

# Error Estimates for Generalized Barycentric Coordinate Finite Element Methods

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# What are *a priori* FEM error estimates?

**Poisson's equation in 2D:** Given a domain  $\mathcal{D} \subset \mathbb{R}^2$  and  $f : \mathcal{D} \rightarrow \mathbb{R}$ , find  $u$  such that

$$\text{strong form} \quad -\Delta u = f \quad u \in H^2(\mathcal{D})$$

$$\text{weak form} \quad \int_{\mathcal{D}} \nabla u \cdot \nabla \phi = \int_{\mathcal{D}} f \phi \quad \forall \phi \in H^1(\mathcal{D})$$

$$\text{discrete form} \quad \int_{\mathcal{D}} \nabla u_h \cdot \nabla \phi_h = \int_{\mathcal{D}} f \phi_h \quad \forall \phi_h \in V_h \subset H^1(\mathcal{D})$$

Typical **Lagrange method:**

→ Mesh  $\mathcal{D}$  by polygons  $\{\Omega\}$  with vertices  $\{\mathbf{v}_i\}$ .

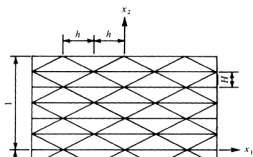
→ Let  $\lambda_i := \text{barycentric coordinate}$  defined piecewise on those  $\Omega$  incident to  $\mathbf{v}_i$ .

→ Let  $u_h := \mathcal{I}_\ell u := \sum_i u(\mathbf{v}_i) \lambda_i$ , reduce discrete form to a linear system & solve.

In this case, the associated **optimal *a priori* error estimate** is **linear** and has the form

$$\underbrace{\left\| u - \sum_{i=1}^n u(\mathbf{v}_i) \lambda_i \right\|_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C \text{diam}(\Omega)}_{\text{optimal error bound}} |u|_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega).$$

# Error estimates depend on mesh geometry



→ Triangular meshes require a maximum angle condition for the estimate to hold.

**BABUŠKA, AZIZ** *On the angle condition in the finite element method*, SIAM J. Num. An., 1976.

→ *A priori* error estimates are based on **affine** maps from a reference element.

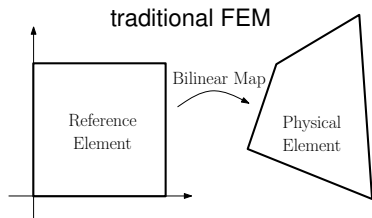
|             |  |  |
|-------------|--|--|
| Lagrange    |  | $\ u - \mathcal{I}_\ell u\ _{H^1(\Omega)} \leq c h  u _{H^2(\Omega)}, \quad \forall u \in H^2(\Omega)$ |
| quadratic   |  | $\ u - \mathcal{I}_q u\ _{H^1(\Omega)} \leq c h^2  u _{H^3(\Omega)}, \quad \forall u \in H^3(\Omega)$  |
| serendipity |  | $\ u - \mathcal{I}_s u\ _{H^1(\Omega)} \leq c h  u _{H^3(\Omega)}, \quad \forall u \in H^3(\Omega)$    |

→ The non-affinely mapped serendipity element converges at a **linear** rate.

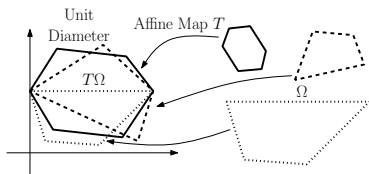
**ARNOLD, BOFFI, FALK** *Approximation by Quadrilateral Finite Elements*, Math. Comp., 2002

# The generalized barycentric coordinate approach

Use **Generalized Barycentric Coordinates (GBCs)** to create polygonal finite elements.

