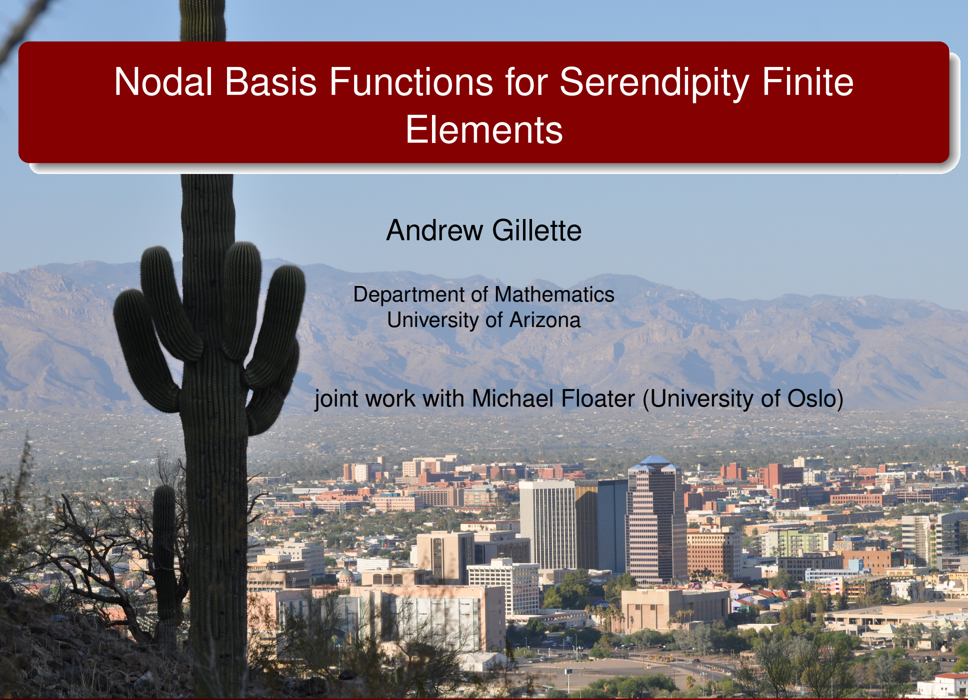

Nodal Basis Functions for Serendipity Finite Elements

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Outline

What is a serendipity finite element method?

Goal: Efficient, accurate approximation of the solution to a PDE over $\Omega \subset \mathbb{R}^n$.

Standard $O(h^r)$ **tensor product** finite element method in $\mathbb{R}^n$:

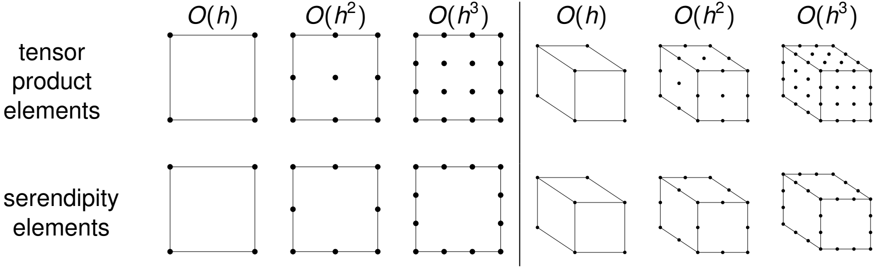
→ Mesh Ω by n-dimensional cubes of side length h.

→ Set up a linear system involving $(r+1)^n$ degrees of freedom (DoFs) per cube.

