Decompositions of (Trimmed) Serendipity Spaces

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joint work with
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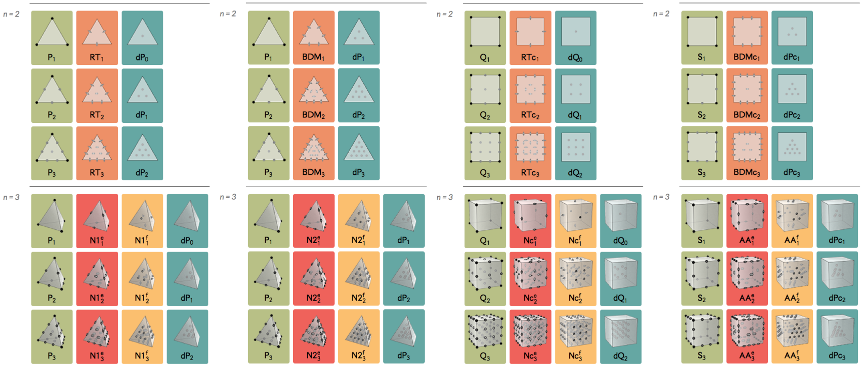

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Outline

1. Four well-known families of finite elements
2. Trimmed serendipity spaces: a new “smaller” family of elements
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Classification of conforming methods

Conforming finite element method types can be broadly classified by three integers:

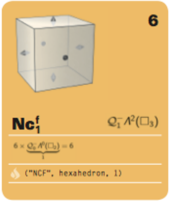
n → the spatial dimension of the domain
r → the order of error decay
k → the differential form order of the solution space

An element type is defined in part by its **degrees of freedom**. Typically:

the more degrees of freedom, the greater the computational cost of the method

Ex: $\mathcal{Q}_1^-\Lambda^2(\square_3)$ is an element for

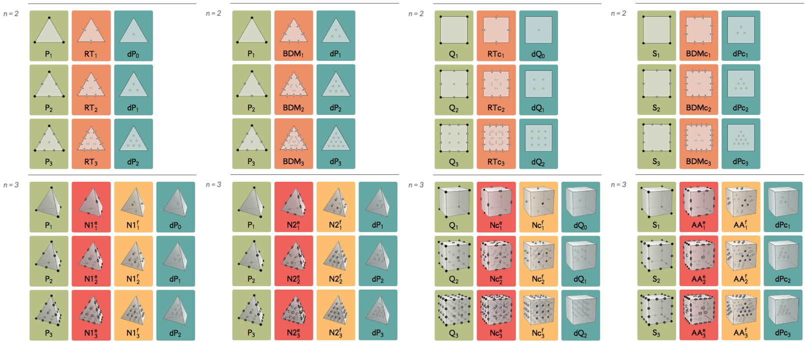
$n = 3$ → domains in $\mathbb{R}^3$
$r = 1$ → linear order of error decay
$k = 2$ → conformity in $\Lambda^2(\mathbb{R}^3) \rightsquigarrow H(\mathrm{div})$



$\mathcal{Q}_1^-\Lambda^2(\square_3)$ is part of the $\mathcal{Q}^-$ 'column' of elements,
is defined on geometry $\square_3$ (i.e. a cube),
has a **6** dimensional space of test functions,
and has an associated set of **6** degrees of freedom
that are unisolvent for the test function space.

The ‘Periodic Table of the Finite Elements’

ARNOLD, LOGG, “Periodic table of the finite elements,” *SIAM News*, 2014.



Classification of many common conforming finite element types.

$n \rightarrow$ Domains in $\mathbb{R}^2$ (top half) and in $\mathbb{R}^3$ (bottom half)
$r \rightarrow$ Order $1, 2, 3$ of error decay (going down columns)
$k \rightarrow$ Conformity type $k = 0, \ldots, n$ (going across a row)

Geometry types: Simplices (left half) and cubes (right half).

An abbreviated reading list (50 years of theory!)

