Real Algebraic Finite Elements and de Rham Cohomology

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joint work with

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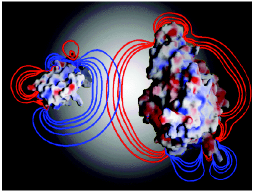
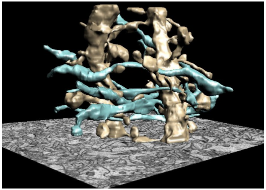
Motivation

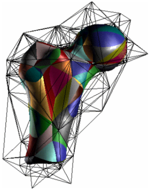
Biological modeling requires **robust** computational methods for solving integral and differential equations over domains constructed from imaging and physical data.



Electrodynamics



Electromagnetics/ Electrodiffusion



Elasticity

These methods must accommodate

- complicated domain geometry and topology
- multiple variables and operators

Outline

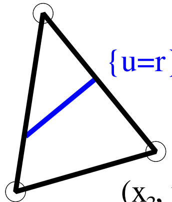
Real Barycentric Coordinates in 2D

- Let T be a triangle in the plane with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3).
- Transform (x, y) coordinates to **real barycentric coordinates** $(\lambda_1, \lambda_2, \lambda_3)$ via

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

- Let $u(x, y)$ be a bivariate **real** linear polynomial with coefficients $c_{ij} \in \mathbb{R}$.

$$u(x, y) = \sum_{i+j \leq 1} c_{ij} x^i y^j$$



- Then there are barycentric coefficients $\gamma_i \in \mathbb{R}$ such that

$$u(x, y) = U(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{3} \gamma_i \lambda_i$$

Real level sets of u are easily described by this implicit formulation.

Real Higher Order Polynomials

- Transform (x, y) coordinates to degree $n \geq 1$ **Bernstein-Bezier (BB) coordinates** $\{\lambda_1^i \lambda_2^j \lambda_3^k\}_{i+j+k=n}$ by computing the trinomial expansion of

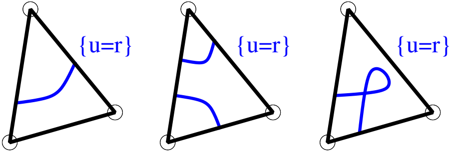
$$(\lambda_1 + \lambda_2 + \lambda_3)^n = 1$$

- Let $u(x, y)$ be a bivariate Real polynomial of **degree** n with coefficients $c_{ij} \in \mathbb{R}$

$$u(x, y) = \sum_{i+j \leq n} c_{ij} x^i y^j$$

- Then there are real Bernstein-Bezier coefficients $\gamma_{ijk} \in \mathbb{R}$ such that

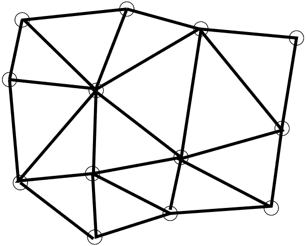
$$u(x, y) = U(\lambda_1, \lambda_2, \lambda_3) = \sum_{i+j+k=n} \gamma_{ijk} \frac{n!}{i!j!k!} \lambda_1^i \lambda_2^j \lambda_3^k$$



Real level sets with complicated geometry and topology are now possible, including, unfortunately, singularities.

C^k Continuity of Piecewise Polynomials or Finite Elements

- Let $\mathcal{T}$ be a triangulated domain or finite element mesh.
- Let $u(x, y)$ be a function on $\mathcal{T}$ such that $u|_T$ is a real polynomial of degree n on each element $T \in \mathcal{T}$.
- To get C^1 (or higher) continuity of u, we must enforce constraints at each boundary facet $F = T_1 \cap T_2$ in $\mathcal{T}$.



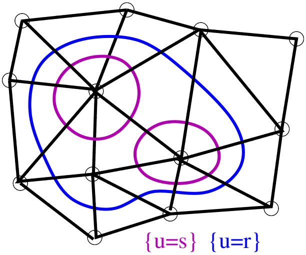
$$C^1 := \left\{ u \quad \text{s.t.} \quad \begin{array}{rcl} \lim_{\vec{x}\to p} u(\vec{x}) & = & u(p), \\ \lim_{\vec{x}\to p} \partial_{x_i} u(\vec{x}) & = & \partial_{x_i} u(p), \\ i = 1, 2; & & \forall p \in \mathcal{T} \end{array} \right\}$$

$$u \in C^1 \iff \begin{array}{c} u|_{T_1} \equiv u|_{T_2}, \\ \nabla(u|_{T_1}) \equiv \nabla(u|_{T_2}) \end{array} \quad \forall F \in \mathcal{T}$$

- This type of continuity is sometimes called **parametric continuity** since the conditions say that parameterizations of u on adjacent triangles must agree at the interface.

