What is a Good Linear Finite Element... On a Generic Polytope?

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Harmonic and triangulation interpolants

Let P be a polygon and $u : P \to \mathbb{R}$ a function known only at the vertices of P.

The **harmonic interpolant of u over P** is the solution to $\begin{cases} \Delta(I_P u) = 0, & \text{on } P, \\ I_P u = g_u, & \text{on } \partial P. \end{cases}$

where $g_u : \partial\Omega \to \mathbb{R}$ is the piecewise linear function equal to u at the vertices of P.

The **triangulation interpolant of u over P**, denoted $I_{\mathcal{T}} u$, is the piecewise linear interpolation of g_u with respect to a triangulation $\mathcal{T}$ of the vertices of P.

Motivating question: Under what conditions on P and $\mathcal{T}$ is $I_P u \approx I_{\mathcal{T}} u$?

We examine this in the context of *a priori* error estimates for finite element methods:

$$\underbrace{|u - Iu|_{H^1(P)}}_{\text{error between function and its interpolation}} \leq \underbrace{C\,\text{diam}(P)\,|u|_{H^2(P)}}_{\text{bound depends on size and geometry of } P}, \qquad \underbrace{\forall u \in H^2(P)}_{\text{estimate holds for any } u \text{ with bounded 2nd derivatives}}$$

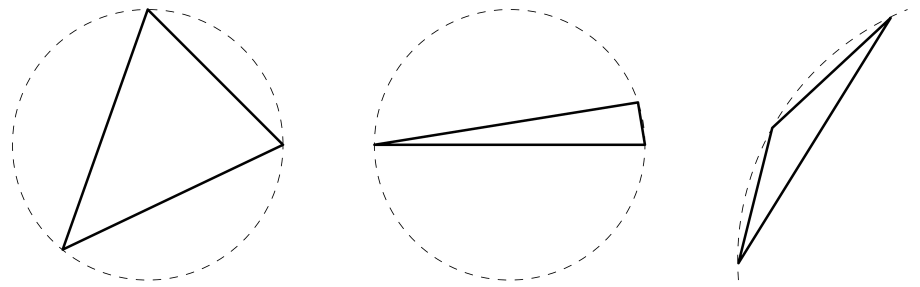
Scaling P so that $\text{diam}(P) = 1$, we examine how C depends on the geometry of P.

Element shape quality measures

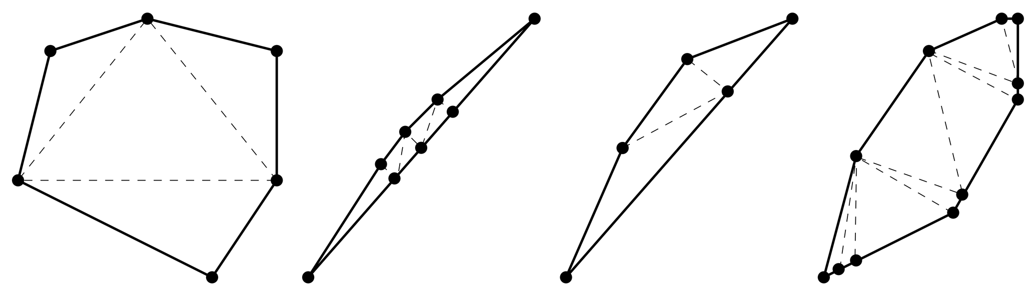
The **aspect ratio** of a polygon P is $\gamma_P := \dfrac{\text{diam}(P)}{\text{max radius of an inscribed circle}}$.

The **circumradius** of a triangle T is $R_T :=$ radius of circle through the vertices of T.

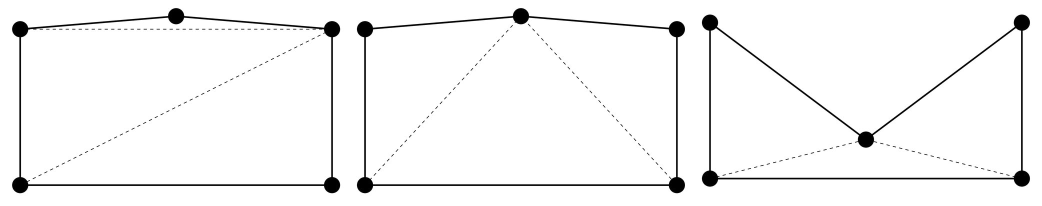
On triangles, a large aspect ratio does not necessarily imply that C is large, but a large circumradius always does.



On polygons, a large aspect ratio does not correlate in a clear way with the circumradii of triangulations of its vertices. . .