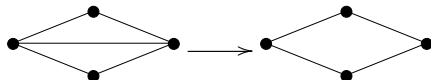


family of **GBC** reference elements

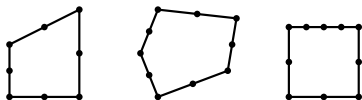


## Advantages of the approach

- Builds on rich theory of **GBCs** (as exhibited at this workshop!)
- Some **GBC** elements are not sensitive to large angles, allowing remeshing:



- Serendipity elements with **quadratic** *a priori* error estimates can be constructed:



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- 4 Future Directions

# Outline

- 1 Background on GBCs
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# Definition

Let  $\Omega$  be a convex polygon in  $\mathbb{R}^2$  with vertices  $\mathbf{v}_1, \dots, \mathbf{v}_n$ . Functions  $\lambda_i : \Omega \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n$  are called **barycentric coordinates** on  $\Omega$  if they satisfy two properties:

- 1 **Non-negative:**  $\lambda_i \geq 0$  on  $\Omega$ .
- 2 **Linear Completeness:** For any linear function  $L : \Omega \rightarrow \mathbb{R}$ ,  $L = \sum_{i=1}^n L(\mathbf{v}_i)\lambda_i$ .

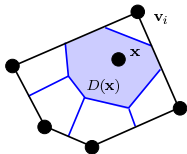
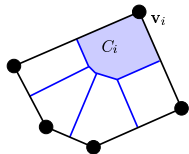
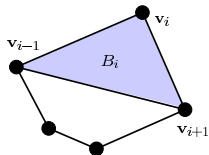
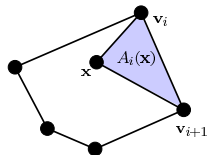
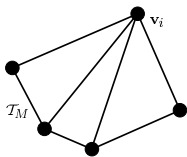
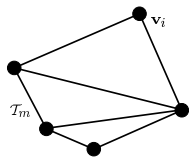
Any set of barycentric coordinates under this definition also satisfies:

- 3 **Partition of unity:**  $\sum_{i=1}^n \lambda_i \equiv 1$ .
- 4 **Linear precision:**  $\sum_{i=1}^n \mathbf{v}_i \lambda_i(\mathbf{x}) = \mathbf{x}$ .
- 5 **Interpolation:**  $\lambda_i(\mathbf{v}_j) = \delta_{ij}$ .

## Theorem [Warren, 2003]

If the  $\lambda_i$  are rational functions of degree  $n - 2$ , then they are unique.

# Many generalizations to choose from . . .



- Triangulation

⇒ [FLOATER, HORMANN, KÓS](#), *A general construction of barycentric coordinates over convex polygons*, 2006

$$0 \leq \lambda_i^{T_m}(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq \lambda_i^{T_M}(\mathbf{x}) \leq 1$$

- Wachspress

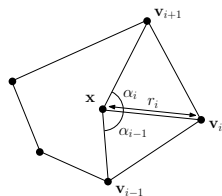
⇒ [WACHSPRESS](#), *A Rational Finite Element Basis*, 1975.

- Sibson / Laplace

⇒ [SIBSON](#), *A vector identity for the Dirichlet tessellation*, 1980.

⇒ [HIYOSHI, SUGIHARA](#), *Voronoi-based interpolation with higher continuity*, 2000.

# Many generalizations to choose from . . .



- Mean value

⇒ FLOATER, *Mean value coordinates*, 2003.

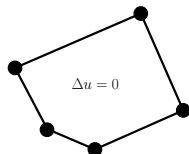
⇒ FLOATER, KÓS, REIMERS, *Mean value coordinates in 3D*, 2005.

- Harmonic

⇒ WARREN, *Barycentric coordinates for convex polytopes*, 1996.

⇒ WARREN, SCHAEFER, HIRANI, DESBRUN, *Barycentric coordinates for convex sets*, 2007.

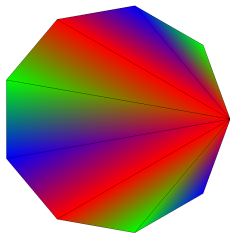
⇒ CHRISTIANSEN, *A construction of spaces of compatible differential forms on cellular complexes*, 2008.



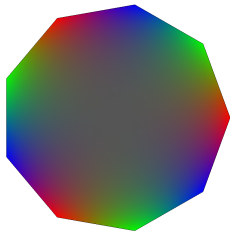
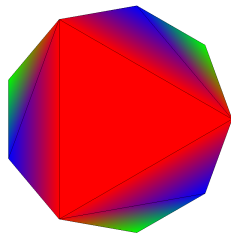
Many more papers could be cited (maximum entropy coordinates, moving least squares coordinates, etc...)