→ For unknown continuous solution u and computed discrete approximation u_h:

$$\underbrace{||u - u_h||_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C\, h^r\, |u|_{H^{r+1}(\Omega)}}_{\text{optimal error bound}}, \qquad \forall u \in H^{r+1}(\Omega).$$

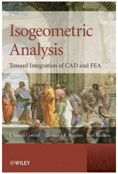
A $O(h^r)$ **serendipity** FEM converges at the **same rate** with **fewer DoFs** per element:

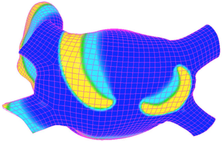


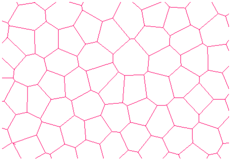
Example: For $O(h^3)$, $n = 3$, 50% fewer DoFs → $\boxed{\approx 50\% \text{ smaller linear system}}$

Motivations and Related Topics

Serendipity elements are an essential tool in modern efforts to robustly implement and accelerate high order computational methods.







- **Isogeometric analysis:** Finding basis functions suitable for both domain description and PDE approximation avoids the expensive computational bottleneck of re-meshing.

  COTTRELL, HUGHES, BAZILEVS *Isogeometric Analysis: Toward Integration of CAD and FEA*, Wiley, 2009.

- **Modern mathematics:** Finite Element Exterior Calculus, Discrete Exterior Calculus, Virtual Element Methods. . .

  ARNOLD, AWANOU *The serendipity family of finite elements*, Found. Comp. Math, 2011.
  DA VEIGA, BREZZI, CANGIANI, MANZINI, RUSSO *Basic Principles of Virtual Element Methods*, M3AS, 2013.

- **Flexible Domain Meshing:** Serendipity type elements for Voronoi meshes provide computational benefits without need of tensor product structure.

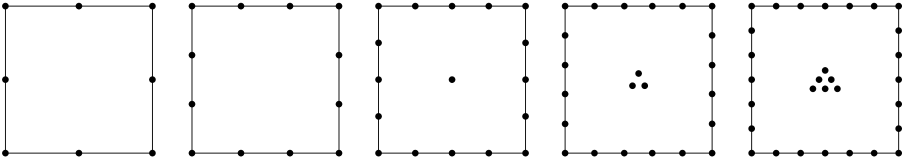
  RAND, GILLETTE, BAJAJ *Quadratic Serendipity Finite Elements on Polygons Using Generalized Barycentric Coordinates*, Mathematics of Computation, in press.

Mathematical challenges

→ Basis functions must be constructed to implement serendipity elements.
→ Current constructions lack key mathematical properties, limiting their broader usage

Goal: Construct basis functions for serendipity elements satisfying the following:

- **Symmetry:** Accommodate interior degrees of freedom that grow according to triangular numbers on square-shaped elements.
- **Tensor product structure:** Write as linear combinations of standard tensor product functions.
- **Dimensional nesting:** Generalize to methods on n-cubes for any $n \geq 2$, allowing restrictions to lower-dimensional faces.

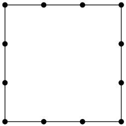


$O(h^2)$	$O(h^3)$	$O(h^4)$	$O(h^5)$	$O(h^6)$
8	12	17	23	30

Outline

Which monomials do we need?

$O(h^3)$ serendipity element:



$$\Big\{\underbrace{\overbrace{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2}^{\substack{\text{total degree at most cubic}\\ \text{(req. for } O(h^3) \text{ approximation)}}}, x^3y, xy^3, x^2y^2, x^3y^2, x^2y^3, x^3y^3}_{\substack{\text{at most cubic in each variable}\\ \text{(used in } O(h^3) \text{ tensor product methods)}}}\Big\}$$

We need an intermediate set of 12 monomials!

The **superlinear** degree of a polynomial ignores linearly-appearing variables.

Example: $\mathrm{sldeg}(xy^3) = 3$, even though $\deg(xy^3) = 4$

Definition: $\mathrm{sldeg}(x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}) := \left(\sum_{i=1}^{n} e_i\right) - \#\{e_i \;:\; e_i = 1\}$

$$\{\underbrace{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, x^3y, xy^3}_{\text{superlinear degree at most 3 (dim=12)}}, x^2y^2, x^3y^2, x^2y^3, x^3y^3\}$$

ARNOLD, AWANOU *The serendipity family of finite elements*, Found. Comp. Math, 2011.