G^k Continuity of Level Sets

- Often C^k (parametric) continuity is too restrictive for domain modeling.
- It suffices to have **geometric continuity** of the level sets of u.



Given a level set $\{u = r\}$, let p be a point where $\{u = r\}$ crosses a face $F = T_1 \cap T_2$ of $\mathcal{T}$. Fix any parameterization

$$q_1 : [0, 1] \to \{u = r\} \cap T_1 \quad \text{s.t. } q_1(1) = p$$

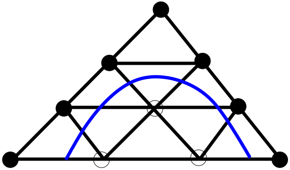
$$q_2 : [0, 1] \to \{u = r\} \cap T_2 \quad \text{s.t. } q_2(0) = p$$

Then $\{u = r\} \in G^1$ at p iff there exists $\alpha \in \mathbb{R}$

$$q'_{i,1}(1) = \alpha q'_{i,2}(0)$$

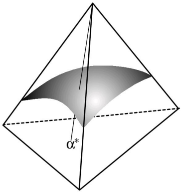
In words, the derivatives of different parameterizations need agree up to a non-zero re-scaling or re-parameterization α.

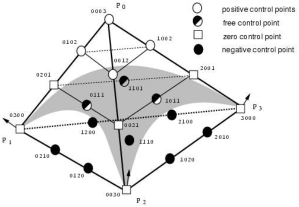
Real Algebraic Splines / Algebraic Finite Elements

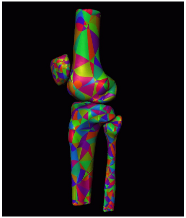


- If $\{u = r\}$ is G^k continuous at every boundary facet, it still might not still be non-singular.
- We can impose semi-algebraic conditions to ensure the real level set is always a manifold.
- A basis for these G^k continuous real algebraic finite elements can be constructed by imposing algebraic conditions.

We can also extend these notions to $\mathbb{R}^n$ and to create real algebraic **vector** finite elements.







Algebraic Spline Finite Elements

Theorem 1

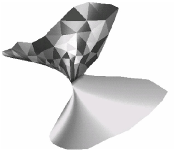
Degree d A-splines with G^{2d-3} continuity can be constructed.

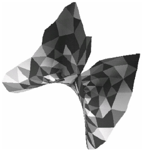
BAJAJ, XU *A-Splines: Local Interpolation and Approximation using G^k Continuous Piecewise Real Algebraic Curves*, CAGD, 1999.

Theorem 2

(a) We can construct a G^1 basis of degree 3 tetrahedral A-spline finite elements
(b) We can construct a G^2 basis of degree 5 tetrahedral A-spline finite elements.

BAJAJ *Implicit Surface Patches*, Intro. to Implicit Surfaces, Morgan Kaufman Publishers, 1997.





Cartan Surface $f = x^2 - yz^2 = 0$

BAJAJ, XU *Spline Approximations of Real Algebraic Surfaces*, Journal of Symbolic Computation, Special Isssue on Parametric Algebraic Curves and Applications, 1997

Splines and Geometric Modeling

Domain Modeling:

- Many spline types exist for domain modeling, including A-, B-, NURB- , X, and Z-splines.
  [SCHONBERG, 1958] and many many others
- Real algebraic finite elements provide an efficient method for creating smooth manifold computational domains with complicated geometry and topology.
- The approximation error between a discretization of such domains and the analytical domain can also be bounded.

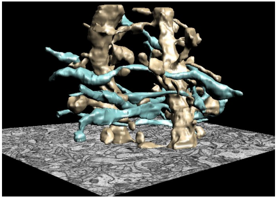
Function Modeling:

- Solutions to PDEs have different, less restrictive continuity requirements than the domains that support them.
- Measuring the error of a function discretized on a discretized domain requires new, additional theory.