RAVIART, THOMAS, “A mixed finite element method for 2nd order elliptic problems” *Lecture Notes in Mathematics*, 1977 ← *3172 citations, including 150 from 2017!*

NÉDÉLEC, “Mixed finite elements in $\mathbb{R}^3$,” *Numerische Mathematik*, 1980

BREZZI, DOUGLAS JR., MARINI, “Two families of mixed finite elements for second order elliptic problems,” *Numerische Mathematik*, 1985

NÉDÉLEC, “A new family of mixed finite elements in $\mathbb{R}^3$,” *Numerische Mathematik*, 1986

ARNOLD, FALK, WINTHER “Finite element exterior calculus, homological techniques, and applications,” *Acta Numerica*, 2006

CHRISTIANSEN, “Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension,” *Numerische Mathematik*, 2007

ARNOLD, FALK, WINTHER “Finite element exterior calculus: from hodge theory to numerical stability,” *Bulletin of the AMS*, 2010

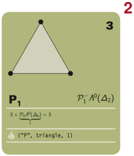
ARNOLD, AWANOU “The serendipity family of finite elements ”, *Found. Comp Math*, 2011

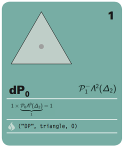
ARNOLD, AWANOU “Finite element differential forms on cubical meshes”, *Math Comp.*, 2013

ARNOLD, BOFFI, BONIZZONI “Finite element differential forms on curvilinear meshes and their approximation properties,” *Numerische Mathematik*, 2014

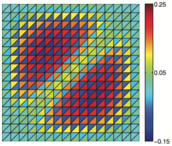
Stable pairs of elements for mixed methods

Picking elements from the table for a mixed method for the Poisson problem:

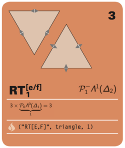


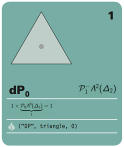


$\subset H^1 \times H^1$ $\subset L^2$ $\Rightarrow$

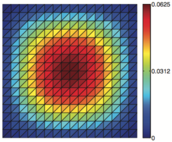


Unstable method





$\subset H(\mathrm{div})$ $\subset L^2$ $\Rightarrow$



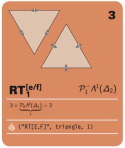
Provably stable method

converges to
u = $x(1-x)y(1-y)$

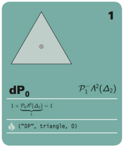
Example and images on right from:

ARNOLD, FALK, WINTHER “Finite Element Exterior Calculus. . .” *Bulletin of the AMS*, 47:2, 2010.

Method selection and cochain complexes

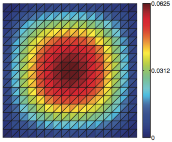


$\subset H(\mathrm{div})$



$\subset L^2$

$\Rightarrow$



Provably stable method

converges to
$\mathrm{u} = x(1-x)y(1-y)$

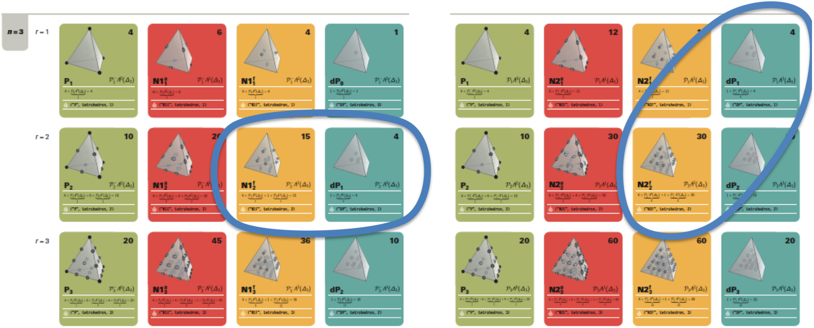
Stable pairs of elements for mixed Hodge-Laplacian problems are found by choosing consecutive spaces in compatible discretizations of the L^2 deRham Diagram.

$$\begin{tikzcd}
H^1 \arrow[r, "\nabla", "\mathrm{grad}"'] & H(\mathrm{curl}) \arrow[r, "\nabla\times", "\mathrm{curl}"'] & H(\mathrm{div}) \arrow[r, "\nabla\cdot", "\mathrm{div}"'] & L^2
\end{tikzcd}$$