. . . however, over all possible triangulations of the vertices, the **constrained Delaunay triangulation** will have the minimal maximum circumradius.



not Delaunay   Delaunay   constrained Delaunay

Comparing interpolants on polygons in 2D

The following ***a priori* error estimate** relates harmonic and triangulation interpolants.

Theorem: There exists a constant $C > 0$ such that for any polygon P, possibly non-convex, all functions $u \in H^2(P)$, and all triangulations $\mathcal{T}$ of P,

$$|u - I_P u|_{H^1(P)} \leq C \left(\max_{T \in \mathcal{T}} R_T \right) |u|_{H^2(P)}, \qquad \forall u \in H^2(P),$$

where R_T denotes the circumradius of triangle T.

To examine the sharpness of this estimate, consider a sequence of polygons $\{P_i\}$ where

P_i has a vertex $\mathbf{v}_i$ and edge e_i such that $\text{dist}(\mathbf{v}_i, e_i) \to 0$ as $i \to \infty$

and there exists $K > 0$ independent of i such that $d(\mathbf{v}_i, \partial e_i) > K$.

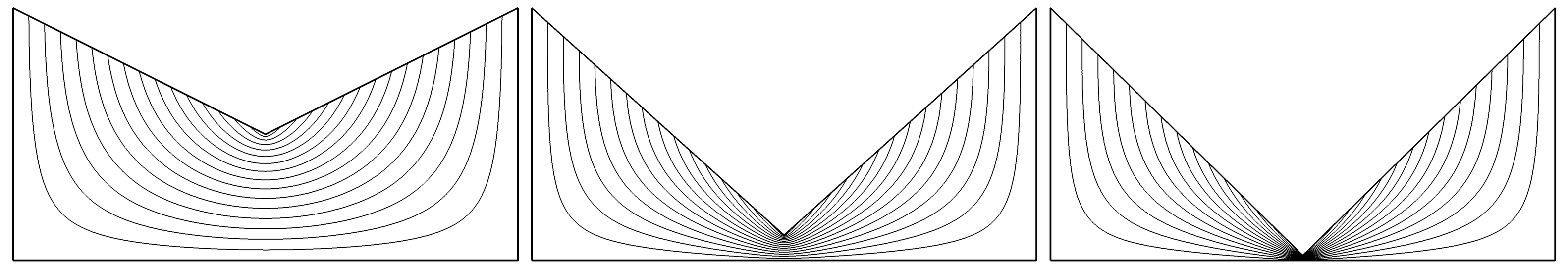
Without loss of generality, by translation, scaling, rotation and reflection, we assume that $\text{diam}(P_i) = 1$, edge e_i lies on the x-axis, and $\mathbf{v}_i$ lies on the positive y-axis.

Lemma: Set $u(x, y) := x^2$ and consider $I_{P_i} u$ for a sequence of polygon-vertex-edge tuples $(P_i, \mathbf{v}_i, e_i)_{i=1}^{\infty}$ as above. If all the P_i are **convex** then

$$\lim_{i \to \infty} \frac{|u - I_{P_i} u|_{H^1(P_i)}}{|u|_{H^2(P_i)}} = \infty.$$

Further, there exists a sequence with **non-convex** P_i such that

$$\lim_{i \to \infty} \frac{|u - I_{P_i} u|_{H^1(P_i)}}{|u|_{H^2(P_i)}} = C < \infty.$$



Contour plots of $I_{P_i} u$ for a sequence of non-convex $\{P_i\}$ used in the proof of the lemma. Large gradients are concentrated at a point; the total gradient remains square integrable.

Relation between the Theorem and Lemma: On polygons:

Convex element: The harmonic interpolant in general has no sharper *a priori* estimate than estimates derived by triangulation of the domain.

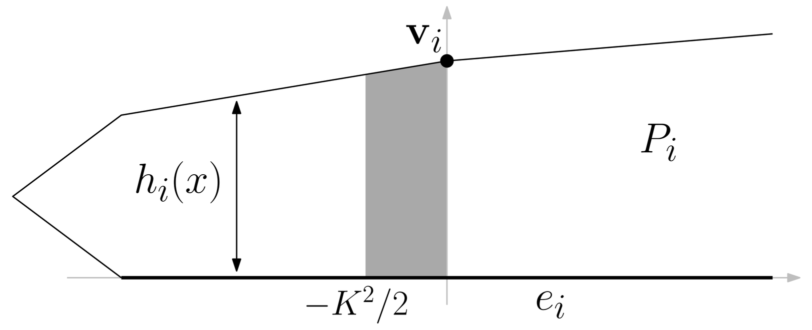
Non-convex element: The H^1-error in the harmonic interpolant can remain bounded even when all triangulations of the domain involve large angles.

