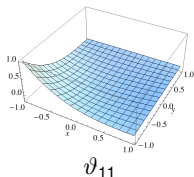
Cubic Hermite Serendipity Geom. Decomp: 2D

Theorem [G, 2012]: A Hermite-like geometric decomposition of $S_3([0, 1]^2)$ exists.



Approximation: $x^r y^s = \sum_{\ell m} \varepsilon_{r,i} \varepsilon_{s,j} \vartheta_{\ell m}$, for **superlinear degree** $(x^r y^s) \leq 3$

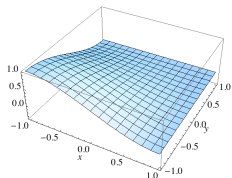
Geometry:



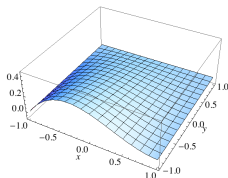
$$\begin{aligned}
 a(x, y) &= a|_{(0,0)} \vartheta_{11} \\
 &+ \partial_x a|_{(0,0)} \vartheta_{21} \\
 &+ \partial_y a|_{(0,0)} \vartheta_{12} \\
 &+ \dots
 \end{aligned}$$

Hermite Style Serendipity Functions (2D)

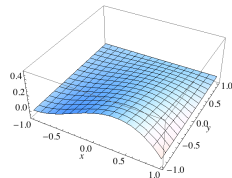
$$[\vartheta_{\ell m}] = \begin{bmatrix} \vartheta_{11} \\ \vartheta_{14} \\ \vartheta_{41} \\ \vartheta_{44} \\ \vartheta_{12} \\ \vartheta_{13} \\ \vartheta_{42} \\ \vartheta_{43} \\ \vartheta_{21} \\ \vartheta_{31} \\ \vartheta_{24} \\ \vartheta_{34} \end{bmatrix} = \begin{bmatrix} -(x-1)(y-1)(-2+x+x^2+y+y^2) \\ (x-1)(y+1)(-2+x+x^2-y+y^2) \\ (x+1)(y-1)(-2-x+x^2+y+y^2) \\ -(x+1)(y+1)(-2-x+x^2-y+y^2) \\ -(x-1)(y-1)^2(y+1) \\ (x-1)(y-1)(y+1)^2 \\ (x+1)(y-1)^2(y+1) \\ -(x+1)(y-1)(y+1)^2 \\ -(x-1)^2(x+1)(y-1) \\ (x-1)(x+1)^2(y-1) \\ (x-1)^2(x+1)(y+1) \\ -(x-1)(x+1)^2(y+1) \end{bmatrix} \cdot \frac{1}{8}$$



ϑ_{11}



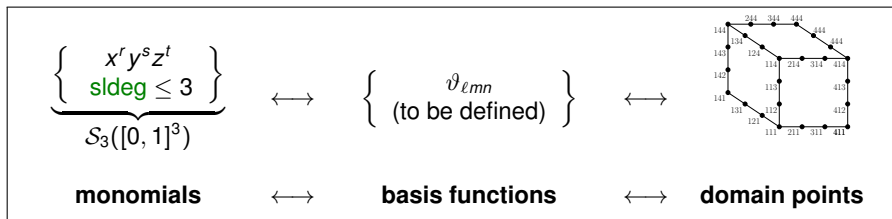
ϑ_{21}



ϑ_{31}

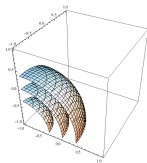
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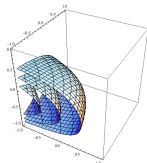


Approximation: $x^r y^s z^t = \sum_{\ell mn} \varepsilon_{r,i} \varepsilon_{s,j} \varepsilon_{t,k} \vartheta_{\ell mn}$, for **superlinear degree**($x^r y^s z^t$) ≤ 3

Geometry:



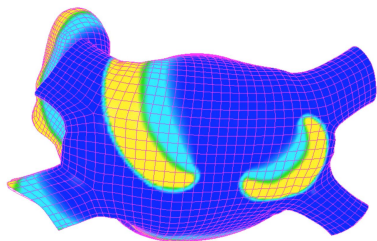
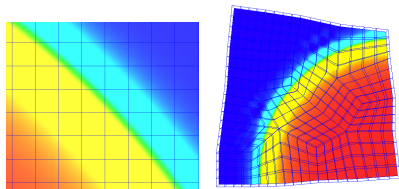
ϑ_{111}



ϑ_{112}

$$\begin{aligned} a(x, y, z) &= a|_{(0,0,0)} \vartheta_{111} \\ &+ \partial_x a|_{(0,0,0)} \vartheta_{211} \\ &+ \partial_y a|_{(0,0,0)} \vartheta_{121} \\ &+ \partial_z a|_{(0,0,0)} \vartheta_{112} \\ &+ \dots \end{aligned}$$

Application: Cardiac Electrophysiology



→ Cubic Hermite serendipity functions recently incorporated into Continuity software package for cardiac electrophysiology models.

→ Used to solve the *monodomain* equations, a type of reaction-diffusion equation

→ Initial results show agreement of serendipity and standard tricubics on benchmark problem with a

4x computational speedup in 3D.

→ Fast computation essential to clinical applications and 'real time' simulations

GONZALEZ, VINCENT, G., McCULLOCH *High Order Interpolation Methods in Cardiac Electrophysiology Simulation*, in preparation, 2013.

Acknowledgments

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Thanks for the opportunity to speak!

Slides and pre-prints: <http://ccom.ucsd.edu/~agillette>

