Fewer Degrees of Freedom, Same Convergence Rate: Recent progress in serendipity finite element methods

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Aerospace Engineering Seminar

What is a serendipity finite element method?

Goal: Efficient, accurate approximation of the solution to a PDE over $\mathcal{D} \subset \mathbb{R}^d$.

Standard $O(h^r)$ **tensor product** finite element method in $\mathbb{R}^d$:

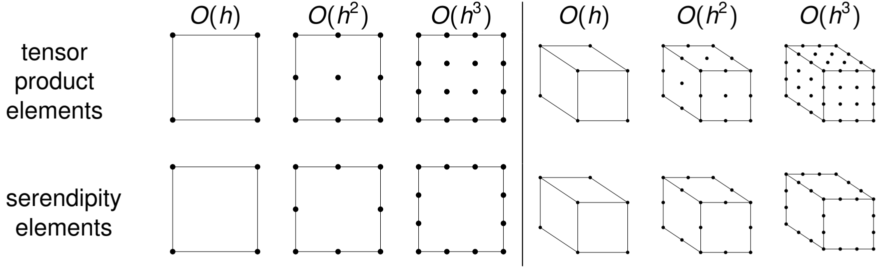
$\rightarrow$ Mesh $\mathcal{D}$ by d-dimensional cubes of side length h.

$\rightarrow$ Set up a linear system involving $(r+1)^d$ degrees of freedom (DoFs) per cube.

$\rightarrow$ For unknown continuous solution u and computed discrete approximation u_h:

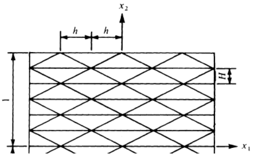
$$\underbrace{||u - u_h||_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C\, h^r\, |u|_{H^2(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^2(\Omega).$$

A $O(h^r)$ **serendipity** FEM converges at the **same rate** with **fewer DoFs** per element:



Example: For $O(h^3)$, $d = 3$, 50% fewer DoFs $\rightarrow$ $\boxed{\approx 50\% \text{ smaller linear system}}$

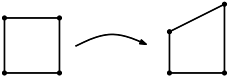
Error estimates depend on mesh geometry

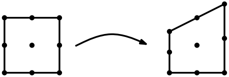


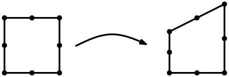
→ Triangular meshes require a maximum angle condition for the estimate to hold.

BABUŠKA, AZIZ *On the angle condition in the finite element method*, SIAM J. Num. An., 1976.

→ *A priori* error estimates are based on **affine** maps from a reference element.

linear  $||u - u_h||_{H^1(\Omega)} \leq c\, h\, |u|_{H^2(\Omega)}, \qquad \forall u \in H^2(\Omega)$

quadratic  $||u - u_h||_{H^1(\Omega)} \leq c\, h^2\, |u|_{H^3(\Omega)}, \qquad \forall u \in H^3(\Omega)$

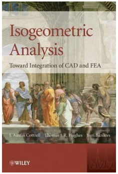
"serendipity"  $||u - u_h||_{H^1(\Omega)} \leq c\, h\, |u|_{H^3(\Omega)}, \qquad \forall u \in H^3(\Omega)$

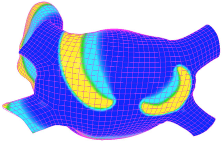
→ The non-affinely mapped serendipity element converges at a linear rate.

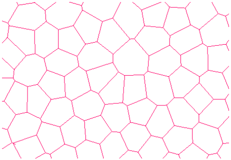
ARNOLD, BOFFI, FALK *Approximation by Quadrilateral Finite Elements*, Math. Comp., 2002

Motivations and Related Topics

Goal: Find geometry-sensitive basis functions for serendipity methods.







- **Isogeometric analysis:** Finding basis functions suitable for both domain description and PDE approximation avoids the expensive computational bottleneck of re-meshing.

  COTTRELL, HUGHES, BAZILEVS *Isogeometric Analysis: Toward Integration of CAD and FEA*, Wiley, 2009.

- **Minimal Effort, Maximal Efficiency:** Modern applications such as patient-specific cardiac electrophysiology needs efficient, stable, error-bounded 'real-time' methods

- **Flexible Domain Meshing:** FEM for polygons and polytopes allows Voronoi domain meshing and avoids dealing with tetrahedral slivers

- **New mathematics:** Finite Element Exterior Calculus, Discrete Exterior Calculus, Virtual Element Methods. . .

  ARNOLD, AWANOU *The serendipity family of finite elements*, Found. Comp. Math, 2011.

  DA VEIGA, BREZZI, CANGIANI, MANZINI, RUSSO *Basic Principles of Virtual Element Methods*, M3AS, 2013.