vector Poisson	σ	μ		
Maxwell's eqn's		h	b	
Darcy / Poisson			$\mathbf{u}$	p

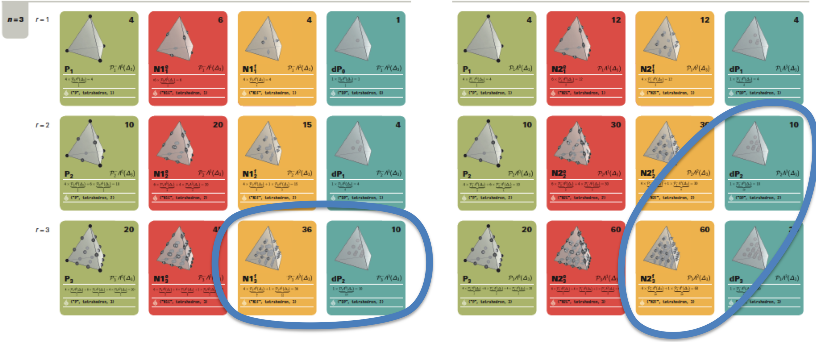
The Periodic Table of Finite Elements lets us 'read off' stable pairs visually.

Stable pairs for tetrahedral meshes



Problem:	Darcy / Poisson
Dimension:	$n = 3$
Mesh type:	tetrahedral
Convergence:	quadratic ($r = 2$)

Stable pairs for tetrahedral meshes



Problem: Darcy / Poisson

Dimension: $n = 3$

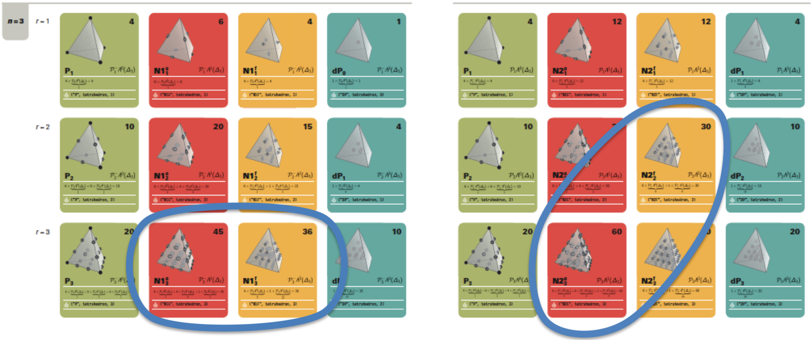
Mesh type: tetrahedral

Convergence: cubic ($r = 3$)

Stable pairs for tetrahedral meshes

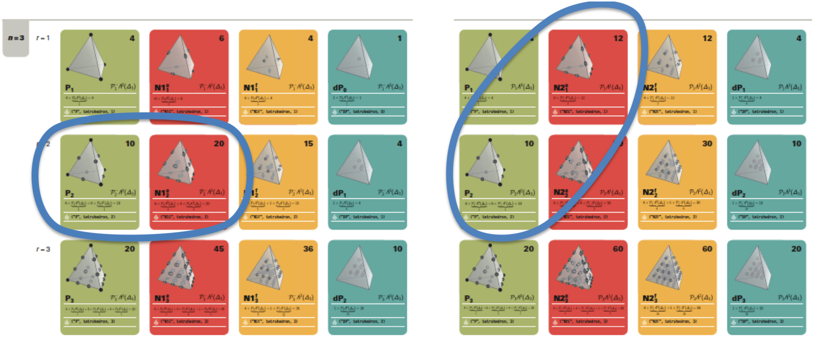
Problem: Maxwell's
Dimension: $n = 3$
Mesh type: tetrahedral
Convergence: quadratic ($r = 2$)

Stable pairs for tetrahedral meshes



Problem:	Maxwell's
Dimension:	$n = 3$
Mesh type:	tetrahedral
Convergence:	cubic ($r = 3$)