Outline

Continuity of Solution Spaces of PDEs

The continuity requirements of solutions to a PDE are often weaker than even the G^1 continuity we require for domains.



- The relation of an electric field to electric potential:

$$\vec{E} = \nabla \phi$$

$\Rightarrow$ *The potential ϕ must have a well-defined gradient.*

- The Maxwell-Faraday equation: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
  $\Rightarrow$ *The electric field E must have a well-defined curl.*
- Gauss' law for magnetism: $\nabla \cdot \vec{B} = 0$
  $\Rightarrow$ *The magnetic field B must have a well-defined divergence.*

The partial derivatives of ϕ, E, and B need not be well-defined, so long as the appropriate differential operator applied to the function is well-defined.

H^1, $H(\text{curl})$, $H(\text{div})$

- Like domain continuity, functional continuity can be enforced on a mesh $\mathcal{T}$ by imposing certain constraints at each face $F = T_1 \cap T_2$, involving the normals to the two tetrahedra T_1, T_2.

$$H^1 := \left\{ u \in L^2(\Omega) \quad \text{s.t.} \quad \nabla u \in \left(L^2(\Omega)\right)^3 \right\}$$

$$u \in H^1 \iff u_1 \hat{n}_1 + u_2 \hat{n}_2 = 0, \quad \forall F \in \mathcal{T}$$

$$H(\text{curl}) := \left\{ \vec{v} \in \left(L^2(\Omega)\right)^3 \quad \text{s.t.} \quad \nabla \times \vec{v} \in \left(L^2(\Omega)\right)^3 \right\}$$

$$\vec{v} \in H(\text{curl}) \iff \vec{v}_1 \times \hat{n}_1 + \vec{v}_2 \times \hat{n}_2 = 0, \quad \forall F \in \mathcal{T}$$

$$H(\text{div}) := \left\{ \vec{v} \in \left(L^2(\Omega)\right)^3 \quad \text{s.t.} \quad \nabla \cdot \vec{v} \in L^2(\Omega) \right\}$$

$$\vec{v} \in H(\text{div}) \iff \vec{v}_1 \cdot \hat{n}_1 + \vec{v}_2 \cdot \hat{n}_2 = 0, \quad \forall F \in \mathcal{T}$$

Robust PDE Methods

A robust computational method for solving PDEs should exhibit

- **Discretization Conformity:** Computed solutions are found in a subspace of the solution space for the continuous problem
  Criterion: Discrete solution spaces replicate the deRham sequence.
- **Discretization Stability:** The true error between the discrete and continuous solutions is bounded by a multiple of the best approximation error
  Criterion: The discrete inf-sup condition is satisfied.
- **Bounded Roundoff Error:** Accumulated numerical errors due to machine precision do not compromise the computed solution
  Criterion: Matrices inverted by the linear solver are well-conditioned.

Problem Statement

Unite the theory of algebraic spline finite elements with Discrete Exterior Calculus to provide robust domain and function discretization.

Selected Prior Work

- Importance of differential geometry in computational methods for electromagnetics:

BOSSAVIT *Computational Electromagnetism* Academic Press Inc. 1998

- Primer on DEC theory and program of work:

DESBRUN, HIRANI, LEOK, MARSDEN *Discrete Exterior Calculus* arXiv:math/0508341v2 [math.DG], 2005

- Generalization of deRham diagram criteria for model conformity:

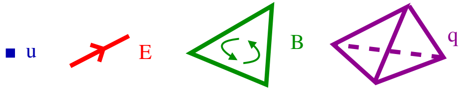
ARNOLD, FALK, WINTHER *Finite element exterior calculus, homological techniques, and applications* Acta Numerica, 15:1-155, 2006.

- Applications of DEC to elasticity problems:

YAVARI *On geometric discretization of elasticity* Journal of Mathematical Physics, 49(2):022901-1–36, 2008

(Smooth) Exterior Calculus

- Differential *k*-forms model *k*-dimensional physical phenomena.



- The exterior derivative *d* generalizes common differential operators.

$$\Lambda^0(\Omega) \xrightarrow[\text{grad}]{d_0} \Lambda^1(\Omega) \xrightarrow[\text{curl}]{d_1} \Lambda^2(\Omega) \xrightarrow[\text{div}]{d_2} \Lambda^3(\Omega)$$

- The Hodge Star transfers information between complementary dimensions of primal and dual spaces.