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Outline

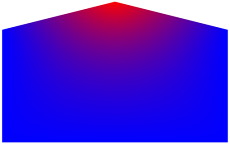
1. Brief introduction to barycentric coordinates
2. Quadratic serendipity elements on polygons
3. Bernstein and Hermite style serendipity elements on cubes
4. Applications and Future Directions

Definition of generalized barycentric coordinates

Let Ω be a convex polygon in $\mathbb{R}^2$ with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$ are called **barycentric coordinates** on Ω if they satisfy two properties:

1. **Non-negative**: $\lambda_i \geq 0$ on Ω.
2. **Linear Completeness**: For any linear function $L : \Omega \to \mathbb{R}$, $L = \sum_{i=1}^{n} L(\mathbf{v}_i)\lambda_i$.

Any set of barycentric coordinates under this definition also satisfies:

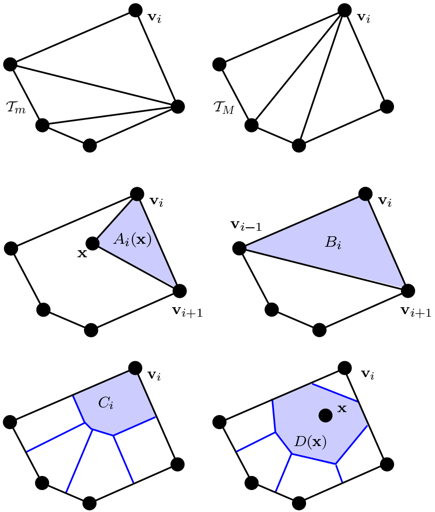


3. **Partition of unity:** $\sum_{i=1}^{n} \lambda_i \equiv 1$.
4. $\boxed{\textbf{Linear precision: } \sum_{i=1}^{n} \mathbf{v}_i \lambda_i(\mathbf{x}) = \mathbf{x}.}$
5. **Interpolation:** $\lambda_i(\mathbf{v}_j) = \delta_{ij}$.

Theorem [Warren, 2003]

If the λ_i are rational functions of degree $n-2$, then they are unique.

Many generalizations to choose from . . .



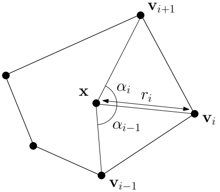
- **Triangulation**
  ⇒ Floater, Hormann, Kós, *A general construction of barycentric coordinates over convex polygons*, 2006

  $$0 \leq \lambda_i^{\mathcal{T}_m}(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq \lambda_i^{\mathcal{T}_M}(\mathbf{x}) \leq 1$$

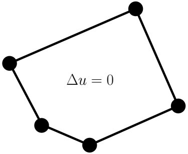
- **Wachspress**
  ⇒ Wachspress, *A Rational Finite Element Basis*, 1975.

- **Sibson / Laplace**
  ⇒ Sibson, *A vector identity for the Dirichlet tessellation*, 1980.
  ⇒ Hiyoshi, Sugihara, *Voronoi-based interpolation with higher continuity*, 2000.

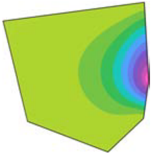
Many generalizations to choose from . . .



- **Mean value**
  $\Rightarrow$ Floater, *Mean value coordinates*, 2003.
  $\Rightarrow$ Floater, Kós, Reimers, *Mean value coordinates in 3D*, 2005.



- **Harmonic**
  $\Rightarrow$ Warren, *Barycentric coordinates for convex polytopes*, 1996.
  $\Rightarrow$ Warren, Schaefer, Hirani, Desbrun, *Barycentric coordinates for convex sets*, 2007.
  $\Rightarrow$ Christiansen, *A construction of spaces of compatible differential forms on cellular complexes*, 2008.

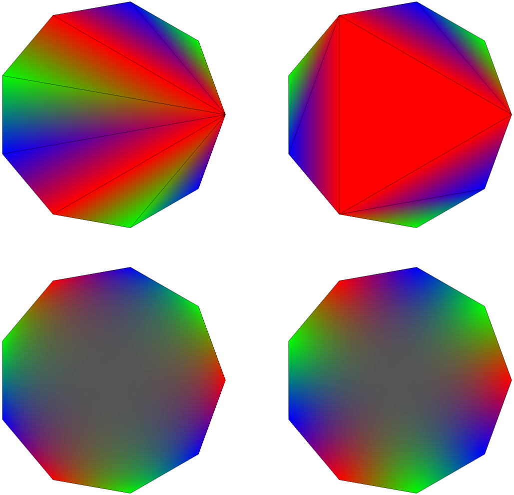


- **Maximum Entropy** $\Rightarrow$ Hormann, Sukumar, *Maximum Entropy Coordinates for Arbitrary Polytopes*, 2008.