Stable pairs for tetrahedral meshes



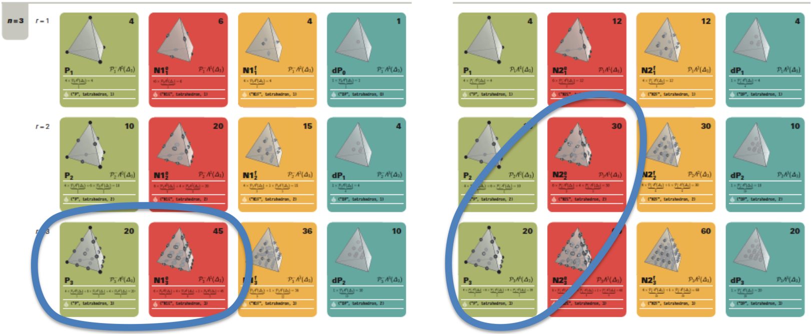
Problem: vector Poisson

Dimension: $n = 3$

Mesh type: tetrahedral

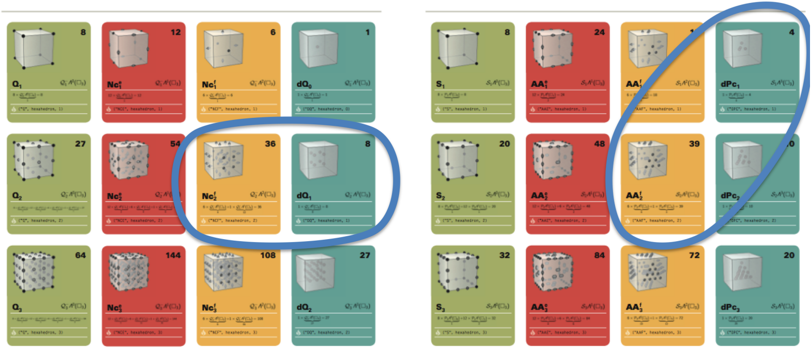
Convergence: quadratic ($r = 2$)

Stable pairs for tetrahedral meshes



Problem: vector Poisson
Dimension: $n = 3$
Mesh type: tetrahedral
Convergence: cubic ($r = 3$)

Stable pairs for cubical meshes



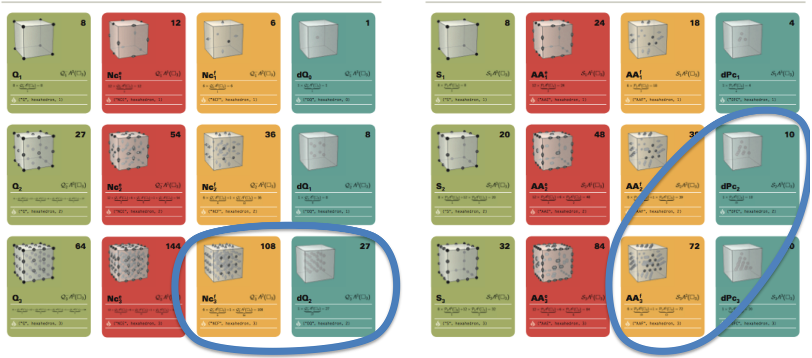
Problem: Darcy / Poisson

Dimension: $n = 3$

Mesh type: cubes

Convergence: quadratic ($r = 2$)

Stable pairs for cubical meshes



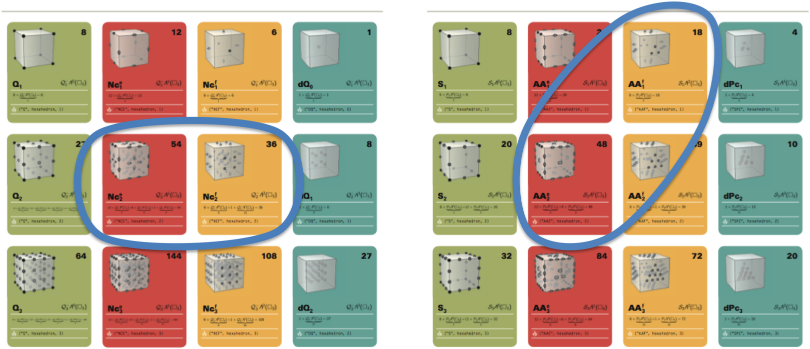
Problem: Darcy / Poisson

Dimension: $n = 3$

Mesh type: cubes

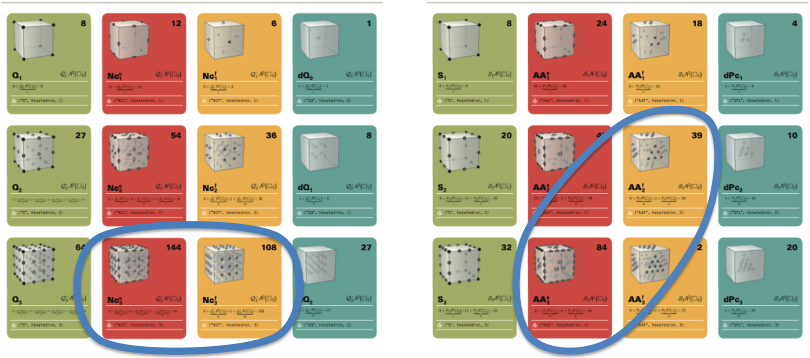
Convergence: cubic ($r = 3$)