Superlinear polynomials form a lower set

Given a monomial $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$,

associate the multi-index of n non-negative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$.

Define the superlinear norm of α as $|\alpha|_{sprlin} := \sum_{\substack{j=1 \\ \alpha_j \geq 2}}^{n} \alpha_j$,

so that the superlinear multi indices are $S_r = \left\{\alpha \in \mathbb{N}_0^n : |\alpha|_{sprlin} \leq r\right\}$.

Observe that S_r has a partial ordering $\mu \leq \alpha$ means $\mu_i \leq \alpha_i$.

Thus S_r is a **lower set**, meaning $\boxed{\alpha \in S_r, \mu \leq \alpha \implies \mu \in S_r}$

We can thus apply the following recent result.

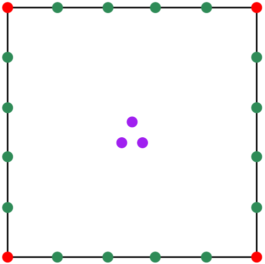
Theorem (Dyn and Floater, 2013)

Fix a lower set $L \subset \mathbb{N}_0^n$ and points $z_\alpha \in \mathbb{R}^n$ for all $\alpha \in L$. For any sufficiently smooth n-variate real function f, there is a *unique polynomial p* in $\mathrm{span}\{x^\alpha : \alpha \in L\}$ that *interpolates f at the points z_α*, with partial derivative interpolation for repeated z_α.

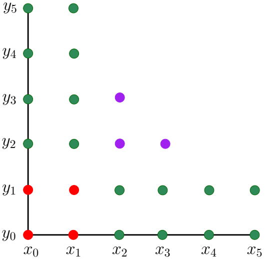
DYN AND FLOATER *Multivariate polynomial interpolation on lower sets*, J. Approx. Th., to appear.

Partitioning and reordering the multi-indices

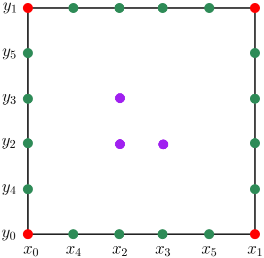
By a judicious choice of the interpolation points $z_\alpha = (x_i, y_j)$, we recover the dimensionality associations of the degrees of freedom of serendipity elements.



The order 5 serendipity element, with degrees of freedom color-coded by dimensionality.



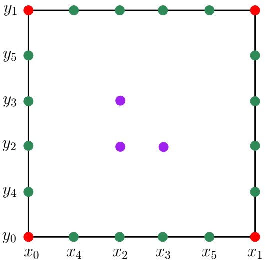
The lower set S_5, with equivalent color coding.



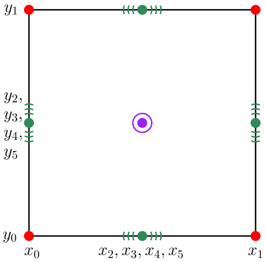
The lower set S_5, with domain points z_α reordered.

Symmetrizing the multi-indices

By collecting the re-ordered interpolation points $z_\alpha = (x_i, y_j)$, at midpoints of the associated face, we recover the dimensionality associations of the degrees of freedom of serendipity elements.



The lower set S_5, with domain points z_α reordered.