$\leftarrow$ (this figure is from the above paper)

Comparison via ‘eyeball’ norm

Triangulated Triangulated



Wachspress Mean Value

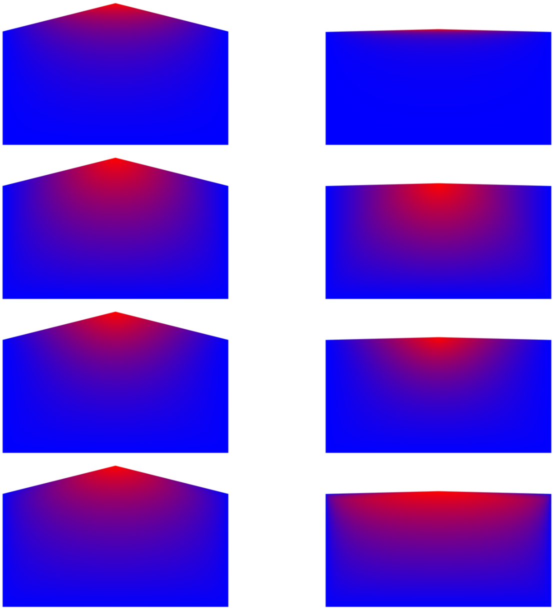
Comparison via 'eyeball' norm

Wachspress

Sibson

Mean Value

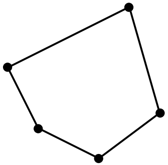
Discrete Harmonic



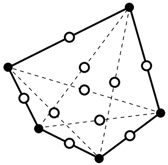
Outline

From linear to quadratic elements

A naïve quadratic element is formed by products of linear element basis functions:



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\}$$



Why is this naïve?

- For an n-gon, this construction gives $n + \binom{n}{2}$ basis functions $\lambda_a\lambda_b$
- The space of quadratic polynomials is only dimension 6: $\{1, x, y, xy, x^2, y^2\}$
- Conforming to a quadratic function on the boundary requires 1 DoF per edge and 1 DoF per vertex $\Rightarrow$ *only $2n$ functions needed!*

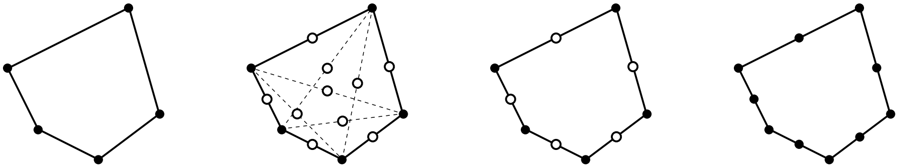
Problem Statement

Construct $2n$ basis functions associated to the vertices and edge midpoints of an arbitrary n-gon such that a quadratic convergence estimate is obtained.

Polygonal Quadratic Serendipity Elements

We define matrices $\mathbb{A}$ and $\mathbb{B}$ to reduce the naïve quadratic basis.

$$\begin{aligned}
\textbf{filled dot} &= \text{Lagrangian domain point} \\
&= \text{all functions in the set evaluate to 0} \\
&\qquad \text{except the associated function which evaluates to 1} \\
\textbf{open dot} &= \text{non-Lagrangian domain point} \\
&= \text{partition of unity satisfied, but not Lagrange property}
\end{aligned}$$



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\} \xrightarrow{\mathbb{A}} \{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

n	$n + \binom{n}{2}$	$2n$	$2n$
Linear	Quadratic	Serendipity	Lagrange

From quadratic to serendipity

The bases are ordered as follows:

ξ_{ii}	and	$\lambda_a\lambda_a$	= basis functions associated with vertices
$\xi_{i(i+1)}$	and	$\lambda_a\lambda_{a+1}$	= basis functions associated with edge midpoints
		$\lambda_a\lambda_b$	= basis functions associated with interior diagonals, i.e. $b \notin \{a-1, a, a+1\}$

Serendipity basis functions ξ_{ij} are a linear combination of pairwise products $\lambda_a\lambda_b$:

$$\begin{bmatrix} \xi_{ii} \\ \vdots \\ \xi_{i(i+1)} \end{bmatrix} = \mathbb{A} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix} = \begin{bmatrix} c_{11}^{11} & \cdots & c_{ab}^{11} & \cdots & c_{(n-2)n}^{11} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{ij} & \cdots & c_{ab}^{ij} & \cdots & c_{(n-2)n}^{ij} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{n(n+1)} & \cdots & c_{ab}^{n(n+1)} & \cdots & c_{(n-2)n}^{n(n+1)} \end{bmatrix} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix}$$