Stable pairs for cubical meshes



Problem: Maxwell's
Dimension: $n = 3$
Mesh type: cubes
Convergence: quadratic ($r = 2$)

Stable pairs for cubical meshes



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Mesh type: cubes
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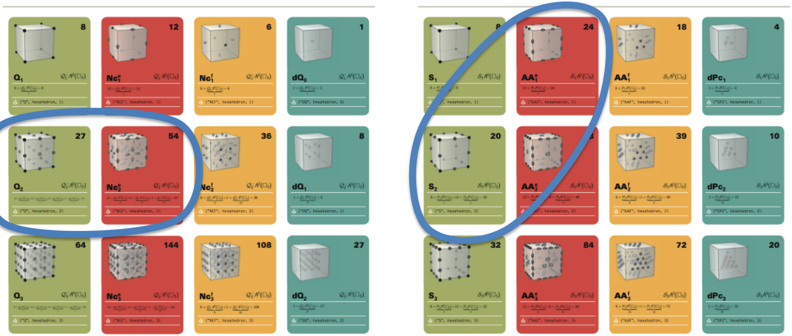
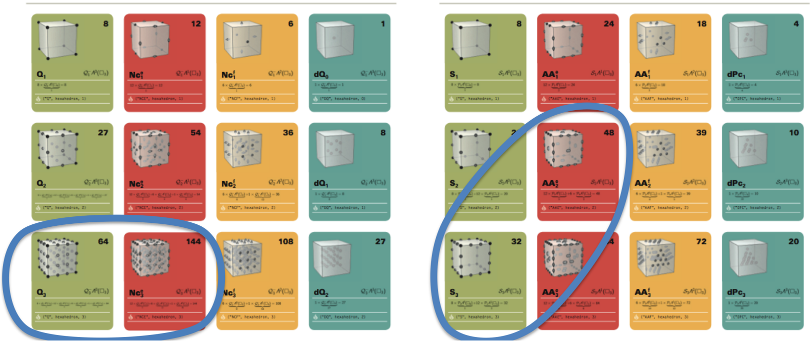
Stable pairs for cubical meshes



Problem:	vector Poisson
Dimension:	$n = 3$
Mesh type:	cubes
Convergence:	quadratic ($r = 2$)

Stable pairs for cubical meshes



Problem: vector Poisson

Dimension: $n = 3$

Mesh type: cubes

Convergence: cubic ($r = 3$)

Exact cochain complexes found in the table

On an *n*-simplex in $\mathbb{R}^n$:

$$\mathcal{P}_r^-\Lambda^0 \to \mathcal{P}_r^-\Lambda^1 \to \cdots \to \mathcal{P}_r^-\Lambda^{n-1} \to \mathcal{P}_r^-\Lambda^n$$

‘trimmed’ polynomials

$$\mathcal{P}_r\Lambda^0 \to \mathcal{P}_{r-1}\Lambda^1 \to \cdots \to \mathcal{P}_{r-n+1}\Lambda^{n-1} \to \mathcal{P}_{r-n}\Lambda^n$$

polynomials

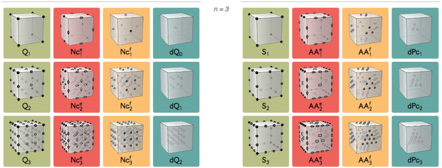
On an *n*-dimensional cube in $\mathbb{R}^n$:

$$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n$$

tensor product

$$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n$$

serendipity



The ‘minus’ spaces proceed across rows of the PToFE (***r* is fixed**) while the ‘regular’ spaces proceed along diagonals (***r* decreases**)

Mysteriously, the degree of freedom count for mixed methods from the $\mathcal{P}_r^-$ spaces is smaller than those from the $\mathcal{P}_r$ spaces, while the opposite is true for the $\mathcal{Q}_r^-$ and $\mathcal{S}_r$ spaces.