# Comparison via 'eyeball' norm

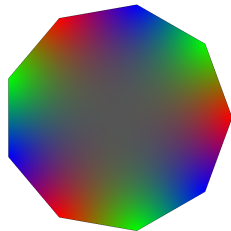
Triangulated



Triangulated



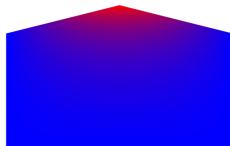
Wachspress



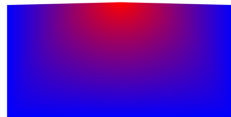
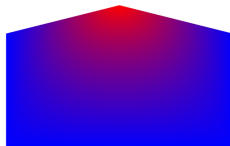
Mean Value

# Comparison via 'eyeball' norm

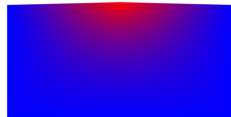
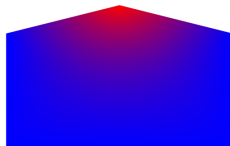
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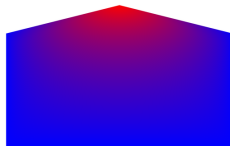
Sibson



Mean Value



Discrete Harmonic



# Outline

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- 2 Error Estimates for Linear Elements**
- 3 Error Estimates for Quadratic 'Serendipity' Elements
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# Optimal Convergence Estimates on Polygons

Let  $\Omega$  be a convex polygon with vertices  $\mathbf{v}_1, \dots, \mathbf{v}_n$ .

For **linear** elements, an **optimal convergence estimate** has the form

$$\underbrace{\left\| u - \sum_{i=1}^n u(\mathbf{v}_i) \lambda_i \right\|_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C \text{diam}(\Omega)}_{\text{optimal error bound}} \|u\|_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega). \quad (1)$$

The **Bramble-Hilbert lemma** in this context says that any  $u \in H^2(\Omega)$  is close to a first order polynomial in  $H^1$  norm.

VERFÜRTH, *A note on polynomial approximation in Sobolev spaces*, Math. Mod. Num. An., 2008.  
DEKEL, LEVIATAN, *The Bramble-Hilbert lemma for convex domains*, SIAM J. Math. An., 2004.

For (1), it suffices to prove an  **$H^1$ -interpolant estimate** over domains of diameter one:

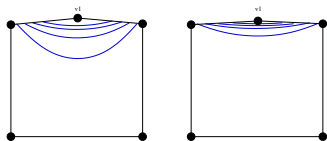
$$\left\| \sum_{i=1}^n u(\mathbf{v}_i) \lambda_i \right\|_{H^1(\Omega)} \leq C_I \|u\|_{H^2(\Omega)}, \quad \forall u \in H^1(\Omega). \quad (2)$$

For (2), it suffices to **bound the gradients** of the  $\{\lambda_i\}$ , i.e. prove  $\exists C_\lambda \in \mathbb{R}$  such that

$$\|\nabla \lambda_i\|_{L^2(\Omega)} \leq C_\lambda. \quad (3)$$

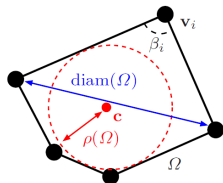
# Geometric Hypotheses for Convergence Estimates

To bound the gradients of the coordinates, we need control of the element geometry.



Let  $\rho(\Omega)$  denote the radius of the largest inscribed circle. The **aspect ratio**  $\gamma$  is defined by

$$\gamma = \frac{\text{diam}(\Omega)}{\rho(\Omega)} \in (2, \infty)$$



Three possible geometric conditions on a polygonal mesh:

- G1.** BOUNDED ASPECT RATIO:  $\exists \gamma^* < \infty$  such that  $\gamma < \gamma^*$
- G2.** MINIMUM EDGE LENGTH:  $\exists d_* > 0$  such that  $|\mathbf{v}_i - \mathbf{v}_{i-1}| > d_*$
- G3.** MAXIMUM INTERIOR ANGLE:  $\exists \beta^* < \pi$  such that  $\beta_i < \beta^*$