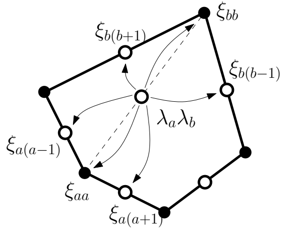
From quadratic to serendipity

We ***require*** the serendipity basis to have quadratic approximation power:

Constant precision: $\quad 1 = \sum_i \xi_{ii} + 2\xi_{i(i+1)}$

Linear precision: $\quad \mathbf{x} = \sum_i \mathbf{v}_i\xi_{ii} + 2\mathbf{v}_{i(i+1)}\xi_{i(i+1)}$

Quadratic precision: $\quad \mathbf{x}\mathbf{x}^T = \sum_i \mathbf{v}_i\mathbf{v}_i^T\xi_{ii} + (\mathbf{v}_i\mathbf{v}_{i+1}^T + \mathbf{v}_{i+1}\mathbf{v}_i^T)\xi_{i(i+1)}$



Theorem

Constants $\{c_{ij}^{ab}\}$ exist for **any** convex polygon such that the resulting basis $\{\xi_{ij}\}$ satisfies constant, linear, and quadratic precision requirements.

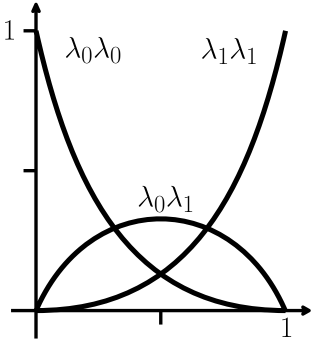
Proof: We produce a coefficient matrix $\mathbb{A}$ with the structure

$$\mathbb{A} := \left[\ \mathbb{I} \ \middle|\ \mathbb{A}'\ \right]$$

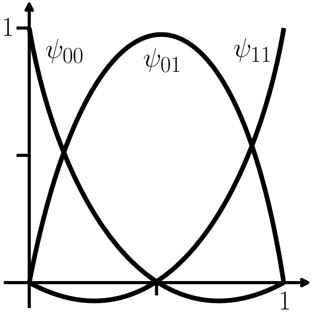
where $\mathbb{A}'$ has only six non-zero entries per column and show that the resulting functions satisfy the six precision equations.

Pairwise products vs. Lagrange basis

Even in 1D, pairwise products of barycentric functions do not form a Lagrange basis at interior degrees of freedom:



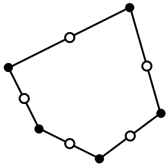
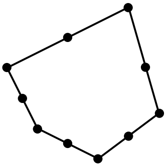
Pairwise products



Lagrange basis

Translation between these two bases is straightforward and generalizes to the higher dimensional case...

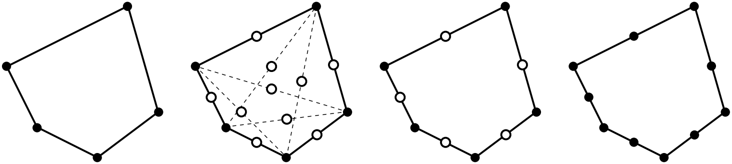
From serendipity to Lagrange

$$\{\xi_{ij}\} \xrightarrow{\quad\mathbb{B}\quad} \{\psi_{ij}\}$$

$$[\psi_{ij}] = \begin{bmatrix} \psi_{11} \\ \psi_{22} \\ \vdots \\ \psi_{nn} \\ \psi_{12} \\ \psi_{23} \\ \vdots \\ \psi_{n1} \end{bmatrix} = \left[\begin{array}{cccc|ccccc} 1 & & & & -1 & & \cdots & & -1 \\ & 1 & & & -1 & -1 & \cdots & & \\ & & \ddots & & & \ddots & \ddots & & \\ & & & \ddots & & & \ddots & \ddots & \\ & & & 1 & & & & -1 & -1 \\ \hline & & & & 4 & & & & \\ & & & & & 4 & & & \\ & 0 & & & & & \ddots & & \\ & & & & & & & \ddots & \\ & & & & & & & & 4 \end{array}\right] \begin{bmatrix} \xi_{11} \\ \xi_{22} \\ \vdots \\ \xi_{nn} \\ \xi_{12} \\ \xi_{23} \\ \vdots \\ \xi_{n1} \end{bmatrix} = \mathbb{B}[\xi_{ij}].$$

Serendipity Theorem



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\} \xrightarrow{\mathbb{A}} \{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

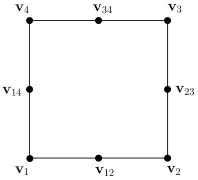
Theorem

For polygons satisfying quality bounds (aspect ratio, edge lengths, interior angles):