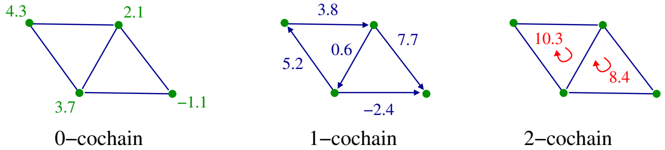
$$\Lambda^0(\Omega) \longleftarrow * \longrightarrow \Lambda^3(\Omega)$$
$$\Lambda^1(\Omega) \longleftarrow * \longrightarrow \Lambda^2(\Omega)$$

Fundamental "Theorem" of Discrete Exterior Calculus

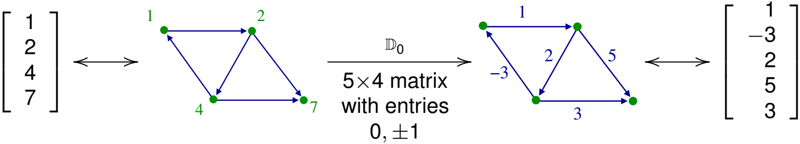
Conforming computational methods must recreate the essential properties of (continuous) exterior calculus on the discrete level.

Discrete Exterior Calculus

- Discrete differential k-forms are k-cochains, i.e. linear functions on k-simplices.



- The discrete exterior derivative $\mathbb{D}$ is the transpose of the boundary operator.



- This creates a discrete analogue of the deRham sequence.

$$\mathcal{C}^0 \xrightarrow[\text{(grad)}]{\mathbb{D}_0} \mathcal{C}^1 \xrightarrow[\text{(curl)}]{\mathbb{D}_1} \mathcal{C}^2 \xrightarrow[\text{(div)}]{\mathbb{D}_2} \mathcal{C}^3$$

The Importance of Cohomology

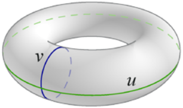
$$\Lambda^0 \xrightarrow[\text{grad}]{d_0} \Lambda^1 \xrightarrow[\text{curl}]{d_1} \Lambda^2 \xrightarrow[\text{div}]{d_2} \Lambda^3$$

$$\mathcal{C}^0 \xrightarrow{\mathbb{D}_0} \mathcal{C}^1 \xrightarrow{\mathbb{D}_1} \mathcal{C}^2 \xrightarrow{\mathbb{D}_2} \mathcal{C}^3$$

Cohomology classes represent the different types of solutions permitted by the topology of the space.

The solution spaces for a discrete method should include representatives from all cohomology classes. Hence **discretization conformity** requires that the top and bottom sequences have the same cohomology group ranks.

Example: The torus has two non-zero cohomology equivalence classes in dim. 1



$$\begin{aligned} \dim(\text{Cohomology at } \Lambda^1) \quad &:= \quad \dim\left(\ker d_1 / \text{im } d_0\right) \\ \| \text{ (if conforming)} & \\ \dim(\text{Cohomology at } \mathcal{C}^1) \quad &:= \quad \dim\left(\ker \mathbb{D}_1 / \text{im } \mathbb{D}_0\right) \end{aligned}$$

Finite element methods

Finite element methods seek solutions in subspaces of the L^2 deRham sequence.

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow[\text{grad}]{d_0} & H(\text{curl}) & \xrightarrow[\text{curl}]{d_1} & H(\text{div}) & \xrightarrow[\text{div}]{d_2} & L^2 \\
\mathcal{I}_0 \uparrow \downarrow \mathcal{P}_0 & & \mathcal{I}_1 \uparrow \downarrow \mathcal{P}_1 & & \mathcal{I}_2 \uparrow \downarrow \mathcal{P}_2 & & \mathcal{I}_3 \uparrow \downarrow \mathcal{P}_3 \\
\mathcal{C}^0 & \xrightarrow{\mathbb{D}_0} & \mathcal{C}^1 & \xrightarrow{\mathbb{D}_1} & \mathcal{C}^2 & \xrightarrow{\mathbb{D}_2} & \mathcal{C}^3
\end{array}
$$

where $\mathcal{I}$ is an interpolation map and $\mathcal{P}$ is a projection map.