# A key geometric proposition

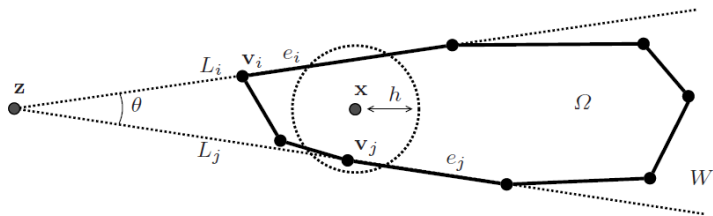
## Proposition

Suppose **G1** (max aspect ratio  $\gamma^*$ ) and **G2** (min edge length  $d_*$ ) hold. Define

$$h_* := \frac{d_*}{2\gamma^*(1 + d_*)} > 0$$

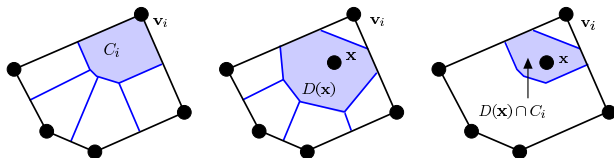
Then for all  $\mathbf{x} \in \Omega$ ,  $B(\mathbf{x}, h_*)$  does not intersect any two non-adjacent edges of  $\Omega$ .

Proof: Suppose  $B(\mathbf{x}, h)$  intersects two non-adjacent edges for some  $h > 0$ . The wedge  $W$  formed by these edges can be used to show that either **G1** fails (contradiction) or  $h > h_*$ .



# Sibson (Natural Neighbor) coordinates

Let  $P$  denote the set of vertices  $\{\mathbf{v}_i\}$  and define  $P' = P \cup \{\mathbf{x}\}$ .



$$\begin{aligned} C_i &:= |V_P(\mathbf{v}_i)| = |\{\mathbf{y} \in \Omega : |\mathbf{y} - \mathbf{v}_i| < |\mathbf{y} - \mathbf{v}_j|, \forall j \neq i\}| \\ &= \text{area of cell for } \mathbf{v}_i \text{ in Voronoi diagram on the points of } P, \end{aligned}$$

$$\begin{aligned} D(\mathbf{x}) &:= |V_{P'}(\mathbf{x})| = |\{\mathbf{y} \in \Omega : |\mathbf{y} - \mathbf{x}| < |\mathbf{y} - \mathbf{v}_i|, \forall i\}| \\ &= \text{area of cell for } \mathbf{x} \text{ in Voronoi diagram on the points of } P'. \end{aligned}$$

By a slight abuse of notation, we also define

$$D(\mathbf{x}) \cap C_i := |V_{P'}(\mathbf{x}) \cap V_P(\mathbf{v}_i)|.$$

The **Sibson coordinates** are defined to be

$$\lambda_i(\mathbf{x}) := \frac{D(\mathbf{x}) \cap C_i}{D(\mathbf{x})} \quad \text{or, equivalently,} \quad \lambda_i(\mathbf{x}) = \frac{D(\mathbf{x}) \cap C_i}{\sum_{j=1}^n D_j(\mathbf{x}) \cap C_j}.$$

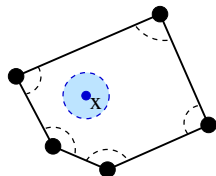
# Gradient bound for Sibson coordinates

## Theorem

Suppose **G1** (max aspect ratio  $\gamma^*$ ) and **G2** (min edge length  $d_*$ ) hold. Let  $\lambda_i$  be the **Sibson** coordinates. Then there exists  $C_\lambda > 0$  such that

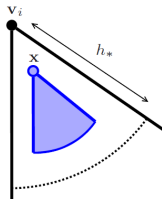
$$\|\lambda_i\|_{H^1(\Omega)} \leq C_\lambda$$

Proof: Recall  $\lambda_i(\mathbf{x}) := \frac{|V_{P'}(\mathbf{x}) \cap V_P(\mathbf{v}_i)|}{|V_{P'}(\mathbf{x})|}$ ; we need a lower bound on the denominator. Either



$\mathbf{x}$  is away from corners and  
a minimum size ball  $\subset V_{P'}(\mathbf{x})$

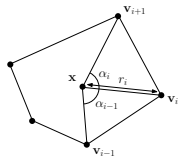
or



$\mathbf{x}$  is near a corner and  
a minimum size wedge  $\subset V_{P'}(\mathbf{x})$

Hence  $V_{P'}(\mathbf{x})$  is bounded uniformly below and there exists  $C_\lambda$  as desired.