- $||\mathbb{A}||$ is uniformly bounded,
- $||\mathbb{B}||$ is uniformly bounded,
- Interpolating with $\{\psi_{ij}\}$ ensures the approximate solution $u_h \in H^1$, and
- span$\{\psi_{ij}\} \supset \mathcal{P}_2(\mathbb{R}^2)$ = quadratic polynomials in x and y

RAND, GILLETTE, BAJAJ *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Submitted, 2011

Special case of a square



Bilinear functions are barycentric coordinates:

$$\begin{aligned}\lambda_1 &= (1-x)(1-y)\\ \lambda_2 &= x(1-y)\\ \lambda_3 &= xy\\ \lambda_4 &= (1-x)y\end{aligned}$$

Compute $[\xi_{ij}] := \left[\ \mathbb{I}\ \middle|\ \mathbb{A}'\ \right][\lambda_a\lambda_b]$

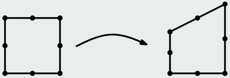
$$\begin{bmatrix}\xi_{11}\\ \xi_{22}\\ \xi_{33}\\ \xi_{44}\\ \xi_{12}\\ \xi_{23}\\ \xi_{34}\\ \xi_{14}\end{bmatrix} = \begin{bmatrix}1 & & 0 & -1 & 0\\ 0 & & 0 & 0 & -1\\ 0 & & 0 & -1 & 0\\ 0 & \cdots & 0 & 0 & -1\\ 0 & \cdots & 0 & 1/2 & 1/2\\ 0 & & 0 & 1/2 & 1/2\\ 0 & & 0 & 1/2 & 1/2\\ 0 & & 1 & 1/2 & 1/2\end{bmatrix}\begin{bmatrix}\lambda_1\lambda_1\\ \lambda_2\lambda_2\\ \lambda_3\lambda_3\\ \lambda_4\lambda_4\\ \lambda_1\lambda_2\\ \lambda_2\lambda_3\\ \lambda_3\lambda_4\\ \lambda_1\lambda_4\end{bmatrix} = \begin{bmatrix}(1-x)(1-y)(1-x-y)\\ x(1-y)(x-y)\\ xy(-1+x+y)\\ (1-x)y(y-x)\\ (1-x)x(1-y)\\ x(1-y)y\\ (1-x)xy\\ (1-x)(1-y)y\end{bmatrix}$$

$$\operatorname{span}\left\{\xi_{ii},\xi_{i(i+1)}\right\} = \operatorname{span}\left\{1,x,y,x^2,y^2,xy,x^2y,xy^2\right\} =: \mathcal{S}_2(I^2)$$

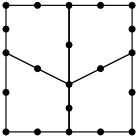
Hence, this provides a computational basis for the serendipity space $\mathcal{S}_2(I^2)$ defined in

ARNOLD, AWANOU *The serendipity family of finite elements*, Found. Comp. Math, 2011.

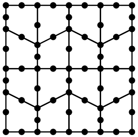
Numerical evidence for non-affine image of a square

Instead of mapping  ,
use quadratic serendipity **GBC** interpolation with mean value coordinates:

$$u_h = I_q u := \sum_{i=1}^{n} u(\mathbf{v}_i)\xi_{ii} + u\left(\frac{\mathbf{v}_i + \mathbf{v}_{i+1}}{2}\right)\xi_{i(i+1)}$$



$n = 2$



$n = 4$

Non-affine bilinear mapping

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	5.0e-2		6.2e-1	
4	6.7e-3	2.9	1.8e-1	1.8
8	9.7e-4	2.8	5.9e-2	1.6
16	1.6e-4	2.6	2.3e-2	1.4
32	3.3e-5	2.3	1.0e-2	1.2
64	7.4e-6	2.1	4.96e-3	1.1

Arnold, Boffi, Falk, Math. Comp., 2002

Quadratic serendipity **GBC** method

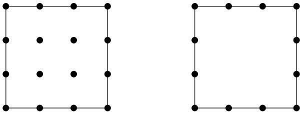
	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	2.34e-3		2.22e-2	
4	3.03e-4	2.95	6.10e-3	1.87
8	3.87e-5	2.97	1.59e-3	1.94
16	4.88e-6	2.99	4.04e-4	1.97
32	6.13e-7	3.00	1.02e-4	1.99
64	7.67e-8	3.00	2.56e-5	1.99
128	9.59e-9	3.00	6.40e-6	2.00
256	1.20e-9	3.00	1.64e-6	1.96

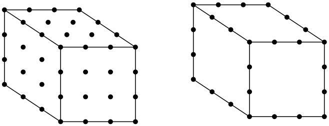
Outline

Cubic and Higher Order Serendipity Bases

→ We've just seen $O(h^2)$ serendipity elements on convex polygons.

