H(curl) and H(div) Elements on Polytopes from Generalized Barycentric Coordinates

Andrew Gillette

Department of Mathematics
University of Arizona

joint work with

Alex Rand (CD-adapco)
Chandrajit Bajaj (UT Austin)

The generalized barycentric coordinate approach

Let P be a convex polytope with vertex set V. We say that

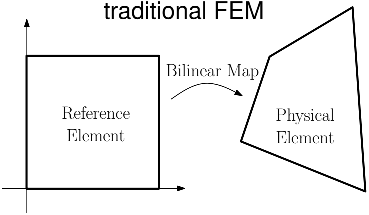
$$\lambda_{\mathbf{v}} : P \to \mathbb{R} \quad \text{are} \quad \textbf{generalized barycentric coordinates (GBCs)} \text{ on } P$$

if they satisfy $\quad \lambda_{\mathbf{v}} \geq 0$ on P and $L = \sum_{\mathbf{v}\in V} L(\mathbf{v}_{\mathbf{v}})\lambda_{\mathbf{v}}, \quad \forall\, L : P \to \mathbb{R}$ linear.

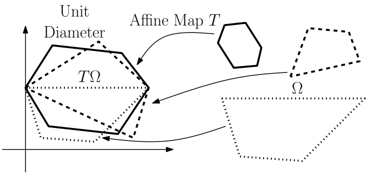
Familiar properties are implied by this definition:

$$\underbrace{\sum_{\mathbf{v}\in V} \lambda_{\mathbf{v}} \equiv 1}_{\text{partition of unity}} \qquad \underbrace{\sum_{\mathbf{v}\in V} \mathbf{v}\lambda_{\mathbf{v}}(\mathbf{x}) = \mathbf{x}}_{\text{linear precision}} \qquad \underbrace{\lambda_{\mathbf{v}_i}(\mathbf{v}_j) = \delta_{ij}}_{\text{interpolation}}$$

traditional FEM

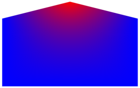


family of **GBC** reference elements

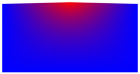


Developments in GBC FEM theory

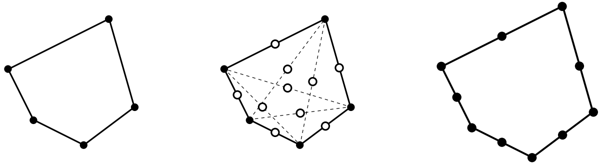
1 Characterization of the dependence of error estimates on polytope geometry.



vs.



2 Construction of higher order scalar-valued methods using $\lambda_{\mathbf{v}}$ functions.



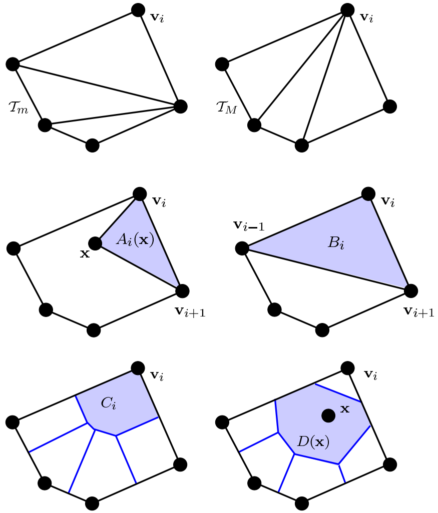
$\{\lambda_i\}$ $\quad$ $\{\lambda_i\lambda_j\}$ $\quad$ $\{\psi_{ij}\}$

3 Construction of H(curl) and H(div) methods using $\lambda_{\mathbf{v}}$ and $\nabla\lambda_{\mathbf{v}}$ functions.

$$H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2$$

$$\{\lambda_i\} \qquad \{\lambda_i\nabla\lambda_j\} \qquad \{\lambda_i\nabla\lambda_j \times \nabla\lambda_k\} \qquad \{\chi_P\}$$

