# Dual Interpolants for Finite Element Methods

Andrew Gillette

joint work with

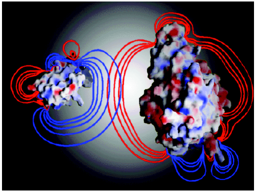
Chandrajit Bajaj and Alexander Rand

Department of Mathematics
Institute of Computational Engineering and Sciences
University of Texas at Austin, USA

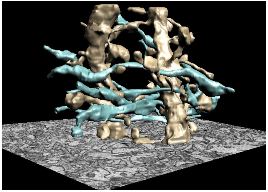
http://www.math.utexas.edu/users/agillette

# Motivation

Biological modeling requires robust computational methods to solve integral and differential equations over spatially realistic domains.



Electrostatics



Electromagnetics/ Electrodiffusion

Elasticity

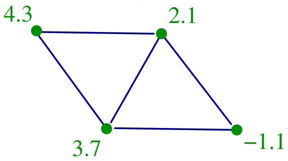
**Dual interpolation** is a new, flexible technique which can accommodate

- complicated domain geometry and topology
- multiple variables and operators
- scalar-valued and vector-valued PDEs

# Scalar Field vs. Vector Field Continuity

PDE with a scalar-valued solution

$\Downarrow$




$\Downarrow$

solve for values of the **scalar field** at vertices of a mesh

$\Downarrow$

use interpolation functions on each triangle which agree on these values

$\Downarrow$

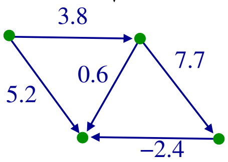
**local $C^0$ continuity along edges**

$\Downarrow$

**global $H^1$ conformity**

PDE with a vector-valued solution

$\Downarrow$




$\Downarrow$

solve for integrals of the **vector field** along edges of a mesh

$\Downarrow$

use interpolation functions on each triangle which agree on these values

$\Downarrow$

**local tangential component continuity along edges**

$\Downarrow$

**global $H(\text{curl})$ conformity**

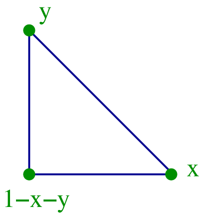
# Scalar Interpolation: Barycentric Functions

Let $\Omega$ be a convex polygon in $\mathbb{R}^2$ with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$ are called **barycentric coordinates** on $\Omega$ if they satisfy two properties:

1. **Non-negative**: $\lambda_i \geq 0$ on $\Omega$.
2. **Linear Completeness**: For any linear function $L : \Omega \to \mathbb{R}$, $L = \sum_{i=1}^{n} L(\mathbf{v}_i)\lambda_i$.

It can be shown that any set of barycentric coordinates under this definition also satisfy:

3. **Partition of unity:** $\sum_{i=1}^{n} \lambda_i \equiv 1$.
4. **Linear precision:** $\sum_{i=1}^{n} \mathbf{v}_i \lambda_i(\mathbf{x}) = \mathbf{x}$.
5. **Interpolation:** $\lambda_i(\mathbf{v}_j) = \delta_{ij}$.



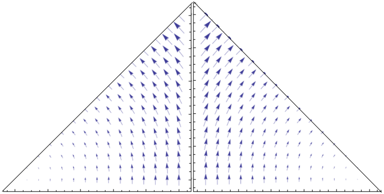

Barycentric functions written next to their associated vertices for the unit triangle

# Vector Interpolation: Whitney 1-forms

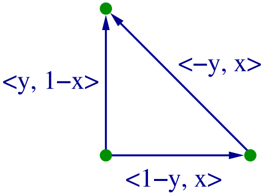
- Barycentric functions are leveraged to construct vector-valued functions called **Whitney 1-forms**
- The Whitney 1-form associated to edge $(\mathbf{v}_i, \mathbf{v}_j)$ is

$$\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$$

These functions have the desired tangential component continuity along edges.

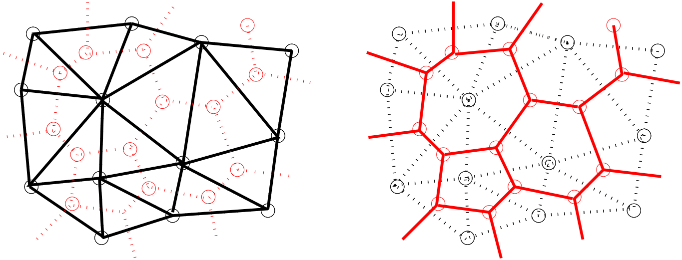


Whitney 1-forms written next to their associated edges for the unit triangle



# Dual Interpolation

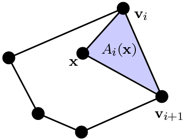
- The premise of **dual interpolation** is to create robust computational schemes analogously based on the dual domain mesh.

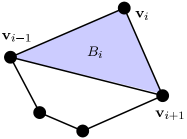


- This is a natural construct arising from the theory of **Discrete Exterior Calculus**: DESBRUN, HIRANI, LEOK, MARSDEN *Discrete Exterior Calculus* arXiv:math/0508341v2 [math.DG], 2005
- We examine three common definitions of generalized barycentric functions over polygons: Wachspress functions, Sibson functions, and Optimal functions.