Theorem [Arnold, Falk, Winther]

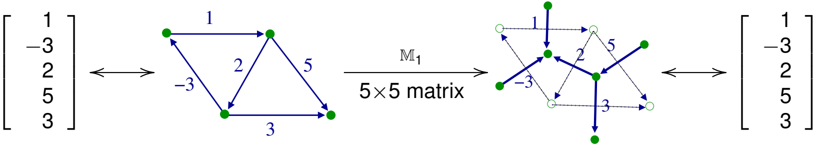
If $\mathcal{I}_k$ is Whitney interpolation and $\mathcal{P}_{k+1} d_k = \mathbb{D}_k \mathcal{P}_k$ then the top and bottom sequences have **isomorphic** cohomology.

Proof: The cohomology induced by Whitney interpolation is the simplicial cohomology [Whitney 1957] which is isomorphic to the deRham cohomology [deRham]. □

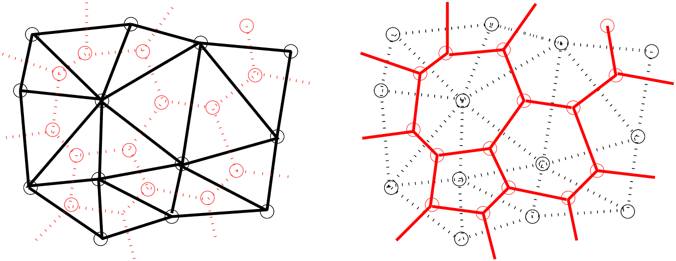
Whitney interpolation provides for discretization conformity in simple cases.

Discrete Exterior Calculus

- The discrete Hodge Star $\mathbb{M}$ transfers information between complementary dimensions on **dual** meshes. In this example, we use the identity matrix for $\mathbb{M}_1$; proper definitions will be discussed later.

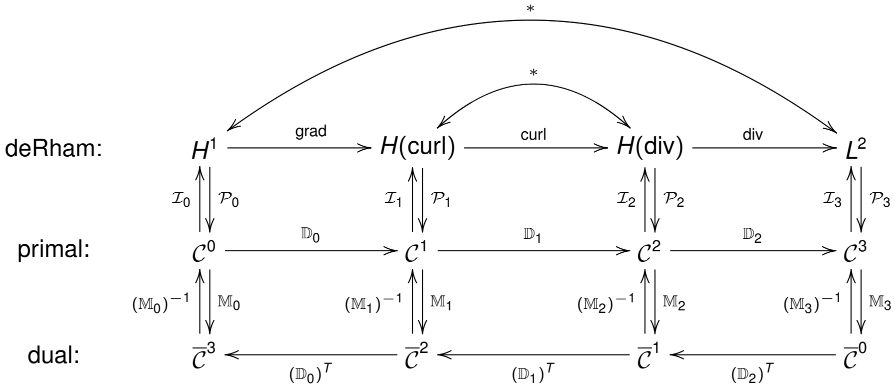


- Interpolants can be defined from the Delaunay primal mesh (black) or Voronoi dual mesh (red):



The DEC-deRham Diagram for $\mathbb{R}^3$

We combine the Discrete Exterior Calculus maps with the L^2 deRham sequence.

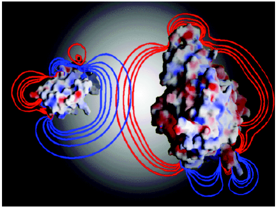


The combined diagram helps elucidate primal and dual formulations of finite element methods.

Poisson Boltzmann Electrostatics

The A-patch molecular surface model is used as the domain for the (linear) Poisson-Boltzman problem.

$$\text{div}\left(\epsilon(\vec{x})\nabla\phi(\vec{x})\right) \quad = \quad \rho_c(\vec{x}) + \bar{\kappa}(\vec{x})\phi(\vec{x}) \quad \text{in } \mathbb{R}^3$$



$$\begin{aligned}
\phi(\vec{x}) &= \text{electrostatic potential} \\
\epsilon(\vec{x}) &= \text{dielectric coefficient} = \begin{cases} \epsilon_I, & \vec{x} \in \Omega \\ \epsilon_E, & \vec{x} \in \mathbb{R}^3 - \Omega \end{cases} \\
\rho_c(\vec{x}) &= \text{charge density from atomic charges} \\
\bar{\kappa}(\vec{x}) &= \text{modified Debye-Huckel parameter}
\end{aligned}$$