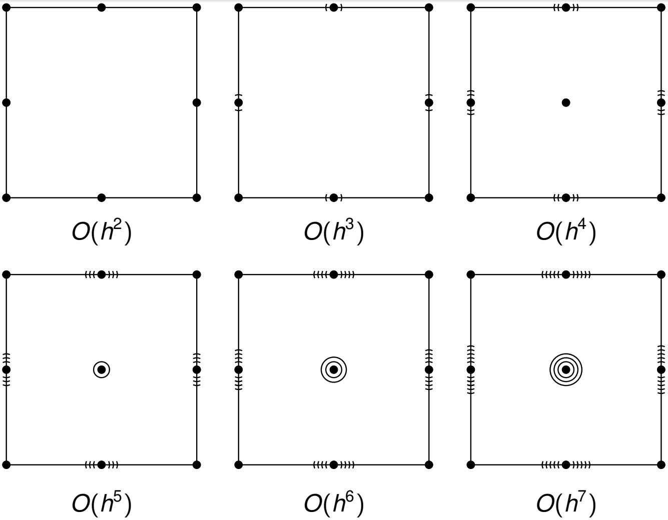


A symmetric reordering, with multiplicity. The associated interpolant recovers values at dots, three partial derivatives at edge midpoints, and two partial derivatives at the face midpoint.

Outline

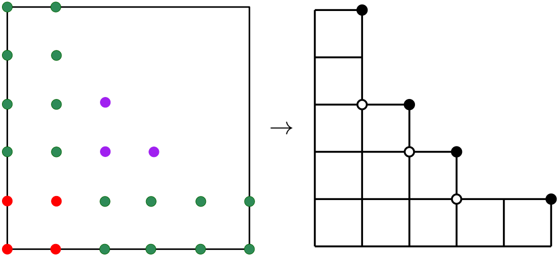
2D symmetric serendipity elements

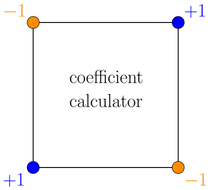
Symmetry: Accommodate interior degrees of freedom that grow according to triangular numbers on square-shaped elements.



Tensor product structure

The Dyn-Floater interpolation scheme is expressed in terms of tensor product interpolation over ‘maximal blocks’ in the set using an inclusion-exclusion formula.





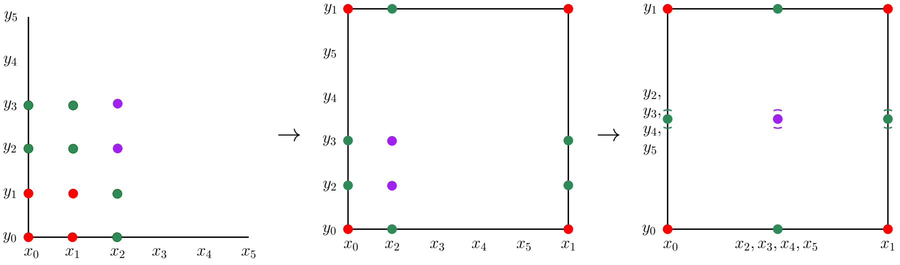
Put differently, the linear combination is the sum over *all* blocks within the lower set with coefficients determined as follows:

→ Place the coefficient calculator at the extremal block corner.
→ Add up all values appearing in the lower set.
→ The coefficient for the block is the value of the sum.

Hence: black dots → +1;   white dots → -1;   others → 0.

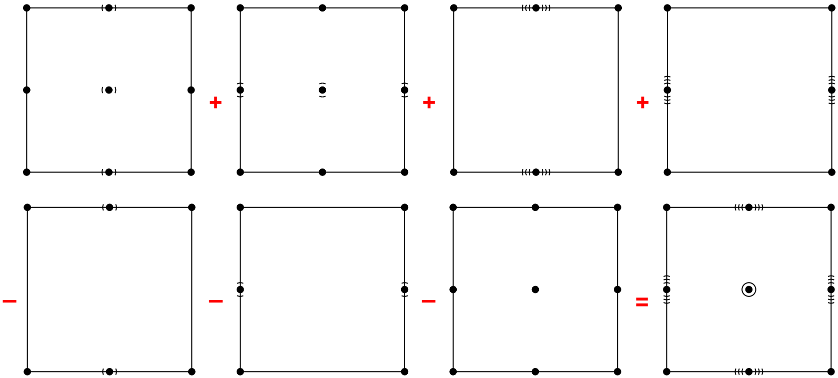
Tensor product structure

Thus, using our symmetric approach, each maximal block in the lower set becomes a standard tensor-product interpolant.