→ What about $O(h^3)$ serendipity elements on squares or cubes?

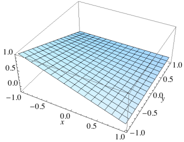




$\{\beta_{ij}\} \xrightarrow{\mathbb{B}} \{\xi_{ij}\}$ 16 12

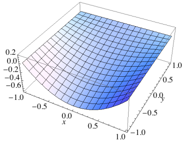
$\{\beta_{ijk}\} \xrightarrow{\mathbb{U}} \{\xi_{ijk}\}$ 64 32

→ Only previously known basis has no symmetric relation to domain points



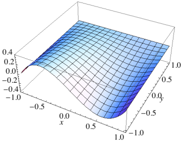
$\frac{1}{4}(x-1)(y-1)$

vertex



$-\frac{1}{4}\sqrt{\frac{3}{2}}\left(x^2-1\right)(y-1)$

edge (quadratic)



$-\frac{1}{4}\sqrt{\frac{5}{2}}x\left(x^2-1\right)(y-1)$

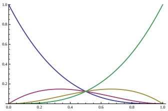
edge (cubic)

SZABÓ AND I. BABUŠKA *Finite element analysis*, Wiley Interscience, 1991.

Cubic Bernstein and Hermite Functions

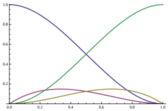
Two well known bases for cubic polynomials used in FEM:

Bernstein basis



$[\beta_i]$

Hermite basis



$[\psi_i]$

$$\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\\\beta_4\end{bmatrix} := \begin{bmatrix}(1-x)^3\\(1-x)^2x\\(1-x)x^2\\x^3\end{bmatrix}$$

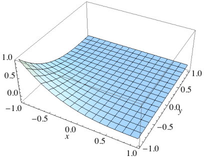
$$\begin{bmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{bmatrix} := \begin{bmatrix}1-3x^2+2x^3\\x-2x^2+x^3\\x^2-x^3\\3x^2-2x^3\end{bmatrix}$$

$\rightarrow$ The Hermite functions interpolate derivative values ($\psi_2'(0)=1$, $\psi_3'(1)=-1$), making them useful for geometric design

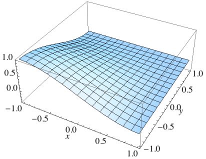
$\rightarrow$ Taking tensor products yields bases for cubic polynomials in 2D and 3D:

$$[\beta_{ij}] = \{\beta_i(x)\beta_j(y)\}$$
$$[\beta_{ijk}] = \{\beta_i(x)\beta_j(y)\beta_k(z)\}$$

$$[\psi_{ij}] = \{\psi_i(x)\psi_j(y)\}$$
$$[\psi_{ijk}] = \{\psi_i(x)\psi_j(y)\psi_k(z)\}$$



β_{11}



ψ_{11}

Cubic Serendipity Spaces

$\rightarrow$ Tensor products span the **full bicubic / tricubic** polynomial spaces

$\rightarrow$ Serendipity methods need only span the **superlinear cubic** polynomials.

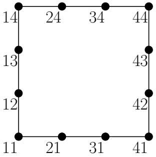
The **superlinear** degree of a polynomial ignores linearly-appearing variables.

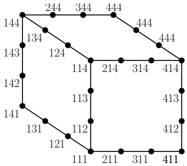
ARNOLD, AWANOU *Finite Element Differential Forms on Cubical Meshes* arXiv:1204.2595, 2012.

$$2D: \quad \overbrace{\{\underbrace{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, \quad x^3y, xy^3}_{\text{superlinear cubics (dim=12)}}, x^2y^2, x^3y^2, x^2y^3, x^3y^3\}}^{\text{all bicubics (dim=16)}}$$

$$3D: \quad \overbrace{\{\underbrace{1, \ldots, xyz, \quad x^3y, x^3z, y^3z, \ldots, x^3yz, xy^3z, xyz^3}_{\text{superlinear cubics (dim=32)}}, x^3y^2, \ldots, x^3y^3z^3\}}^{\text{all tricubics (dim=64)}}$$

The number of superlinear cubics coincides with the domain points
$\Rightarrow$ another linear algebra problem!