# Gradient bound for mean value coordinates



The **mean value coordinates** are defined by

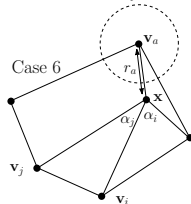
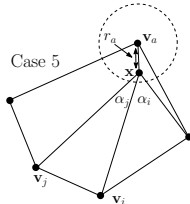
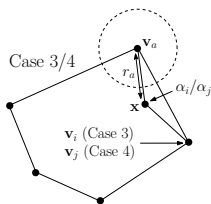
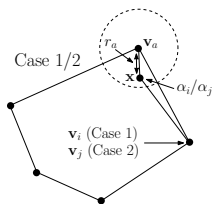
$$\lambda_i^{\text{MV}}(\mathbf{x}) := \frac{w_i(\mathbf{x})}{\sum_{j=1}^n w_j(\mathbf{x})} \quad w_i(\mathbf{x}) := \frac{\tan\left(\frac{\alpha_i(\mathbf{x})}{2}\right) + \tan\left(\frac{\alpha_{i-1}(\mathbf{x})}{2}\right)}{\|\mathbf{v}_i - \mathbf{x}\|}$$

## Theorem

Suppose **G1** (max aspect ratio  $\gamma^*$ ) and **G2** (min edge length  $d_*$ ) hold. Let  $\lambda_i$  be the **mean value coordinates**. Then there exists  $C_\lambda > 0$  such that

$$\|\lambda_i\|_{H^1(\Omega)} \leq C_\lambda$$

Proof: Divide analysis into six cases based on proximity to  $\mathbf{v}_a$  and size of  $\alpha_i$  and  $\alpha_j$



# Polygonal Finite Element Optimal Convergence

## Theorem

In the table, any necessary geometric criteria to achieve the **a priori linear error estimate** are denoted by N. The set of geometric criteria denoted by S in each row **taken together** are sufficient to guarantee the estimate.

|              |                         | <b>G1</b><br>(aspect<br>ratio) | <b>G2</b><br>(min edge<br>length) | <b>G3</b><br>(max interior<br>angle) |
|--------------|-------------------------|--------------------------------|-----------------------------------|--------------------------------------|
| Triangulated | $\lambda^{\text{Tri}}$  | -                              | -                                 | S,N                                  |
| Wachspress   | $\lambda^{\text{Wach}}$ | S                              | S                                 | S,N                                  |
| Sibson       | $\lambda^{\text{Sibs}}$ | S                              | S                                 | -                                    |
| Mean Value   | $\lambda^{\text{MV}}$   | S                              | S                                 | -                                    |
| Harmonic     | $\lambda^{\text{Har}}$  | S                              | -                                 | -                                    |

GILLETTE, RAND, BAJAJ *Error Estimates for Generalized Barycentric Interpolation*  
Advances in Computational Mathematics, to appear, 2011

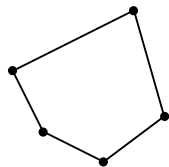
RAND, GILLETTE, BAJAJ *Interpolation Error Estimates for Mean Value Coordinates*,  
submitted, 2011.

# Outline

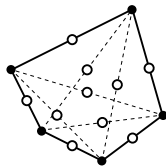
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# From linear to quadratic elements

A naïve quadratic element is formed by products of linear element basis functions:



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a \lambda_b\}$$



Why is this naïve?

- For an  $n$ -gon, this construction gives  $n + \binom{n}{2}$  basis functions  $\lambda_a \lambda_b$
- The space of quadratic polynomials is only dimension 6:  $\{1, x, y, xy, x^2, y^2\}$
- Conforming to a linear function on the boundary requires 2 degrees of freedom per edge  $\Rightarrow$  *only 2n functions needed!*