Many choices of generalized barycentric coordinates



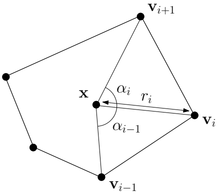
- Triangulation
  $\Rightarrow$ Floater, Hormann, Kós, *A general construction of barycentric coordinates over convex polygons*, 2006

  $$0 \leq \lambda_i^{\mathcal{T}_m}(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq \lambda_i^{\mathcal{T}_M}(\mathbf{x}) \leq 1$$

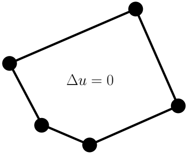
- Wachspress
  $\Rightarrow$ Wachspress, *A Rational Finite Element Basis*, 1975.
  $\Rightarrow$ Warren, *Barycentric coordinates for convex polytopes*, 1996.

- Sibson / Laplace
  $\Rightarrow$ Sibson, *A vector identity for the Dirichlet tessellation*, 1980.
  $\Rightarrow$ Hiyoshi, Sugihara, *Voronoi-based interpolation with higher continuity*, 2000.

Many choices of generalized barycentric coordinates



- Mean value
  ⇒ FLOATER, *Mean value coordinates*, 2003.
  ⇒ FLOATER, KÓS, REIMERS, *Mean value coordinates in 3D*, 2005.



- Harmonic
  ⇒ WARREN, SCHAEFER, HIRANI, DESBRUN, *Barycentric coordinates for convex sets*, 2007.
  ⇒ CHRISTIANSEN, *A construction of spaces of compatible differential forms on cellular complexes*, 2008.

Many more papers could be cited (maximum entropy coordinates, moving least squares coordinates, surface barycentric coordinates, etc...)

From scalar to vector elements

The classical finite element sequences for a domain $\Omega \subset \mathbb{R}^n$ are written:

$$n = 2: \qquad H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xleftrightarrow{\text{rot}} H(\text{div}) \xrightarrow{\text{div}} L^2$$

$$n = 3: \qquad H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2$$

These correspond to the L^2 deRham diagrams from differential topology:

$$n = 2: \qquad H\Lambda^0 \xrightarrow{d_0} H\Lambda^1 \xleftrightarrow{\cong} H\Lambda^1 \xrightarrow{d_1} H\Lambda^2$$

$$n = 3: \qquad H\Lambda^0 \xrightarrow{d_0} H\Lambda^1 \xrightarrow{d_1} H\Lambda^2 \xrightarrow{d_2} H\Lambda^3$$

Conforming finite element subspaces of $H\Lambda^k$ are of two types:

$$\mathcal{P}_r\Lambda^k := k\text{-forms with degree } r \text{ polynomial coefficients}$$

$$\mathcal{P}_r^-\Lambda^k := \mathcal{P}_{r-1}\Lambda^k \oplus \{\text{certain additional } k\text{-forms}\}$$

This notation, from Finite Element Exterior Calculus, can be used to describe many well-known finite element spaces.

ARNOLD, FALK, WINTHER *Finite Element Exterior Calculus*, Bulletin of the AMS, 2010.

Classical finite element spaces on simplices

n=2 (triangles)

k	dim	space	type	classical description
0	3	$\mathcal{P}_1\Lambda^0$	H^1	Lagrange elements of degree ≤ 1
	3	$\mathcal{P}_1^-\Lambda^0$	H^1	Lagrange elements of degree ≤ 1
1	6	$\mathcal{P}_1\Lambda^1$	$H(\text{div})$	Brezzi-Douglas-Marini $H(\text{div})$ elements of degree ≤ 1
	3	$\mathcal{P}_1^-\Lambda^1$	$H(\text{div})$	Raviart-Thomas elements of order 0
2	3	$\mathcal{P}_1\Lambda^2$	L^2	discontinuous linear
	1	$\mathcal{P}_1^-\Lambda^2$	L^2	discontinuous piecewise constant

n=3 (tetrahedra)