Bernstein Style Serendipity Functions (2D)

We define a 12x16 matrix $\mathbb{B}$ so that

$$[\xi_{\ell m}] = \mathbb{B}[\beta_{ij}]$$

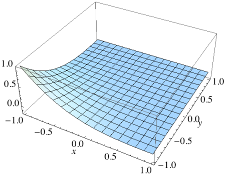
where $\mathbb{B} := \left[\ \mathbb{I} \ \middle|\ \mathbb{B}'\ \right]$, $\mathbb{I}$ is the 12×12 identity matrix, and $\mathbb{B}'$ is a fixed 12×4 matrix of integers whose entries range from -4 to 2, yielding:

$$[\xi_{\ell m}] = \begin{bmatrix} \xi_{11} \\ \xi_{14} \\ \xi_{41} \\ \xi_{44} \\ \xi_{12} \\ \xi_{13} \\ \xi_{42} \\ \xi_{43} \\ \xi_{21} \\ \xi_{31} \\ \xi_{24} \\ \xi_{34} \end{bmatrix} = \begin{bmatrix} (x-1)(y-1)(-2-2x+x^2-2y+y^2) \\ -(x-1)(y+1)(-2-2x+x^2+2y+y^2) \\ -(x+1)(y-1)(-2+2x+x^2-2y+y^2) \\ (x+1)(y+1)(-2+2x+x^2+2y+y^2) \\ -(x-1)(y-1)^2(y+1) \\ (x-1)(y-1)(y+1)^2 \\ (x+1)(y-1)^2(y+1) \\ -(x+1)(y-1)(y+1)^2 \\ -(x-1)^2(x+1)(y-1) \\ (x-1)(x+1)^2(y-1) \\ (x-1)^2(x+1)(y+1) \\ -(x-1)(x+1)^2(y+1) \end{bmatrix} \cdot \frac{1}{16}$$

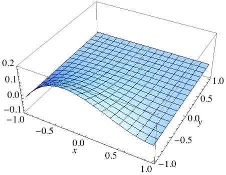
Theorem

$[\xi_{\ell m}]$ is a basis for the superlinear cubics and $\xi_{\ell m} \equiv \beta_{\ell m}$ on all edges.

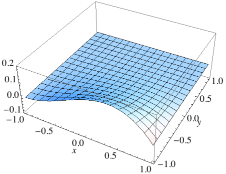
GILLETTE *Hermite and Bernstein Style Basis Functions for Cubic Serendipity Spaces on Squares and Cubes* arXiv:1208.5973, 2012



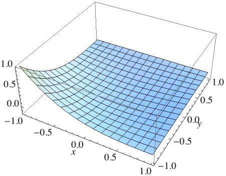
β_{11}



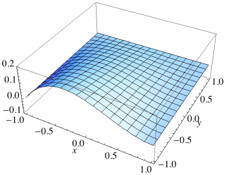
β_{21}



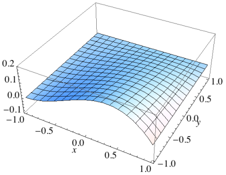
β_{31}



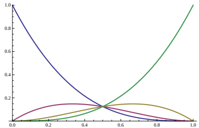
ξ_{11}



ξ_{21}



ξ_{31}



Bicubic Bernstein functions (top) and Bernstein-style serendipity functions (bottom).

$\rightarrow$ Note boundary agreement with Bernstein functions.

Hermite Style Serendipity Functions (2D)

Similarly, define a 12x16 matrix $\mathbb{H}$ so that

$$[\vartheta_{\ell m}] = \mathbb{H}[\psi_{ij}]$$

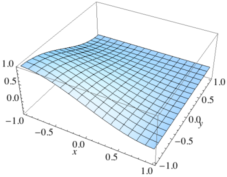
where $\mathbb{H} := \left[\begin{array}{c|c} \mathbb{I} & \mathbb{H}' \end{array}\right]$ and entries of $\mathbb{H}$ are in $\{-1, 0, 1\}$.

$$[\vartheta_{\ell m}] = \begin{bmatrix} \vartheta_{11} \\ \vartheta_{14} \\ \vartheta_{41} \\ \vartheta_{44} \\ \vartheta_{12} \\ \vartheta_{13} \\ \vartheta_{42} \\ \vartheta_{43} \\ \vartheta_{21} \\ \vartheta_{31} \\ \vartheta_{24} \\ \vartheta_{34} \end{bmatrix} = \begin{bmatrix} -(x-1)(y-1)(-2+x+x^2+y+y^2) \\ (x-1)(y+1)(-2+x+x^2-y+y^2) \\ (x+1)(y-1)(-2-x+x^2+y+y^2) \\ -(x+1)(y+1)(-2-x+x^2-y+y^2) \\ -(x-1)(y-1)^2(y+1) \\ (x-1)(y-1)(y+1)^2 \\ (x+1)(y-1)^2(y+1) \\ -(x+1)(y-1)(y+1)^2 \\ -(x-1)^2(x+1)(y-1) \\ (x-1)(x+1)^2(y-1) \\ (x-1)^2(x+1)(y+1) \\ -(x-1)(x+1)^2(y+1) \end{bmatrix} \cdot \frac{1}{8}$$

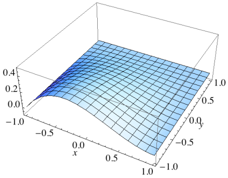
Theorem

$[\vartheta_{\ell m}]$ is a basis for the superlinear cubics and $\vartheta_{\ell m} \equiv \psi_{\ell m}$ on all edges.