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1. Four well-known families of finite elements
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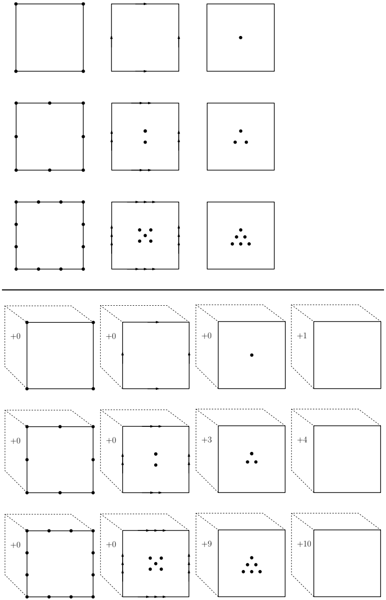
The 5th column: Trimmed serendipity spaces



A new column for the PToFE:
the **trimmed serendipity** elements.

$\boxed{\mathcal{S}_r^- \Lambda^k(\square_n)}$ denotes

approximation order r,
subset of k-form space $\Lambda^k(\Omega)$,
use on meshes of n-dim'l cubes.

Defined for any $n \geq 1$, $0 \leq k \leq n$, $r \geq 1$

Identical or analogous properties to all the other colummns in the table.

The advantage of the $\mathcal{S}_r^- \Lambda^k$ spaces is that they have fewer degrees of freedom for mixed methods than their tensor product and serendipity counterparts.

Key properties of the trimmed serendipity spaces

$$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n$$ **tensor product**

$$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n$$ **serendipity**

$$\mathcal{S}_r^-\Lambda^0 \to \mathcal{S}_r^-\Lambda^1 \to \cdots \to \mathcal{S}_r^-\Lambda^{n-1} \to \mathcal{S}_r^-\Lambda^n$$ **trimmed serendipity**

Subcomplex: $d\mathcal{S}_r^-\Lambda^k \subset \mathcal{S}_r^-\Lambda^{k+1}$

Exactness: The above sequence is exact.
i.e. the image of incoming map = kernel of outgoing map

Inclusion: $\mathcal{S}_r\Lambda^k \subset \mathcal{S}_{r+1}^-\Lambda^k \subset \mathcal{S}_{r+1}\Lambda^k$

Trace: $\mathrm{tr}_f\mathcal{S}_r^-\Lambda^k(\mathbb{R}^n) \subset \mathcal{S}_r^-\Lambda^k(f)$, for any $(n-1)$-hyperplane f in $\mathbb{R}^n$

Special cases:
$\mathcal{S}_r^-\Lambda^0 = \mathcal{S}_r\Lambda^0$
$\mathcal{S}_r^-\Lambda^n = \mathcal{S}_{r-1}\Lambda^n$
$\mathcal{S}_r^-\Lambda^k + d\mathcal{S}_{r+1}\Lambda^{k-1} = \mathcal{S}_r\Lambda^k$.

Replace '$\mathcal{S}$' by '$\mathcal{P}$' ⇝ key properties about the first two columns for $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$!

Dimension count and comparison

Formula for counting degrees of freedom of $\mathcal{S}_r^-\Lambda^k(\square_n)$:

$$\sum_{d=k}^{\min\{n,\lfloor r/2\rfloor+k\}} 2^{n-d}\binom{n}{d}\left(\binom{r-d+2k-1}{r-d+k-1}\binom{r-d+k-1}{d-k}+\binom{r-d+2k}{k}\binom{r-d+k-1}{d-k-1}\right)$$

	k	r=1	2	3	4	5	6	7
n=2	0	4	8	12	17	23	30	38
	1	4	10	17	26	37	50	65
	2	1	3	6	10	15	21	28
n=3	0	8	20	32	50	74	105	144
	1	12	36	66	111	173	255	360
	2	6	21	45	82	135	207	301
	3	1	4	10	20	35	56	84
n=4	0	16	48	80	136	216	328	480
	1	32	112	216	392	656	1036	1563
	2	24	96	216	422	746	1227	1910
	3	8	36	94	200	375	644	1036
	4	1	5	15	35	70	126	210

Mixed Method dimension comparison 1

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$$

We compare degree of freedom counts among the three families for use on meshes of affinely-mapped squares or cubes, when a conforming method with (at least) order r decay in the approximation of p, $\mathbf{u}$, and div $\mathbf{u}$ is desired.