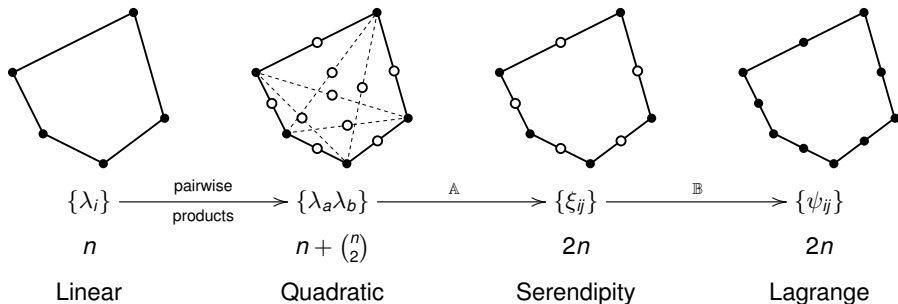
## Problem Statement

Construct  $2n$  basis functions associated to the vertices and edge midpoints of an arbitrary  $n$ -gon such that a quadratic convergence estimate is obtained.

# Polygonal Quadratic Serendipity Elements

We define matrices  $\mathbb{A}$  and  $\mathbb{B}$  to reduce the naïve quadratic basis.

- filled dot** = Lagrangian domain point  
= all functions in the set evaluate to 0  
except the associated function which evaluates to 1
- open dot** = non-Lagrangian domain point  
= partition of unity satisfied, but not Lagrange property





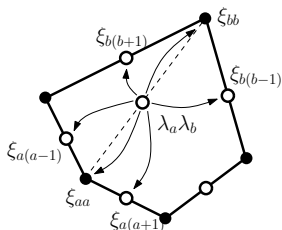
# From quadratic to serendipity

We **require** the serendipity basis to have quadratic approximation power:

$$\text{Constant precision:} \quad 1 = \sum_i \xi_{ii} + 2\xi_{i(i+1)}$$

$$\text{Linear precision:} \quad \mathbf{x} = \sum_i \mathbf{v}_i \xi_{ii} + 2\mathbf{v}_{i(i+1)} \xi_{i(i+1)}$$

$$\text{Quadratic precision:} \quad \mathbf{x}\mathbf{x}^T = \sum_i \mathbf{v}_i \mathbf{v}_i^T \xi_{ii} + (\mathbf{v}_i \mathbf{v}_{i+1}^T + \mathbf{v}_{i+1} \mathbf{v}_i^T) \xi_{i(i+1)}$$



## Theorem

Constants  $\{c_{ij}^{ab}\}$  exist for **any** convex polygon such that the resulting basis  $\{\xi_{ij}\}$  satisfies constant, linear, and quadratic precision requirements.

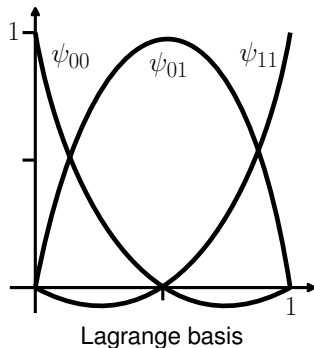
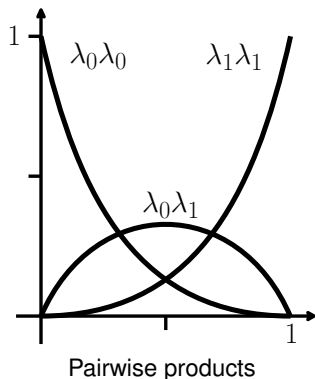
Proof: We produce a coefficient matrix  $\mathbb{A}$  with the structure

$$\mathbb{A} := [ \mathbb{I} \mid \mathbb{A}' ]$$

where  $\mathbb{A}'$  has only six non-zero entries per column and show that the resulting functions satisfy the six precision equations.

# Pairwise products vs. Lagrange basis

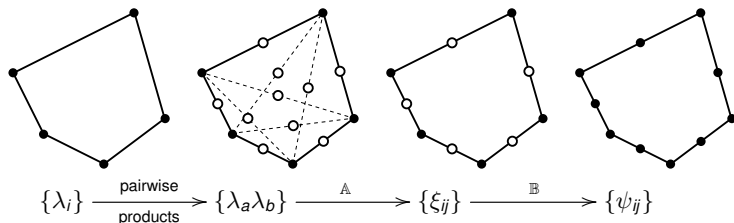
Even in 1D, pairwise products of barycentric functions do not form a Lagrange basis at interior degrees of freedom:



Translation between these two bases is straightforward and generalizes to the higher dimensional case...