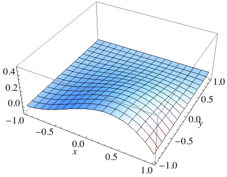
Hermite Style Serendipity Functions (2D)



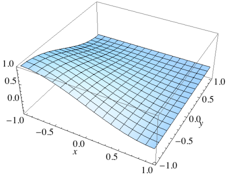
ψ_{11}



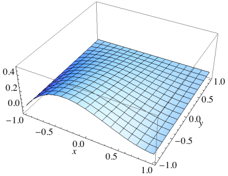
ψ_{21}



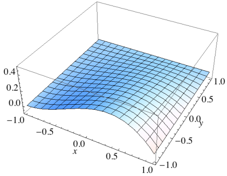
ψ_{31}



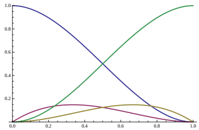
ϑ_{11}



ϑ_{21}



ϑ_{31}



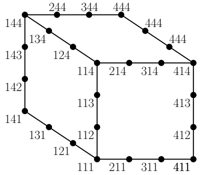
Bicubic Hermite functions (top) and Hermite-style serendipity functions (bottom).

$\rightarrow$ Note boundary agreement with Hermite functions.

Bernstein Style Serendipity Functions (3D)

$$[\xi_{\ell mn}] = \begin{bmatrix} \xi_{111} \\ \xi_{114} \\ \vdots \\ \xi_{442} \\ \xi_{443} \end{bmatrix} = \begin{bmatrix} -(x-1)(y-1)(z-1)(-5-2x+x^2-2y+y^2-2z+z^2) \\ (x-1)(y-1)(z+1)(-5-2x+x^2-2y+y^2+2z+z^2) \\ \vdots \\ (x+1)(y+1)(z-1)^2(z+1) \\ -(x+1)(y+1)(z-1)(z+1)^2 \end{bmatrix} \cdot \frac{1}{32}$$

$$[\vartheta_{\ell mn}] = \begin{bmatrix} \vartheta_{111} \\ \vartheta_{114} \\ \vdots \\ \vartheta_{442} \\ \vartheta_{443} \end{bmatrix} = \begin{bmatrix} (x-1)(y-1)(z-1)(-2+x+x^2+y+y^2+z+z^2) \\ -(x-1)(y-1)(z+1)(-2+x+x^2+y+y^2-z+z^2) \\ \vdots \\ (x+1)(y+1)(z-1)^2(z+1) \\ -(x+1)(y+1)(z-1)(z+1)^2 \end{bmatrix} \cdot \frac{1}{16}$$

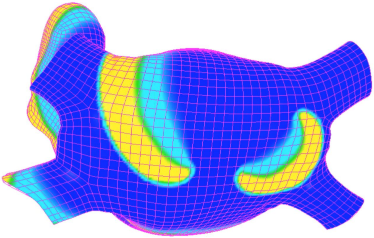


Theorem

$[\xi_{\ell mn}]$ and $[\vartheta_{\ell mn}]$ are bases for the superlinear cubics in 3D and agree with the corresponding tensor product polynomials on all edges and faces.

Outline

Application: Cardiac Electrophysiology



Videos prepared by Matt Gonzalez and others from the Cardiac Mechanics Research Group, led by Dr. Andrew McCulloch, UC San Diego.

Future Directions

- Bases for $O(h^r)$ scalar serendipity spaces on squares / cubes with $r > 3$
  $\rightarrow$*work in progress*
- Bases for k-form scalar serendipity spaces (e.g. vector elements)
  $\rightarrow$*work in progress*
- Generalized barycentric coordinate methods on polytopes in 3D
  $\rightarrow$*work in progress with N. Sukumar (UC Davis) and M. Floater (U. Oslo)*

Acknowledgments

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Alexander Rand	UT Austin / CD-adapco
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Matt Gonzalez	UC San Diego
Andrew McCulloch	UC San Diego
Julian Rimoli	Georgia Tech $\to$ Thanks for the invitation to speak!

Slides and pre-prints: `http://ccom.ucsd.edu/~agillette`