Total # of degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1\rvert + \lvert\mathcal{Q}_r^-\Lambda^2\rvert$	$\lvert\mathcal{S}_r\Lambda^1\rvert + \lvert\mathcal{S}_{r-1}\Lambda^2\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1\rvert + \lvert\mathcal{S}_r^-\Lambda^2\rvert$
1	4+1 = 5	8+1 = 9	4+1 = 5
2	12+4 = 16	14+3 = 17	10+3 = 13
3	24+9 = 33	22+6 = 28	17+6 = 23

Total # of degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2\rvert + \lvert\mathcal{Q}_r^-\Lambda^3\rvert$	$\lvert\mathcal{S}_r\Lambda^2\rvert + \lvert\mathcal{S}_{r-1}\Lambda^3\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2\rvert + \lvert\mathcal{S}_r^-\Lambda^3\rvert$
1	6+1 = 7	18+1 = 19	6+1 = 7
2	36+8 = 44	39+4 = 43	21+4 = 25
3	108+27 = 135	72+10 = 82	45+10 = 55

Mixed Method dimension comparison 2

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$$

The number of interior degrees of freedom is reduced from tensor product, to serendipity, to trimmed serendipity:

\# of **interior** degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1_0\rvert + \lvert\mathcal{Q}_r^-\Lambda^2_0\rvert$	$\lvert\mathcal{S}_r\Lambda^1_0\rvert + \lvert\mathcal{S}_{r-1}\Lambda^2_0\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1_0\rvert + \lvert\mathcal{S}_r^-\Lambda^2_0\rvert$
1	0+1 = 1	0+1 = 1	0+1 = 1
2	4+4 = 8	2+3 = 5	2+3 = 5
3	12+9 = 21	6+6 = 12	5+6 = 11

\# of **interior** degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2_0\rvert + \lvert\mathcal{Q}_r^-\Lambda^3_0\rvert$	$\lvert\mathcal{S}_r\Lambda^2_0\rvert + \lvert\mathcal{S}_{r-1}\Lambda^3_0\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2_0\rvert + \lvert\mathcal{S}_r^-\Lambda^3_0\rvert$
1	0+1 = 1	0+1 = 1	0+1 = 1
2	12+8 = 20	3+4 = 7	3+4 = 7
3	54+27 = 81	12+10 = 22	9+10 = 19

Mixed Method dimension comparison 3

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$$

Assuming interior degrees of freedom could be dealt with efficiently (e.g. by static condensation), trimmed serendipity elements *still* have the fewest DoFs:

of **interface** (edge) degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1(\partial\square_2)\rvert$	$\lvert\mathcal{S}_r\Lambda^1(\partial\square_2)\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1(\partial\square_2)\rvert$
1	4	8	4
2	8	12	8
3	12	16	12

of **interface** (edge+face) degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2(\partial\square_3)\rvert$	$\lvert\mathcal{S}_r\Lambda^2(\partial\square_3)\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2(\partial\square_3)\rvert$
1	6	18	6
2	24	36	18
3	54	60	36

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Decomposition by polynomial subspace

$\mathcal{S}_r^-\Lambda^k(\square_n)$ is a space of differential k-forms whose coefficients are polynomials in $\mathbb{R}^n$.

$$\mathcal{S}_r^-\Lambda^k = \mathcal{P}_r^-\Lambda^k \oplus \mathcal{J}_r\Lambda^k \oplus d\mathcal{J}_r\Lambda^{k-1}$$

Polynomial coefficients in each summand:

$\mathcal{P}_r^-\Lambda^k$:	anything up to degree $r-1$ and some degree r
$\mathcal{J}_r\Lambda^k$:	certain polynomials whose degree is between $r+1$ and $r+n-k-1$
$d\mathcal{J}_r\Lambda^{k-1}$:	certain polynomials whose degree is between r and $r+n-k-2$

The “regular” serendipity space has an analogous decomposition:

$$\mathcal{S}_r\Lambda^k = \mathcal{P}_r\Lambda^k \oplus \mathcal{J}_r\Lambda^k \oplus d\mathcal{J}_{r+1}\Lambda^{k-1}$$

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This decomposition provides a direct sum into some precise but elaborate subspaces:

$$\mathcal{J}_r\Lambda^k(\mathbb{R}^n) := \sum_{l\geq 1} \kappa\mathcal{H}_{r+l-1,l}\Lambda^{k+1}(\mathbb{R}^n),$$

where $\mathcal{H}_{r,l}\Lambda^k(\mathbb{R}^n) := \left\{\omega \in \mathcal{H}_r\Lambda^k(\mathbb{R}^n) \mid \text{ldeg}\,\omega \geq l\right\},$

where $\text{ldeg}(x^\alpha dx_\sigma) := \#\{i \in \sigma^* \ : \ \alpha_i = 1\}.$