# Serendipity Theorem



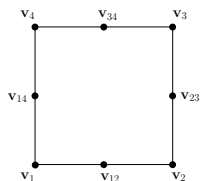
## Theorem

Given bounds on polygon aspect ratio (**G1**), minimum edge length (**G2**), and maximum interior angles (**G3**):

- $\|\mathbb{A}\|$  is uniformly bounded,
- $\|\mathbb{B}\|$  is uniformly bounded, and
- $\text{span}\{\psi_{ij}\} \supset \mathcal{P}_2(\mathbb{R}^2) =$  quadratic polynomials in  $x$  and  $y$

RAND, GILLETTE, BAJAJ *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Submitted, 2011

# Special case of a square



Bilinear functions are barycentric coordinates:

$$\lambda_1 = (1 - x)(1 - y)$$

$$\lambda_2 = x(1 - y)$$

$$\lambda_3 = xy$$

$$\lambda_4 = (1 - x)y$$


Compute  $[\xi_{ij}] := [\mathbb{I} \mid \mathbb{A}'] [\lambda_a \lambda_b]$

$$\begin{bmatrix} \xi_{11} \\ \xi_{22} \\ \xi_{33} \\ \xi_{44} \\ \xi_{12} \\ \xi_{23} \\ \xi_{34} \\ \xi_{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & \dots & 0 & 0 & -1 \\ 0 & \dots & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} \lambda_1 \lambda_1 \\ \lambda_2 \lambda_2 \\ \lambda_3 \lambda_3 \\ \lambda_4 \lambda_4 \\ \lambda_1 \lambda_2 \\ \lambda_2 \lambda_3 \\ \lambda_3 \lambda_4 \\ \lambda_1 \lambda_4 \end{bmatrix} = \begin{bmatrix} (1-x)(1-y)(1-x-y) \\ x(1-y)(x-y) \\ xy(-1+x+y) \\ (1-x)y(y-x) \\ (1-x)x(1-y) \\ x(1-y)y \\ (1-x)xy \\ (1-x)(1-y)y \end{bmatrix}$$

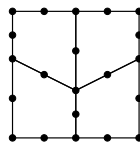
$$\text{span} \{ \xi_{ii}, \xi_{i(i+1)} \} = \text{span} \{ 1, x, y, x^2, y^2, xy, x^2y, xy^2 \} =: \mathcal{S}_2(I^2)$$

Hence, this provides a computational basis for the serendipity space  $\mathcal{S}_2(I^2)$  defined in [ARNOLD, AWANOU](#) *The serendipity family of finite elements*, Found. Comp. Math, 2011.

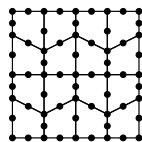
# Numerical evidence for non-affine image of a square

Instead of mapping , use quadratic serendipity **GBC** interpolation with mean value coordinates:

$$u_h = I_q u := \sum_{i=1}^n u(\mathbf{v}_i) \psi_{ii} + u\left(\frac{\mathbf{v}_i + \mathbf{v}_{i+1}}{2}\right) \psi_{i(i+1)}$$



$n = 2$



$n = 4$

Non-affine bilinear mapping

| n  | $\ u - u_h\ _{L^2}$ |            | $\ \nabla(u - u_h)\ _{L^2}$ |            |
|----|---------------------|------------|-----------------------------|------------|
|    | error               | rate       | error                       | rate       |
| 2  | 5.0e-2              |            | 6.2e-1                      |            |
| 4  | 6.7e-3              | 2.9        | 1.8e-1                      | 1.8        |
| 8  | 9.7e-4              | 2.8        | 5.9e-2                      | 1.6        |
| 16 | 1.6e-4              | 2.6        | 2.3e-2                      | 1.4        |
| 32 | 3.3e-5              | 2.3        | 1.0e-2                      | 1.2        |
| 64 | 7.4e-6              | <b>2.1</b> | 4.96e-3                     | <b>1.1</b> |