0	4	$\mathcal{P}_1\Lambda^0$	H^1	Lagrange elements of degree ≤ 1
	4	$\mathcal{P}_1^-\Lambda^0$	H^1	Lagrange elements of degree ≤ 1
1	12	$\mathcal{P}_1\Lambda^1$	$H(\text{curl})$	Nédélec second kind $H(\text{curl})$ elements of degree ≤ 1
	6	$\mathcal{P}_1^-\Lambda^1$	$H(\text{curl})$	Nédélec first kind $H(\text{curl})$ elements of order 0
2	12	$\mathcal{P}_1\Lambda^2$	$H(\text{div})$	Nédélec second kind $H(\text{div})$ elements of degree ≤ 1
	4	$\mathcal{P}_1^-\Lambda^2$	$H(\text{div})$	Nédélec first kind $H(\text{div})$ elements of order 0
3	4	$\mathcal{P}_1\Lambda^3$	L^2	discontinuous linear
	1	$\mathcal{P}_1^-\Lambda^3$	L^2	discontinuous piecewise constant

Basis functions on simplices

n=2 (triangles)

k	dim	space	type	basis functions
0	3	$\mathcal{P}_1\Lambda^0$	H^1	λ_i
1	6	$\mathcal{P}_1\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j$
	6	$\mathcal{P}_1\Lambda^1$	$H(\text{div})$	$\text{rot}(\lambda_i\nabla\lambda_j)$
	3	$\mathcal{P}_1^-\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i$
	3	$\mathcal{P}_1^-\Lambda^1$	$H(\text{div})$	$\text{rot}(\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i)$
2	3	$\mathcal{P}_1\Lambda^2$	L^2	piecewise linear functions
	1	$\mathcal{P}_1^-\Lambda^2$	L^2	piecewise constant functions

n=3 (tetrahedra)

0	4	$\mathcal{P}_1\Lambda^0$	H^1	λ_i
1	12	$\mathcal{P}_1\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j$
	6	$\mathcal{P}_1^-\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i$
2	12	$\mathcal{P}_1\Lambda^2$	$H(\text{div})$	$\lambda_i\nabla\lambda_j \times \nabla\lambda_k$
	4	$\mathcal{P}_1^-\Lambda^2$	$H(\text{div})$	$(\lambda_i\nabla\lambda_j \times \nabla\lambda_k) + (\lambda_j\nabla\lambda_k \times \nabla\lambda_i) + (\lambda_k\nabla\lambda_i \times \nabla\lambda_j)$
3	4	$\mathcal{P}_1\Lambda^3$	L^2	piecewise linear functions
	1	$\mathcal{P}_1^-\Lambda^3$	L^2	piecewise constant functions

Essential properties of basis functions

The vector-valued basis constructions ($0 < k < n$) have two key properties:

1 **Global continuity in $H(\text{curl})$ or $H(\text{div})$**

$\lambda_i \nabla \lambda_j$ agree on **tangential** components at element interfaces $\implies H(\text{curl})$ continuity

$\lambda_i \nabla \lambda_j \times \nabla \lambda_k$ agree on **normal** components at element interfaces $\implies H(\text{div})$ continuity

2 **Reproduction of requisite polynomial differential forms.**

For $i, j \in \{1, 2, 3\}$:

$$\text{span}\{\lambda_i \nabla \lambda_j\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \end{bmatrix}, \begin{bmatrix} y \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\} \cong \mathcal{P}_1 \Lambda^1(\mathbb{R}^2)$$

$$\text{span}\{\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\} \cong \mathcal{P}_1^- \Lambda^1(\mathbb{R}^2)$$

Using generalized barycentric coordinates, we can extend all these results to polygonal and polyhedral elements.

Basis functions on polygons and polyhedra

Theorem [G., Rand, Bajaj, 2014]

Let P be a convex polygon or polyhedron. Given any set of generalized barycentric coordinates $\{\lambda_i\}$ associated to P, the functions listed below have **global continuity and** polynomial differential form **reproduction** properties as indicated.