# Wachspress Coordinates

Let $\mathbf{x} \in \Omega$ and define $A_i(\mathbf{x})$ and $B_i$ as the areas shown.




Define the Wachspress weight function as

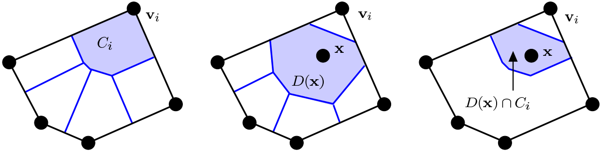
$$w_i^{\text{Wach}}(\mathbf{x}) = B_i \prod_{j \neq i, i-1} A_j(\mathbf{x}).$$

The Wachspress coordinates are then given by the *rational* functions

$$\lambda_i^{\text{Wach}}(\mathbf{x}) = \frac{w_i^{\text{Wach}}(\mathbf{x})}{\sum_{j=1}^{n} w_j^{\text{Wach}}(\mathbf{x})} \tag{1}$$

# Sibson (Natural Neighbor) Coordinates

Let $P$ denote the set of vertices $\{\mathbf{v}_i\}$ and define $P' = P \cup \{\mathbf{x}\}$.




$$C_i := |V_P(\mathbf{v}_i)| = |\{\mathbf{y} \in \Omega : |\mathbf{y} - \mathbf{v}_i| < |\mathbf{y} - \mathbf{v}_j| , \forall j \neq i\}|$$
$$= \text{area of cell for } \mathbf{v}_i \text{ in Voronoi diagram on the points of } P,$$

$$D(\mathbf{x}) := |V_{P'}(\mathbf{x})| = |\{\mathbf{y} \in \Omega : |\mathbf{y} - \mathbf{x}| < |\mathbf{y} - \mathbf{v}_i| , \forall i\}|$$
$$= \text{area of cell for } \mathbf{x} \text{ in Voronoi diagram on the points of } P'.$$

By a slight abuse of notation, we also define

$$D(\mathbf{x}) \cap C_i := |V_{P'}(\mathbf{x}) \cap V_P(\mathbf{v}_i)|.$$

The Sibson coordinates are defined to be

$$\lambda_i^{\text{Sibs}}(\mathbf{x}) := \frac{D(\mathbf{x}) \cap C_i}{D(\mathbf{x})} \qquad \text{or, equivalently,} \qquad \lambda_i^{\text{Sibs}}(\mathbf{x}) = \frac{D(\mathbf{x}) \cap C_i}{\sum_{j=1}^{n} D_j(\mathbf{x}) \cap C_j}.$$

# Optimal Coordinates

Let $g_i : \partial\Omega \to \mathbb{R}$ be the piecewise linear function satisfying

$$g_i(\mathbf{v}_j) = \delta_{ij}, \quad g_i \text{ linear on each edge of } \Omega.$$

The optimal coordinate function $\lambda_i^{\mathrm{Opt}}$ is defined to be the solution of Laplace's equations with $g_i$ as boundary data,

$$\begin{cases} \Delta\left(\lambda_i^{\mathrm{Opt}}\right) = 0, & \text{on } \Omega, \\ \lambda_i^{\mathrm{Opt}} = g_i. & \text{on } \partial\Omega. \end{cases} \tag{2}$$

These coordinates are optimal in the sense that they minimize the norm of the gradient over all functions satisfying the boundary conditions,

$$\lambda_i^{\mathrm{Opt}} = \operatorname{argmin}\left\{ |\lambda|_{H^1(\Omega)} \; : \; \lambda = g_i \text{ on } \partial\Omega \right\}.$$

# Dual Interpolation Results

For each definition considered, we have the following results.

## Theorem [G, Rand, Bajaj]

Assume certain standard geometric quality conditions on the dual mesh can be guaranteed. Then a **dual formulation** of a finite element method for a **scalar-valued** function has the **optimal convergence estimate** on the mesh.

## Theorem

Constructing **Whitney-like 1-forms** analogously to the triangular case produces globally $H$(curl)-conforming **vector fields**.

PROOF: Consider edge $(\mathbf{v}_i, \mathbf{v}_j)$ and $\lambda_k$ associated to a different vertex $\mathbf{v}_k$. Then the edge is part of the zero level set of $\lambda_k$. Hence $\nabla\lambda_k$ must be perpendicular to the edge at all points along it and any summand $\lambda_l\nabla\lambda_k$ has no tangential component on the edge. Therefore, the tangential components only depend on $\lambda_i$ and $\lambda_j$, from which the result follows. □

# Questions?

- Details can be found in our pre-prints:

*Error Estimates for Generalized Barycentric Interpolation.* Submitted, 2010.
*Dual Formulations of Mixed Finite Element Methods.* Submitted, 2010.

- Slides and pre-prints available at http://www.math.utexas.edu/users/agillette
- This talk was presented at the 2011 Joint Mathematics Meetings in New Orleans, LA.