Linear Poisson-Boltzmann Equation

The linearized Poisson-Boltzmann problem on a domain $\Omega \subset \mathbb{R}^3$ is

$$\text{div}\left(\epsilon(\vec{x})\nabla\phi(\vec{x})\right) = \rho_c(\vec{x}) + \bar{\kappa}(\vec{x})\phi(\vec{x}) \quad \text{in } \mathbb{R}^3$$

$$\begin{aligned}
\phi(\vec{x}) &= \text{electrostatic potential} \\
\epsilon(\vec{x}) &= \text{dielectric coefficient} = \begin{cases} \epsilon_I, & \vec{x} \in \Omega \\ \epsilon_E, & \vec{x} \in \mathbb{R}^3 - \Omega \end{cases} \\
\rho_c(\vec{x}) &= \text{charge density from atomic charges} \\
\bar{\kappa}(\vec{x}) &= \text{modified Debye-Huckel parameter}
\end{aligned}$$

Typical primal discretization:

$$\mathbb{D}_0^T \mathbb{M}_1 \epsilon \mathbb{D}_0 u = f$$

Portion of DEC-deRham diagram:

H^1 $\qquad$ $H(\text{curl})$

$$\begin{array}{ccccc}
\mathcal{C}^0 & \phi & \xrightarrow{\mathbb{D}_0} & \epsilon\mathbb{D}_0\phi & \mathcal{C}^1 \\
 & & & \downarrow \mathbb{M}_1 & \\
\bar{\mathcal{C}}^3 & (\mathbb{D}_0)^T\mathbb{M}_1\epsilon\mathbb{D}_0\phi & \xleftarrow[(\mathbb{D}_0)^T]{} & \mathbb{M}_1\epsilon\mathbb{D}_0\phi & \bar{\mathcal{C}}^2
\end{array}$$

Linear Poisson-Boltzmann Equation

$$\text{div}\left(\epsilon(\vec{x})\nabla\phi(\vec{x})\right) \quad = \quad \rho_c(\vec{x}) + \bar{\kappa}(\vec{x})\phi(\vec{x}) \quad \text{in } \mathbb{R}^3$$

- ϕ is a 0-form but it need not be discretized on a primal mesh.
- From the DEC-deRham diagram, we can derive a dual discretization.

Primal discretization:

$$\mathbb{D}_0^T \mathbb{M}_1 \epsilon \mathbb{D}_0 \phi = \rho_c(\vec{x}) + \bar{\kappa}(\vec{x})\phi(\vec{x})$$

$$\begin{array}{ccc} \phi & \xrightarrow{\mathbb{D}_0} & \epsilon\mathbb{D}_0\phi \\ & & \downarrow \mathbb{M}_1 \\ (\mathbb{D}_0)^T\mathbb{M}_1\epsilon\mathbb{D}_0\phi & \xleftarrow[(\mathbb{D}_0)^T]{} & \mathbb{M}_1\epsilon\mathbb{D}_0\phi \end{array}$$

Dual discretization:

$$\mathbb{D}_2 \epsilon (\mathbb{M}_2)^{-1} (\mathbb{D}_2)^T \phi = \rho_c(\vec{x}) + \bar{\kappa}(\vec{x})\phi(\vec{x})$$

$$\begin{array}{ccc} \epsilon(\mathbb{M}_2)^{-1}(\mathbb{D}_2)^T\phi & \xrightarrow{\mathbb{D}_2} & \mathbb{D}_2\epsilon(\mathbb{M}_2)^{-1}(\mathbb{D}_2)^T\phi \\ (\mathbb{M}_2)^{-1} \uparrow & & \\ (\mathbb{D}_2)^T\phi & \xleftarrow[(\mathbb{D}_2)^T]{} & \phi \end{array}$$

Dual discretizations may be more robust than their primal versions.