n=2 (polygons)

k	space	type	functions
1	$\mathcal{P}_1\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j$
	$\mathcal{P}_1\Lambda^1$	$H(\text{div})$	$\text{rot}(\lambda_i\nabla\lambda_j)$
	$\mathcal{P}_1^-\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i$
	$\mathcal{P}_1^-\Lambda^1$	$H(\text{div})$	$\text{rot}(\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i)$

n=3 (polyhedra)

k	space	type	functions
1	$\mathcal{P}_1\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j$
	$\mathcal{P}_1^-\Lambda^1$	$H(\text{curl})$	$\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i$
2	$\mathcal{P}_1\Lambda^2$	$H(\text{div})$	$\lambda_i\nabla\lambda_j \times \nabla\lambda_k$
	$\mathcal{P}_1^-\Lambda^2$	$H(\text{div})$	$(\lambda_i\nabla\lambda_j \times \nabla\lambda_k) + (\lambda_j\nabla\lambda_k \times \nabla\lambda_i) + (\lambda_k\nabla\lambda_i \times \nabla\lambda_j)$

Note: The indices range over ***all*** pairs or triples of vertex indices from P.

Polynomial differential form reproduction identities

Let $P \subset \mathbb{R}^3$ be a convex polyhedron with vertex set $\{\mathbf{v}_i\}$. Let $\mathbf{x} = \begin{bmatrix} x & y & z \end{bmatrix}^T$.

Then for any 3×3 real matrix $\mathbb{A}$,

$$\sum_{i,j} \lambda_i \nabla \lambda_j (\mathbf{v}_j - \mathbf{v}_i)^T = \mathbb{I}$$

$$\sum_{i,j} (\mathbb{A}\mathbf{v}_i \cdot \mathbf{v}_j)(\lambda_i \nabla \lambda_j) = \mathbb{A}\mathbf{x}$$

$$\frac{1}{2} \sum_{i,j,k} \lambda_i \nabla \lambda_j \times \nabla \lambda_k \left((\mathbf{v}_j - \mathbf{v}_i) \times (\mathbf{v}_k - \mathbf{v}_i) \right)^T = \mathbb{I}$$

$$\frac{1}{2} \sum_{i,j,k} (\mathbb{A}\mathbf{v}_i \cdot (\mathbf{v}_j \times \mathbf{v}_k))(\lambda_i \nabla \lambda_j \times \nabla \lambda_k) = \mathbb{A}\mathbf{x}.$$

By appropriate choice of constant entries for $\mathbb{A}$, the column vectors of $\mathbb{I}$ and $\mathbb{A}\mathbf{x}$ span $\mathcal{P}_1\Lambda^1 \subset H(\text{curl})$ or $\mathcal{P}_1\Lambda^2 \subset H(\text{div})$.

$\rightarrow$ Additional identities for the remaining cases are stated in:

G, Rand, Bajaj *Construction of Scalar and Vector Finite Element Families on Polygonal and Polyhedral Meshes*, arXiv:1405.6978, 2014

Reducing the basis

In some cases, it should be possible to reduce the size of the basis constructed by our method, in an analogous fashion to the quadratic scalar case.

n=2 (polygons)

k	space	# construction	# boundary	# polynomial
0	$\mathcal{P}_1\Lambda^0(\mathfrak{m})/\mathcal{P}_0^-\Lambda^1(\mathfrak{m})$	v	v	3
1	$\mathcal{P}_1\Lambda^1(\mathfrak{m})$	$v(v-1)$	$2e$	6
	$\mathcal{P}_1^-\Lambda^1(\mathfrak{m})$	$\binom{v}{2}$	e	3
2	$\mathcal{P}_1\Lambda^2(\mathfrak{m})$	$\frac{v(v-1)(v-2)}{2}$	0	3
	$\mathcal{P}_1^-\Lambda^2(\mathfrak{m})$	$\binom{v}{3}$	0	1

$\rightarrow$ The $n = 3$ (polyhedra) version of this table is given in:

G, RAND, BAJAJ *Construction of Scalar and Vector Finite Element Families on Polygonal and Polyhedral Meshes*, arXiv:1405.6978, 2014

Acknowledgments

Chandrajit Bajaj	UT Austin
Alexander Rand	UT Austin / CD-adapco

Happy birthday, Doug!

Slides and pre-prints: `http://math.arizona.edu/~agillette/`

More on GBCs: `http://www.inf.usi.ch/hormann/barycentric`