Tools used in the proofs

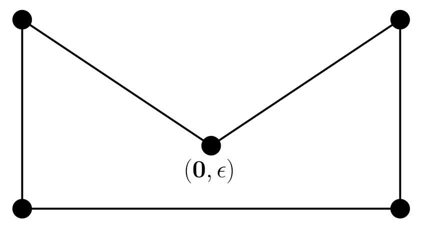
By **Dirichlet's principle**, the harmonic interpolant $I_P u$ is also the minimizer of H^1 semi-norm:

$$I_P u = \text{argmin} \left\{ |v|_{H^1(P)} \,:\, v = g_u \text{ on } \partial P \right\}.$$

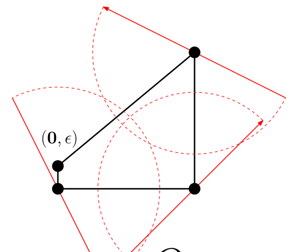
In the convex case for the lemma, we show that the integral of $\partial_y I_{P_i} u$ grows without bound for $u = x^2$ over the shaded region as $d(\mathbf{v}_i, e_i) \to 0$.



In the non-convex case, Dirichlet's principle ensures that $|I_{P_\epsilon} u|_{H^1(P_\epsilon)} \leq C\,||g_\epsilon||^2_{H^{1/2}(\partial Q_\epsilon)}$, by a trace theorem. We show that the domains Q_ϵ can be locally flattened with a uniformly bounded Lipschitz constant.



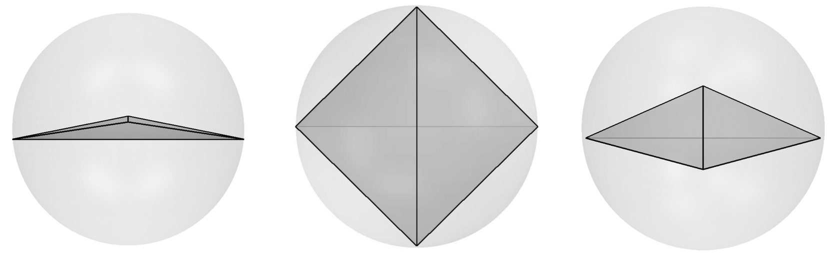
P_ϵ



Q_ϵ

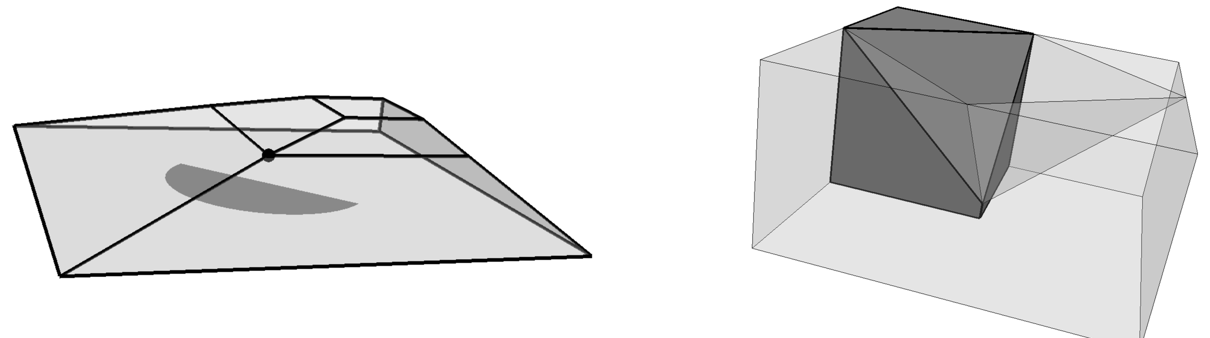
Comparing interpolants on polyhedra in 3D

A **sliver tetrahedron** can have a large aspect ratio but modest circumradius, preventing a direct analogue of the Theorem in terms of circumradii.



Theorem: Fix $\gamma > 2$. There exists a constant $C > 0$ depending only on γ such that for any polyhedron P with aspect ratio less than γ,

$$|u - I_P u|_{H^1(P)} \leq C\,\text{diam}(P)\,|u|_{H^2(P)}, \qquad \forall u \in H^2(P).$$



convex polyhedron   non-convex polyhedron

Conclusions from 2D that generalize to polyhedra:

Convex element: The H^1-error in the harmonic interpolant cannot be bounded if a vertex can be arbitrarily close to a non-adjacent face.

Non-convex element: The H^1-error in the harmonic interpolant can remain bounded even when a vertex and non-adjacent face are close, as in example shown above.

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