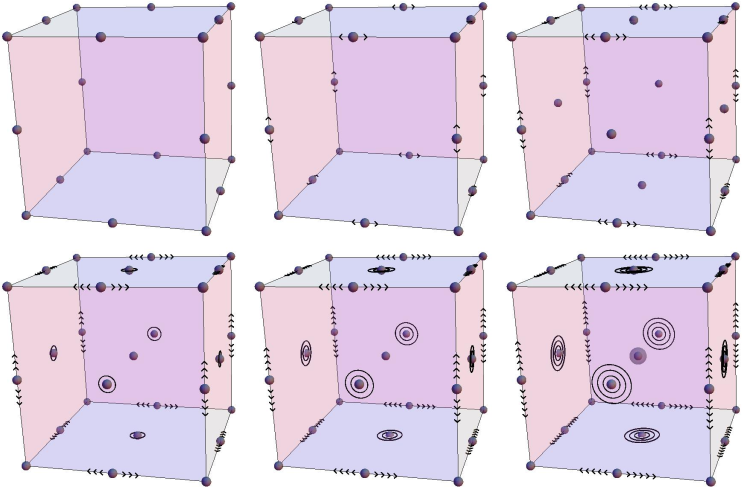
Linear combination of tensor products

Tensor product structure: Write basis functions as linear combinations of standard tensor product functions.



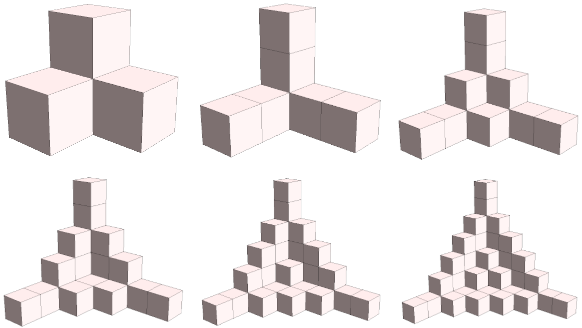
3D elements

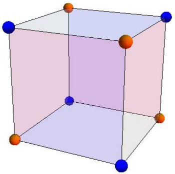
Dimensional nesting: Generalize to methods on *n*-cubes for any $n \geq 2$, allowing restrictions to lower-dimensional faces.



3D coefficient computation

Lower sets for superlinear polynomials in 3 variables:





Decomposition into a linear combination of tensor product interpolants works the same as in 2D, using the 3D coefficient calculator at left. (blue $\to$ +1; orange $\to$ -1).

FLOATER, G, *Nodal bases for the serendipity family of finite elements*, Submitted, 2014. Available as arXiv:1404.6275

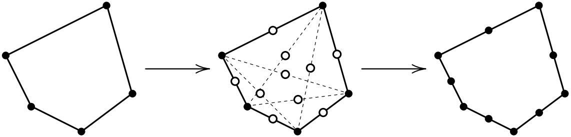
Outline

Future Directions

- Incorporate elements into finite element software packages.
- Analyze speed vs. accuracy trade-offs.

	1	2	3	4	5	6	7	$r \geq 2n$
$n = 2$								
$\dim \mathcal{Q}_r$	4	9	16	25	36	49	64	$r^2 + 2r + 1$
$\dim \mathcal{S}_r$	4	8	12	17	23	30	38	$\frac{1}{2}(r^2 + 3r + 6)$
$n = 3$								
$\dim \mathcal{Q}_r$	8	27	64	125	216	343	512	$r^3 + 3r^2 + 3r + 1$
$\dim \mathcal{S}_r$	8	20	32	50	74	105	144	$\frac{1}{6}(r^3 + 6r^2 + 29r + 24)$

- Expand serendipity results to generic polygons and polyhedra.



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Thanks for your attention!

Slides and pre-prints: `http://math.arizona.edu/~agillette/`