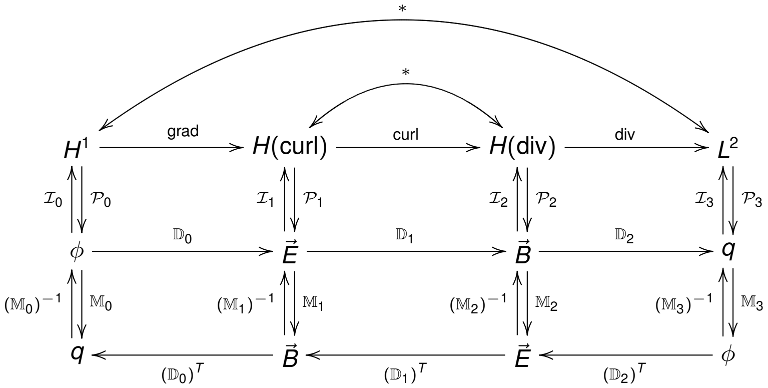
DEC for LPBE

The two discretizations inside the DEC-deRham diagram:

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\text{grad}} & H(\text{curl}) & \xrightarrow{\text{curl}} & H(\text{div}) & \xrightarrow{\text{div}} & L^2 \\
\mathcal{I}_0 \uparrow\downarrow \mathcal{P}_0 & & \mathcal{I}_1 \uparrow\downarrow \mathcal{P}_1 & & \mathcal{I}_2 \uparrow\downarrow \mathcal{P}_2 & & \mathcal{I}_3 \uparrow\downarrow \mathcal{P}_3 \\
\phi & \xrightarrow{\mathbb{D}_0} & \epsilon\mathbb{D}_0\phi & \xrightarrow{\mathbb{D}_1} & \epsilon(\mathbb{M}_2)^{-1}(\mathbb{D}_2)^T\phi & \xrightarrow{\mathbb{D}_2} & \mathbb{D}_2\epsilon(\mathbb{M}_2)^{-1}(\mathbb{D}_2)^T\phi \\
(\mathbb{M}_0)^{-1} \uparrow\downarrow \mathbb{M}_0 & & (\mathbb{M}_1)^{-1} \uparrow\downarrow \mathbb{M}_1 & & (\mathbb{M}_2)^{-1} \uparrow\downarrow \mathbb{M}_2 & & (\mathbb{M}_3)^{-1} \uparrow\downarrow \mathbb{M}_3 \\
(\mathbb{D}_0)^T\mathbb{M}_1\epsilon\mathbb{D}_0\phi & \xleftarrow[(\mathbb{D}_0)^T]{} & \mathbb{M}_1\epsilon\mathbb{D}_0\phi & \xleftarrow[(\mathbb{D}_1)^T]{} & (\mathbb{D}_2)^T\phi & \xleftarrow[(\mathbb{D}_2)^T]{} & \phi
\end{array}
$$

DEC for Maxwell's Equations

Functions involved in Maxwell's Equations fit naturally into the DEC-deRham diagram:



Short Summary

Discrete Exterior Calculus analysis:

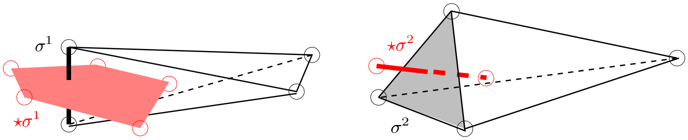
- DEC analysis is a general methodology for elucidating how to discretize PDE problems, especially mixed problems with variables discretized on both primal and dual domain meshes.
- In many cases, including the Poisson-Boltzmann equation and Maxwell's equations, DEC analysis reveals alternative PDE discretization methods.
- The robustness of such methods hinges on the choice of a discrete Hodge star and its inverse.

Outline

Discrete Hodge Stars

A discrete Hodge star transfers information between primal and dual meshes:

primal mesh simplex $\sigma^k \Longleftrightarrow$ **dual mesh cell** $\star\sigma^k$



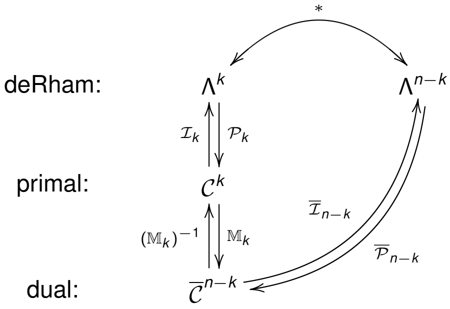
A few discrete Hodge stars from the literature:

DIAGONAL	[Desbrun et al.]	$(\mathbb{M}_k^{Diag})_{ij}$	$:=$	$\frac{\lvert\star\sigma_i^k\rvert}{\lvert\sigma_j^k\rvert}\delta_{ij}$
GEOMETRIC	[Auchmann, Kurz]	$(\mathbb{M}_k^{Geom})_{ij}$	$:=$	$\alpha_{ij}\beta_{ij}\frac{k(n-k)+1}{N_k}\frac{\lvert\star\sigma_i^k\rvert}{\lvert\sigma_j^k\rvert}$
WHITNEY	[Dodziuk],[Bell]	$(\mathbb{M}_k^{Whit})_{ij}$	$:=$	$\left(\eta_{\sigma_i^k},\eta_{\sigma_j^k}\right)_{\mathcal{C}^k}$

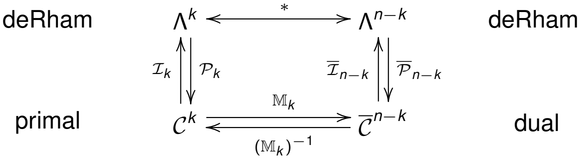
We evaluate the choice of discrete Hodge star via conformity criteria implied from the DEC-deRham diagram

Conformity Criteria from Discrete Hodge Stars

If we have projection to or interpolation from a dual mesh, we have the maps:



More concisely, we expect some commutativity of the diagram:



Commutativity at $\mathcal{C}^k$

$$\begin{array}{ccccc}
\text{deRham} & \Lambda^k & \xleftrightarrow{\;\;*\;\;} & \Lambda^{n-k} & \text{deRham} \\
 & \mathcal{I}_k \uparrow \downarrow \mathcal{P}_k & & \overline{\mathcal{I}}_{n-k} \uparrow \downarrow \overline{\mathcal{P}}_{n-k} & \\
\text{primal} & \mathcal{C}^k & \underset{(\mathbb{M}_k)^{-1}}{\overset{\mathbb{M}_k}{\rightleftarrows}} & \overline{\mathcal{C}}^{n-k} & \text{dual}
\end{array}$$

Strong commutativity at $\mathcal{C}^k$: $\quad *\mathcal{I}_k = \overline{\mathcal{I}}_{n-k}\mathbb{M}_k$

Weak commutativity at $\mathcal{C}^k$: $\quad \int_{\mathcal{T}} \alpha \wedge *\mathcal{I}_k = \int_{\mathcal{T}} \alpha \wedge \overline{\mathcal{I}}_{n-k}\mathbb{M}_k, \quad \forall \alpha \in \Lambda^k$

Criterion for weak commutativity at primal *k*-cochains ($\mathcal{C}^k$)

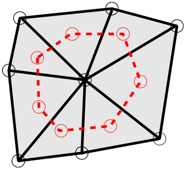
$$|\mathcal{T}| \, (\alpha, \mathcal{I}_k)_{\Lambda^k} = \int_{\mathcal{T}} \alpha \wedge \overline{\mathcal{I}}_{n-k}\mathbb{M}_k, \quad \forall \alpha \in \Lambda^k$$

Commutativity Condition

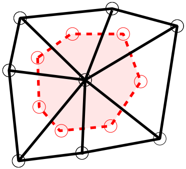
$$\text{Using } \mathbb{M}_0^{Diag}: \quad |\mathcal{T}|\,(\alpha, \lambda_i)_{H^1} = |\star\,\sigma_i^0| \int_{\star\sigma_i^0} \alpha\mu, \quad \forall \alpha \in H^1$$

$$\text{Using } \mathbb{M}_0^{Whit}: \quad |\mathcal{T}|\,(\alpha, \lambda_i)_{H^1} = \sum_{\text{vertex } j} (\lambda_i, \lambda_j)_{H^1} \int_{\star\sigma_i^0} \alpha\mu, \quad \forall \alpha \in H^1$$

The criteria required by existing discrete Hodge stars are difficult to satisfy since they relate integrals over different domains.



support of left side



support of right side

Conjecture

The weak commutativity condition can only be satisfied by a discrete Hodge star which incorporates the domain interpolation.

Thank You / Shukran

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