Decomposition by degrees of freedom

The **degrees of freedom** associated to a d-dimensional sub-face f of an n-dimensional cube $\square_n$ are (for any $k \leq d \leq \min\{n, \lfloor r/2 \rfloor + k\}$):

$$u \longmapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r-2(d-k)-1}\Lambda^{d-k}(f) \;\oplus\; d\mathcal{H}_{r-2(d-k)+1}\Lambda^{d-k-1}(f),$$

These degrees of freedom are **unisolvent** for $\mathcal{S}_r^-\Lambda^k(\square_n)$.

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The direct sum decomposition of the indexing space gives some insight:

$$\underbrace{\mathcal{P}_{r-2(d-k)-1}\Lambda^{d-k}(f)}_{\text{indexing space for } \mathcal{S}_{r-1}\Lambda^k(f)} \;\oplus\; \underbrace{d\mathcal{H}_{r-2(d-k)+1}\Lambda^{d-k-1}(f)}_{\text{subspace of } \mathcal{H}_{r-2(d-k)}\Lambda^{d-k}(f)}$$

. . . but this characterization only seems to aid in the proof of unisolvence.

Decomposition by Cubical Geometry

We can also decompose $\mathcal{S}_r^- \Lambda^k(\square_n)$ by the subspace of "zero trace":

$$\mathcal{S}_r^- \Lambda^k = \mathcal{S}_r^- \Lambda^k_{\mathbf{0}} \oplus \left(\mathcal{S}_r^- \Lambda^k_{\mathbf{0}} \right)^\perp$$

We use this decomposition to prove that $\mathcal{S}_r^- \Lambda^\bullet(\square_n)$ is a **minimal compatible finite element system** containing $\mathcal{P}_{r-1}\Lambda^\bullet(\square_n)$.

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A computational basis for $\mathcal{S}_r^- \Lambda^k_{\mathbf{0}}$ would aid in the construction of bases for $\mathcal{S}_r^- \Lambda^k$.

Building such a basis is non-trvial. Consider

$$\alpha := \begin{array}{r} (z-1)(y^2-1)\ dx \\ +y(x+1)(z-1)\ dy \\ +(x+1)(y^2-1)\ dz \end{array} \qquad \beta := \begin{array}{r} (z-1)(y^2-1)\ dx \\ -2y(x+1)(z-1)\ dy \\ +(x+1)(y^2-1)\ dz \end{array}$$

Both α and β have a natural association to the approximation of y on the edge

$\{x=1, z=-1\}$, and both are elements of $\mathcal{Q}_2\Lambda^1(\square_3)$. But *only* β is in $\mathcal{S}_1\Lambda^1(\square_3)$!

Decompositions shared insight

Why is $\beta \in \mathcal{S}_1\Lambda^1(\square_3)$?

$$\beta \;=\; \begin{array}{r} (z-1)(y^2-1)\;dx \\ -2y(x+1)(z-1)\;dy \\ +(x+1)(y^2-1)\;dz \end{array}$$

$$= \underbrace{\begin{array}{r} y^2z\;dx \\ -2xyz\;dy \\ xy^2\;dz \end{array} + \begin{array}{r} -y^2\;dx \\ 2xy\;dy \\ 0\;dz \end{array} + \begin{array}{r} 0\;dx \\ -2yz\;dy \\ y^2\;dz \end{array}}_{\text{basis elements for } d\mathcal{J}_2\Lambda^0} + \underbrace{\begin{array}{r} (-z+1)\;dx \\ 2y\;dy \\ (-x-1)\;dz \end{array}}_{\in\mathcal{P}_1\Lambda^1}.$$

From the polynomial subspace decompositions:

$$\beta \in d\mathcal{J}_2\Lambda^0 \oplus \mathcal{P}_1\Lambda^1 \quad \subset \mathcal{S}_1\Lambda^1$$

$$\beta \in d\mathcal{J}_2\Lambda^0 \oplus \mathcal{P}_1\Lambda^1 \quad \subset \mathcal{S}_2^-\Lambda^1$$

Full report on this approach coming soon!

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Slides and Pre-prints

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