ARNOLD, BOFFI, FALK, Math. Comp., 2002

Quadratic serendipity **GBC** method

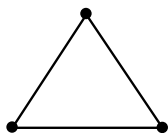
| n   | $\ u - u_h\ _{L^2}$ |             | $\ \nabla(u - u_h)\ _{L^2}$ |             |
|-----|---------------------|-------------|-----------------------------|-------------|
|     | error               | rate        | error                       | rate        |
| 2   | 2.34e-3             |             | 2.22e-2                     |             |
| 4   | 3.03e-4             | 2.95        | 6.10e-3                     | 1.87        |
| 8   | 3.87e-5             | 2.97        | 1.59e-3                     | 1.94        |
| 16  | 4.88e-6             | 2.99        | 4.04e-4                     | 1.97        |
| 32  | 6.13e-7             | 3.00        | 1.02e-4                     | 1.99        |
| 64  | 7.67e-8             | 3.00        | 2.56e-5                     | 1.99        |
| 128 | 9.59e-9             | 3.00        | 6.40e-6                     | 2.00        |
| 256 | 1.20e-9             | <b>3.00</b> | 1.64e-6                     | <b>1.96</b> |

# Outline

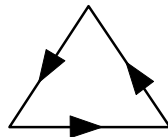
- 1 Background on GBCs
- 2 Error Estimates for Linear Elements
- 3 Error Estimates for Quadratic 'Serendipity' Elements
- 4 Future Directions**

# From scalar to vector elements

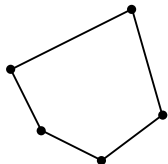
Barycentric functions are used to define  $H(\text{curl})$  vector elements on triangles:



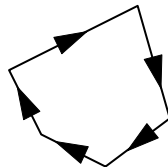
$$\{\lambda_j\} \xrightarrow[\text{construction}]{\text{Whitney}} \{\lambda_a \nabla \lambda_b - \lambda_b \nabla \lambda_a\}$$



Generalized barycentric functions provide  $H(\text{curl})$  elements on polygons:



$$\{\lambda_j\} \xrightarrow[\text{construction}]{\text{Whitney}} \{\lambda_a \nabla \lambda_b - \lambda_b \nabla \lambda_a\}$$



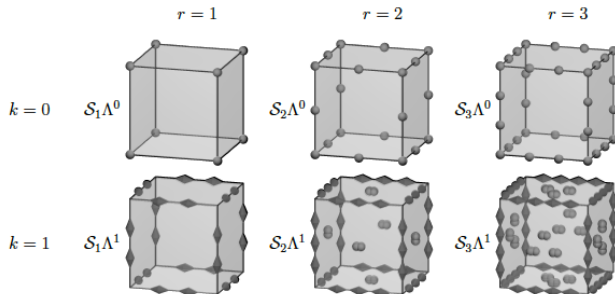
This idea fits naturally into the framework of **Discrete Exterior Calculus** and suggests a wide range of applications. . .

*...work in progress*

GILLETTE, BAJAJ *Dual Formulations of Mixed Finite Element Methods with Applications*  
Computer-Aided Design 43:10, pages 1213-1221, 2011.

# Basis functions for serendipity spaces

Recent work characterized serendipity spaces in  $n$  dimensions for scalar fields, vector fields, and their generalization to differential  $k$ -forms:



**ARNOLD, AWANOU** *Finite Element Differential Forms on Cubical Meshes*  
arXiv:1204.2595, 2012.

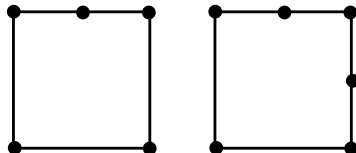
The results of that paper coupled with the techniques used here suggest a variety of new research directions. . . *...work in progress*

# GBCs for adaptive algorithms

Recall from the table of geometric dependencies for gradient bounds:

|            |                       | <b>G1</b><br>(aspect ratio) | <b>G2</b><br>(min edge length) | <b>G3</b><br>(max interior angle) |
|------------|-----------------------|-----------------------------|--------------------------------|-----------------------------------|
| Mean Value | $\lambda^{\text{MV}}$ | S                           | S                              | -                                 |

Thus, the quadratic serendipity construction with mean value coordinates can still allow **quadratic** convergence for elements with interior angles of  $\pi$ , including typical elements from adaptive methods:



# References

- Thank to the organizers for the invitation and for organizing!
- Slides and pre-prints are available at:

<http://ccom.ucsd.edu/~agillette>

